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Extensions of Lattice Set Functions to Regular Borel Measures

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Abstract. This paper deals with the unique extension of a finite regular set function from the δ-lattice of all compact G_δ-subsets of a locally compact Hausdorff space to a finite regular measure on the δ-ring of all relatively compact Borel subsets of the space. This extension is a *two-step method* because it is performed (without density assumptions) via the δ-ring of all relatively compact Baire subsets of the space.

1. Introduction

Throughout the paper $(X,\mathscr{G})$ denotes a completely regular Hausdorff space, and $C(X)$ the set of all continuous functions $f: X \to [0,1]$. The classes

$$\mathscr{G}^* = \{\{x \in X : f(x) \neq 0\} : f \in C(X)\}$$

and

$$\mathscr{F}^* = \{\{x \in X : f(x) = 0\} : f \in C(X)\}$$

are the classes of the co-zero (or, functionally open) and zero (or, functionally closed) subsets of X, respectively.

The σ-algebra generated by $\mathscr{G}$ is denoted by $\mathscr{B}$ and the σ-algebra generated by $\mathscr{G}^*$ is denoted by $\mathscr{B}^*$. The elements of $\mathscr{B}$ and $\mathscr{B}^*$ are the *Borel* and *Baire* subsets of X, respectively. (In sections 2 and 3, the terms Borel and Baire will be used in a more restrictive sense.) Of course, $\mathscr{B}^* \subseteq \mathscr{B}$, and $\mathscr{B}^* \neq \mathscr{B}$ in general, see [15, p. 108]. This definition of Baire sets is apparently due to Hewitt [8]. Unless otherwise specified, measures are non-negative and countably additive. A *Baire* (*Borel*) measure on X is a measure defined on the σ-algebra $\mathscr{B}^*$ ($\mathscr{B}$). The Borel extension problem asks: Given a Baire measure, when can it be extended to a Borel measure?

Recall that a topological space X is called *countably paracompact* if X is a Hausdorff space and every countable open cover of X has a locally finite open refinement, see [6, p. 392]. In 1956, Mař k [9] proved that all normal countably

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paracompact spaces have the following property: Every finite Baire measure on X admits a unique extension to a regular Borel measure on X. (The main results of that paper appeared in 1957 in a shorter English version [10]). In [15, § 9], Wheeler fully reviewed a number of interesting topics relating to the problem of when a Baire measure can be extended to a regular Borel measure. He defined ([15, p. 131, Definition 9.1]) a completely regular Hausdorff space X to be a *Mař k space* if every Baire measure on X admits an extension to a regular Borel measure, and asked several questions thereupon. In [12], Ohta and Tamano answered the questions about Mař k spaces asked by Wheeler [15] and studied their topological properties. While answering these questions, Ohta and Tamano [12, p. 401], introduced the notion of *quasi-Mař k spaces*: A space X is called a *quasi-Mař k space* if each Baire measure on X admits an extension to a (not necessarily regular) Borel measure on X. Neglecting the regularity of the extension allowed Ohta and Tamano to get much stronger results, and they also posed the following open question, [12, p. 401]: Is every quasi-Mař k space a Mař k space? In [1, p. 277, Theorem 2.1], Aldaz proved that a countably metacompact quasi-Mař k space is Mař k, where a space X is called *countably metacompact* if X is Hausdorff and every countable open cover of X has a point-finite open refinement, see also [6, p. 399]. Every paracompact space is metacompact, but not conversely; when normal, countable metacompactness is equivalent to countable paracompactness, [6, p. 399].

In order to search for an example of a quasi-Mař k space which is not Mař k, Aldaz [1] worked among the class of spaces that are not countably metacompact. A space X is called a *Dowker space* if X is normal a nd not countably paracompact, see [14, p. 765, Theorem 1.2]. Such a space is, of course, also not countably metacompact. A space is *almost Dowker* if it is regular but not countably metacompact. In order to give an example of a quasi-Mař k space which is not Mař k, Aldaz [1] used the special set-theoretic hypothesis ♣ from [14, p. 768]: For every limit ordinal $\alpha < \omega_1$, there is a sequence S_α (order isomorphic with ω), cofinal in α, such that every uncountable subset of ω_1 contains some S_α. Then, Aldaz used the Dowker space appearing in [14, pp. 768–769], to prove that, under the assumption ♣, there exists a normal quasi-Mař k space which is not Mař k.

