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Robin's Inequality for Sum of Divisors Function and the Riemann Hypothesis

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Dedicated to the memory of Professor Włodzimierz Staś

Abstract. Let $\sigma(n)$ denote the sum of divisors function. In this paper we give a simple proof of the Robin inequality **(R)**: $\sigma(n) < e^{\gamma} n \log\log n$, for all positive integers $n \geq 5041$.

The Robin inequality **(R)** implies Riemann Hypothesis.

1. **Introduction**

The Riemann zeta function $\zeta(s)$ for $s = \sigma + it$ is defined by the Dirichlet series $\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which converges for $\sigma > 1$ and it has analytic continuation to the complex plane with one singularity a simple pole with residue 1.

The Riemann Hypothesis ([8]), is intimately connected with the distribution of prime numbers.The Riemann zeta-function is a special case of L-functions.

These L-functions are connected with many important and difficult problems in number theory, algebraic geometry, topology, representation theory and modern physics, see; Berry and Keating [1], Katz and Sarnak [5], and Murty [7].

In 1984 Robin [9] proved very interesting and important criterion:

Robin's criterion. *The Riemann Hypothesis is true if and only if*

$$\textbf{(R)} \qquad \sigma(n) < e^{\gamma} n \log\log n,$$

for all positive integers $n \geq 5041$*, where* $\sigma(n) = \sum_{d|n} d$ *and* γ *is Euler's constant.*

In 2002 Lagarias [6] proved some extension of the Robin criterion. Many others criteria and important results connected with the Riemann Hypothesis have been

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proved and these results have been described by Conrey in his elegant article [3]. Recently, Choie, Lichardopol, Moree and Sole' in the paper [2] proved that if $n \geq 37$ does not satisfy Robin's criterion it must be even and is neither squarefree nor squarefull, moreover that n must be divisible by fifth power > 1.

In our paper [4]; see, Theorem 1, p. 70 has been proved of the following result:

R.1. *Let $n = 2m$, $(2, m) = 1$. Then for all odd positive integers $m > \frac{3^9}{2}$ we have*

$$\sigma(2m) < \frac{39}{40} e^{\gamma} 2m \log\log 2m, \tag{1.1}$$

and

$$\sigma(m) < e^{\gamma} m \log\log m. \tag{1.2}$$

It is easy to see that from the result of **R.1** follows that for complete proof of the Robin inequality **(R)** it suffices to prove that inequality **(R)** is true for all positive integers n such that $n = 2^{\alpha} m$, $(2, m) = 1$ and $\alpha \geq 2$.

In this connection in the paper [4] has been proved the following theorem (see, Theorem 2, p. 71):

R.2. *If there exists an positive integer m_0 such that for every odd positive integer $m > m_0$, we have*

$$\sigma(2m) < \frac{3}{4} e^{\gamma} 2m \log\log 2m, \tag{1.3}$$

then for all integers $n = 2^{\alpha} m$, $(2, m) = 1$, $m > m_0$ and every fixed integer $\alpha \geq 2$ we have

$$\sigma(2^{\alpha} m) < e^{\gamma} 2^{\alpha} m \log\log 2^{\alpha} m. \tag{1.4}$$

The result described in **R.1** is cited by M. Weber in his monograph [11] on page 541.

From the result of **R.2** follows that for complete proof of the Robin inequality **(R)** it suffices to prove the following theorem:

Theorem. *For all integers $n = 2m$, $(2, m) = 1$ such that $m > \frac{1}{2} e^{e^9}$ we have*

$$\sigma(2m) < \frac{3}{4} e^{\gamma} 2m \log\log 2m. \tag{$*$}$$

We prove this Theorem in part 3 of this paper by using basic lemmas given in next part of our paper.

2. **Basic Lemmas**

Lemma 1. *Let $n = 2^\alpha m$, $(2, m) = 1$ and let $\omega(m)$ denote the number of distinct primes divisors of m. If $\omega(m) = 1$, then for every odd positive integer $m > \frac{1}{4}e^{e^2}$ and each fixed integer $\alpha \geq 2$, we have*

$$\sigma(2^\alpha m) < e^\gamma 2^\alpha m \log\log 2^\alpha m. \tag{2.1}$$

Proof. By the assumption of Lemma 1 it follows that $m = p^\beta$, where $p \geq 3$ is an odd prime and $\beta \geq 1$ is positive integer.