In [12], Ohta and Tamano gave an example of a locally compact space which is not quasi-Mař k. The result is that the Borel extension problem, as phrased above, is not in general applicable to a locally compact space X, unless one deals with Borel and Baire subsets of X in a more restrictive sense. That is what we shall be dealing with in the remainder of this paper.

2. Terminology and Basic Results

We shall need the following detail from [5, §5], about the extension of measures. Let $\mathscr{D}$ be a ring of subsets of X and $\rho : \mathscr{D} \to \mathbb{R}$ a finite measure. Extend ρ from $\mathscr{D}$ to the hereditary σ-ring $\mathscr{H}(\mathscr{D})$ generated by $\mathscr{D}$ by the usual Carath odory method to obtain an outer measure ρ^* on $\mathscr{H}(\mathscr{D})$. Let $\mathscr{T}(\rho)$ be the σ-ring of all sets $E \in \mathscr{H}(\mathscr{D})$ having the property $\rho^*(A) = \rho^*(A \cap E) + \rho^*(A \setminus E)$ for every set $A \in \mathscr{H}(\mathscr{D})$. Then ρ^* is a unique σ-finite measure on $\mathscr{T}(\rho)$ and $\mathscr{D} \subseteq \mathscr{T}(\rho)$. Let $\mathscr{M}(\rho)$ be the class of all sets $E \in \mathscr{P}$ that are locally in $\mathscr{T}(\rho)$, that is, $E \in \mathscr{M}(\rho)$ if and only if $E \cap A \in \mathscr{T}(\rho)$ for every set $A \in \mathscr{T}(\rho)$. Then $\mathscr{M}(\rho)$ is the σ-algebra of all ρ-measurable subsets of X, and $\mathscr{T}(\rho) \subseteq \mathscr{M}(\rho)$. The following extension of the measure ρ^* on $\mathscr{T}(\rho)$ to a measure on $\mathscr{M}(\rho)$ is due to I. Segal. Let $\nu = \rho^*|\mathscr{T}(\rho)$. Then ν is a measure on $\mathscr{T}(\rho)$ and its variation $\overline{\nu}$ is a measure on $\mathscr{M}(\rho)$, and by [5, p. 39, Corollary 2], satisfies $\overline{\nu}(E) = \nu(E) = \rho^*(E)$ for $E \in \mathscr{T}(\rho)$ and, for $E \in \mathscr{P}$,

$$\overline{\nu}(E) = \sup\{\rho^*(A) : A \subseteq E \text{ and } A \in \mathscr{T}(\rho)\}.$$

Extend the measure ρ^* on $\mathscr{T}(\rho)$ to a measure, again denoted by ρ^*, on $\mathscr{M}(\rho)$:

$$\rho^*(E) = \sup\{\rho^*(A) : A \subseteq E \text{ and } A \in \mathscr{T}(\rho)\}$$

for every set $E \in \mathscr{M}(\rho)$.

Unless specifically stated otherwise, $(X, \mathscr{G})$ will henceforth be a locally compact Hausdorff space. Of course, $(X, \mathscr{G})$ is then a completely regular space. The symbols $\mathscr{P}, \mathscr{F}$ and $\mathscr{K}$ denote the classes of all subsets, all closed subsets and all compact subsets of X, respectively. If $A \in \mathscr{P}$, then its $\mathscr{G}$-closure will be denoted by $\overline{A}$. Clearly, $\mathscr{G}^* \subseteq \mathscr{G}$ and $\mathscr{F}^* \subseteq \mathscr{F}$. The class $\mathscr{G}^*$ (which is a lattice) is a base for $\mathscr{G}$, since $(X, \mathscr{G})$ is a completely regular Hausdorff space, see [6, p. 65]. Denote the δ-lattice of all compact G_δ-subsets of X by $\mathscr{K}_\delta$ and let $\mathscr{B}_0$ be the δ-ring generated by $\mathscr{K}_\delta$. Then $\mathscr{K}_\delta \subseteq \mathscr{F}^*$, see [7, p. 217, Theorem C]. The class of all open F_σ-subsets of X is denoted by $\mathscr{G}_\sigma$, and the class of all closed G_δ-subsets of X is denoted by $\mathscr{F}_\delta$. If $A \in \mathscr{F}^*$, then $A = f^{-1}(\{0\}) = \bigcap_{n=1}^{\infty} f^{-1}\big((-\frac{1}{n}, \frac{1}{n})\big)$, which shows that $A \in \mathscr{F}_\delta$. Consequently, $\mathscr{K}_\delta \subseteq \mathscr{F}^* \subseteq \mathscr{F}_\delta \subseteq \mathscr{F}$.

For every pair $(K, G) \in \mathscr{K}_\delta \times \mathscr{G}^*$, let $I(K, G) = \{A \in \mathscr{P} : K \subseteq A \subseteq G\}$ and let $\mathscr{I} = \{I(K, G) : (K, G) \in \mathscr{K}_\delta \times \mathscr{G}^*\}$. Following [5, p. 302], the class $I(K, G)$ is called an interval with origin K and extremity G. Note that: $\mathscr{P} = I(\emptyset, X) \in \mathscr{I}$; $I(K, G) = \emptyset \Leftrightarrow K \nsubseteq G$; $I(K, G) \neq \emptyset \Leftrightarrow K \subseteq G$. It is clear that $\mathscr{I}$ is a base for a topology, say $\mathscr{U}$, on $\mathscr{P}$. A subcollection $\mathscr{C} \subseteq \mathscr{P}$ is $\mathscr{U}$-dense in $(\mathscr{P}, \mathscr{U})$ if for every pair $(K, G) \in \mathscr{K}_\delta \times \mathscr{G}^*$, where $K \subseteq G$, there exists a set $A \in \mathscr{C}$ such that $K \subseteq A \subseteq G$. A set $A \in \mathscr{K}_\delta$ is said to be $\mathscr{U}$-regular on the left (right) with respect to a set function $\alpha : \mathscr{K}_\delta \to \mathbb{R}$ if α is continuous on the left (right) in A for the topology $\mathscr{U}$, that is, if for every number $\varepsilon > 0$ there exists a set $K \in \mathscr{K}_\delta$, where $K \subseteq A$ (a set $G \in \mathscr{G}^*$, where $G \supseteq A$) such that for every set $A' \in \mathscr{K}_\delta$ with $K \subseteq A' \subseteq A$ ($A \subseteq A' \subseteq G$) we have $| \alpha(A) - \alpha(A') | < \varepsilon$. A set $A \in \mathscr{K}_\delta$ is said to be $\mathscr{U}$-regular with respect to a set function $\alpha : \mathscr{K}_\delta \to \mathbb{R}$ if α is continuous in A for the topology $\mathscr{U}$, that is, if for every number $\varepsilon > 0$ there exists an interval $I(K, G) \in \mathscr{I}$ such that $\alpha(I(K, G) \cap \mathscr{K}_\delta) \subseteq (\alpha(A) - \varepsilon, \alpha(A) + \varepsilon)$. The set function $\alpha : \mathscr{K}_\delta \to \mathbb{R}$ is said to be $\mathscr{U}$-regular on $\mathscr{K}_\delta$ if every set $A \in \mathscr{K}_\delta$ is $\mathscr{U}$-regular with respect to α, that is, if α is continuous on $\mathscr{K}_\delta$ for the topology $\mathscr{U}$.

The above can be repeated for the class $\mathscr{J} = \{I(K,G) : (K,G) \in \mathscr{K} \times \mathscr{G}\}$, which in turn is also a base for a topology, say $\mathscr{V}$, on $\mathscr{P}$. Regularity (left, right) of a set function $\alpha : \mathscr{K} \to \mathbb{R}$ with respect to the topology $\mathscr{V}$, is treated as above, with the appropriate changes being made, that is, use $\mathscr{K}$, $\mathscr{G}$ and $\mathscr{V}$ instead of $\mathscr{K}_\delta$, $\mathscr{G}^*$ and $\mathscr{U}$, respectively; this case shall be referred to as $\mathscr{V}$-regularity. We know that $\mathscr{U} \subseteq \mathscr{V}$, therefore a set function $\alpha : \mathscr{K}_\delta \to \mathbb{R}$ that is $\mathscr{U}$-regular on $\mathscr{K}_\delta$ is also $\mathscr{V}$-regular on $\mathscr{K}_\delta$.