Moreover,by the assumption that $m > \frac{1}{4}e^{e^2}$ it follows that

$$m = p^\beta > \frac{1}{4}e^{e^2}. \tag{2.2}$$

On the other hand we have

$$\sigma(2^\alpha m) = \sigma(2^\alpha p^\beta) = \sigma(2^\alpha)\sigma(p^\beta) = (2^{\alpha+1} - 1)\frac{p^{\beta+1} - 1}{p - 1}. \tag{2.3}$$

Since $p - 1 \geq \frac{2}{3}p$ and $2^{\alpha+1} - 1 < 2^{\alpha+1}$, then from (2.3) and (2.2) it follows that

$$\sigma(2^\alpha m) < 2^{\alpha+1} \cdot \frac{p^{\beta+1}}{\frac{2}{3}p} = 3 \cdot 2^\alpha m. \tag{2.4}$$

From the assumption follows that $2^\alpha m \geq 2^2 m > e^{e^2}$, hence

$$e^\gamma \log\log 2^\alpha m > 1.6 \log\log e^{e^2} > 1.6 \times 2 > 3. \tag{2.5}$$

By (2.5) and (2.4) it follows that the inequality (2.1) holds and the proof of Lemma 1 is finished. □

Lemma 2. *Let $n = 2m_1$, $(2, m_1) = 1$ and $\omega(m_1) = 2$. Then for every odd positive integer m_1 such that $m_1 > \frac{1}{2}e^{e^3}$, we have*

$$\frac{\sigma(2m_1)}{2m_1} < \frac{3}{4}e^\gamma \log\log 2m_1. \tag{2.6}$$

Proof. By the assumption of Lemma 2 it follows that $m_1 = p_1^{\alpha_1} p_2^{\alpha_2}$. Therefore by the properties of the function σ we get,

$$\frac{\sigma(2m_1)}{2m_1} = \frac{\sigma(2)\sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2})}{2p_1^{\alpha_1} p_2^{\alpha_2}} = \frac{3}{2}\frac{(p_1^{\alpha_1+1} - 1)(p_2^{\alpha_2+1} - 1)}{p_1^{\alpha_1} p_2^{\alpha_2}(p_1 - 1)(p_2 - 1)}. \tag{2.7}$$

Since $p_1 \geq 3$ and $p_2 \geq 5$ then we have,

$$p_1 - 1 \geq \frac{2}{3}p_1, \quad p_2 - 1 \geq \frac{4}{5}p_2. \tag{2.8}$$

From (2.7) and (2.8) we obtain

$$\frac{\sigma(2m_1)}{2m_1} < \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{3}{4} \cdot \frac{15}{4}. \tag{2.9}$$

Since $e^\gamma > 1.6$ and $2m_1 > e^{e^3}$, then we get

$$e^\gamma \log\log 2m_1 > 1.6 \log\log e^{e^3} = 1.6 \times 3 = 4.8 > \frac{15}{4}. \tag{2.10}$$

From (2.10) and (2.9) we obtain the inequality (2.6) and the proof of Lemma 2 is complete. $\square$

Lemma 3. *Let* $n = \prod_{j=1}^{k} p_j^{\alpha_j}$ *and let* $I(n) = \prod_{j=1}^{k}\left(1 - \frac{1}{p_j^{1+\alpha_j}}\right)$, *where* p_j *are primes and* $\alpha_j \geq 1$ *are integers for* $j = 1, 2, \ldots, k$ *and let* σ, φ *be the sum of divisors function and Euler's totient function, respectively. Then we have*

$$\frac{\sigma(n)}{n} = I(n)\frac{n}{\varphi(n)}. \tag{2.11}$$

The proof of Lemma 3 is given in our paper [4], see; Lemma 2 on page 69.

Lemma 4 (Rosser-Schoenfeld's inequality [1], Theorem 15; Cf. [4], Lemma 1, p. 69)**.** *Let* φ *be the Euler's totient function. Then for all positive integers* $n \geq 3$, *the following inequality is true*

$$\frac{n}{\varphi(n)} \leq e^\gamma\left(\log\log n + \frac{2.5}{e^\gamma \log\log n}\right), \tag{\textbf{R-S}}$$

except, when $n = 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 = 223\ 092\ 870$
and this case the constant $c = 2.5$ *must be replaced by the constant* $c_1 = 2.50637 < 2.51$.

Now, we prove the following Lemma:

Lemma 5. *Let* $n = 2m_1$, $(2, m_1) = 1$ *and* $\omega(m_1) = 2$. *Then for all integers* $2m_1 > e^{e^9}$ *we have*

$$I(m_1) < \frac{50}{51}. \tag{2.12}$$

Proof. From Lemma 3 and the definition of the function $I(n)$ for case $n = 2m_1 = 2 \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2}$, we have

$$I(n) = I(2m_1) = \left(1 - \frac{1}{2^{1+1}}\right)\left(1 - \frac{1}{p_1^{1+\alpha_1}}\right)\left(1 - \frac{1}{p_2^{1+\alpha_2}}\right) = I(2) \cdot I(m_1). \tag{2.13}$$

By (2.13) it follows that

$$0 < I(m_1) < 1 \quad \Leftrightarrow \quad I(m_1) \in (0,1) = \left(0, \frac{50}{51}\right) \cup \left[\frac{50}{51}, 1\right). \tag{2.14}$$

From (2.14) and the assumption of Lemma 5 and Lemma 3 follows that the inequality (2.6) is true, so denote that we have

$$\text{(2.15)}\quad \text{If } \sigma(2m_1) < \frac{3}{4}e^{\gamma}2m_1\log\log 2m_1,\ \text{then } I(m_1) \in \left(0, \frac{50}{51}\right) \text{ or } I(m_1) \in \left[\frac{50}{51}, 1\right).$$

Suppose that

$$\text{(2.16)}\quad \text{If } \sigma(2m_1) < \frac{3}{4}e^{\gamma}2m_1\log\log 2m_1,\ \text{then } I(m_1) \in \left[\frac{50}{51}, 1\right),$$

so denote that (2.16) is equivalent to

$$\text{(2.17)}\quad \text{If } \sigma(2m_1) < \frac{3}{4}e^{\gamma}2m_1\log\log 2m_1,\ \text{then } \frac{50}{51} \le I(m_1) < 1.$$

Applying to (2.17) well-known the law of contraposition we get,

$$\text{if } I(m_1) < \frac{50}{51},\ \text{then } \sigma(2m_1) \ge \frac{3}{4}e^{\gamma}2m_1\log\log 2m_1.$$

From the identity (2.11) of Lemma 3 and the fact that the function $I(n)$ is multiplicative function, we obtain

$$\text{(2.18)}\quad \frac{\sigma(2m_1)}{2m_1} = \left(1 - \frac{1}{2^2}\right)I(m_1)\frac{2m_1}{\varphi(2m_1)} = \frac{3}{4}I(m_1)\frac{2m_1}{\varphi(2m_1)},$$

because, in this case $n = 2m_1$.

Applying to the $\dfrac{2m_1}{\varphi(2m_1)}$ of (2.18) the Rosser-Schoenfeld's inequality (R-S) from Lemma 4, with constant $c_1 < 2.51$ we get,

$$\text{(2.19)}\quad \frac{\sigma(2m_1)}{2m_1} < \frac{3}{4}I(m_1)e^{\gamma}\log\log 2m_1\left(1 + \frac{2.51}{e^{\gamma}(\log\log 2m_1)^2}\right).$$

Since $2m_1 > e^{e^9}$ then we have

$$\text{(2.20)}\quad e^{\gamma}(\log\log 2m_1)^2 > 1.6 \times 81 = 129.6 > 125.5 = 50 \times 2.51.$$

From (2.20) we obtain

$$\text{(2.21)}\quad 1 + \frac{2.51}{e^{\gamma}(\log\log 2m_1)^2} < 1 + \frac{2.51}{50 \times 2.51} = 1 + \frac{1}{50} = \frac{51}{50}.$$

By (2.19) and (2.21) it follows that

$$\text{(2.22)}\quad \frac{\sigma(2m_1)}{2m_1} < \frac{3}{4}I(m_1)\frac{51}{50}e^{\gamma}\log\log 2m_1.$$

From the assumption that $I(m_1) < \dfrac{50}{51}$ and (2.22) we get that the (2.16) is impossible.

The proof of Lemma 5 is complete. □

3. Proof of the Theorem

The proof of the Theorem we give by the induction with respect to $k = \omega(m)$.

From Lemma 2 it follows that the Theorem is true when $k = 2$. Suppose that the Theorem is true for all m such that $\omega(m) < k$.

We prove our Theorem for m, when $\omega(m) = k$.

Let $n = 2m = 2m_1 \cdot M$, $(2, m) = 1$, $m = m_1 \cdot M$; $(m_1, M) = 1$, where $\omega(m_1) = 2, \omega(m) = \omega(m_1 \cdot M) = k > 2$.

Then from the identity (2.11) of Lemma 3 and the Rosser-Schoenfeld's inequality (R-S) of Lemma 4 we obtain

$$\begin{aligned} \text{(3.1)} \qquad \frac{\sigma(n)}{n} &= \frac{\sigma(2m_1 \cdot M)}{2m_1 \cdot M} \\ &< \frac{3}{4} \cdot I(m_1 \cdot M) e^{\gamma} \log\log 2m_1 \cdot M \left(1 + \frac{2,51}{e^{\gamma}(\log\log 2m_1 \cdot M)^2}\right). \end{aligned}$$

By Lemma 5 and the formula (2.11) of Lemma 3 it follows that

$$\text{(3.2)} \qquad I(m) = I(m_1 \cdot M) = I(m_1) \cdot I(M) < \frac{50}{51},$$

because the function $I(m)$ is multiplicative function. Hence, from (3.1) and (3.2) follows that

$$\text{(3.3)} \qquad \frac{\sigma(n)}{n} = \frac{\sigma(2m)}{2m} < \frac{3}{4} e^{\gamma} \cdot \frac{50}{51} \left(1 + \frac{2,51}{e^{\gamma}(\log\log 2m)^2}\right) \log\log 2m.$$

From (3.3) and (2.21) it follows that for all integers m, such that $2m > e^{e^9}$, we have

$$\text{(3.4)} \qquad \frac{\sigma(2m)}{2m} < \frac{3}{4} e^{\gamma} \log\log 2m,$$

and the proof of the Theorem is complete.

Remark. By Proposition 1 and 4 of [9] it is enough, to prove RH, to derive Robin inequality (R) for n large enough and therefore no need of computer check.

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