Denote the δ-ring generated by the lattice $\mathscr{K}$ by the symbol $\mathscr{B}_1$. If $A \in \mathscr{B}_1$ then $A \subseteq \cup_{i=1}^{n} K_i$, with $K_i \in \mathscr{K}$ for $i = 1, 2, \ldots, n$; see [5, p. 6, Proposition 10]. Since every set $A \in \mathscr{B}_1$ is contained in a compact set, $\mathscr{B}_1$ consists of relatively compact subsets of X. The δ-ring $\mathscr{B}_1$ also contains all the relatively compact open subsets of X, see [5, p. 287, Corollary]; the class $\mathscr{B}_1$ will be called the δ-ring of the *relatively compact Borel subsets* of X. Obviously, $\mathscr{B}_0 \subseteq \mathscr{B}_1$, hence the sets of $\mathscr{B}_0$ are relatively compact Borel sets, and $\mathscr{B}_0$ will be called the δ-ring of the *relatively compact Baire subsets* of X. (In Halmos [7], the Borel sets of X are the sets belonging to the σ-ring generated by $\mathscr{K}$, and the Baire sets of X are the sets belonging to the σ-ring generated by $\mathscr{K}_0$. The definition of Baire sets used by Ross and Stromberg [13, p. 151] is consistent with that of Halmos whenever X is σ-compact and locally compact.)

The following results will be employed in section 3.

Lemma 2.1 ([5, p. 294, Proposition 11]). *Let X be a locally compact Hausdorff space, $K \in \mathscr{K}$ and $G \in \mathscr{G}$ with $K \subseteq G$. There exists a set $K_0 \in \mathscr{K}_\delta$ and a set $G_0 \in \mathscr{G}_\sigma \cap \mathscr{B}_0$ such that*

$$K \subseteq G_0 \subseteq K_0 \subseteq G.$$

Proof. By the hypothesis on X, there is a set $U \in \mathscr{G}$ with $\overline{U}$ compact that satisfies $K \subseteq U \subseteq \overline{U} \subseteq G$. Since X is Tychonoff, the exists a function $f \in C(X)$ such that $f(x) = 1$ on K and $f(x) = 0$ on $X \setminus U$. Put $G_0 = \left\{x \in X : f(x) > \frac{1}{2}\right\}$. Then $G_0 = \cup_{n=1}^{\infty} \left\{x \in X : f(x) \geq 2 + \frac{1}{n}\right\}$, so that $G_0 \in \mathscr{G}_\sigma$. Let $K_0 = \left\{x \in X : f(x) \geq \frac{1}{2}\right\}$. Then K_0 is a closed G_δ-set, see [7, p. 217, Theorem C]. Clearly, $K \subseteq G_0 \subseteq K_0 \subseteq U \subseteq G$. This means that K_0 is compact. The sets $C_n = \left\{x \in X : f(x) \geq \frac{1}{2} + \frac{1}{n}\right\}$ are closed G_δ-sets for every $n \in \mathbb{N}$, and because $C_n \subseteq K_0$ for every $n \in \mathbb{N}$, it follows that $C_n \in \mathscr{K}_\delta$ for every $n \in \mathbb{N}$. Since $C_n \in \mathscr{B}_0$ for every $n \in \mathbb{N}$ and $C_n \subseteq K_0 \in \mathscr{K}_\delta \subseteq \mathscr{B}_0$, it follows from [5, p. 4, Definition 3(4)] that $G_0 \in \mathscr{B}_0$. This completes the proof. □

Lemma 2.2 ([11, p. 360; 3.2]). *Let X be a completely regular Hausdorff space and $G_n \in \mathscr{G}^*$ for $n = 1, 2, 3, \ldots$. Then $\cup_{n=1}^{\infty} G_n \in \mathscr{G}^*$.*

Proof. Let $f_n \in C(X)$ and $G_n = f_n^{-1}(\mathbb{R} \setminus \{0\})$ for $n = 1, 2, 3, \ldots$. Let $f = \sum_{n=1}^{\infty} 2^{-n} f_n$. Then $f \in C(X)$ and $\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} f_n^{-1}((-\infty, 0) \cup (0, \infty)) = f_n^{-1}(\mathbb{R} \setminus \{0\}) \in \mathscr{G}^*$. Consequently, $\mathscr{F}^*$ is closed under countable intersections. □

3. **Extensions of Regular Set Functions**

The two results of this section are based on [5, p. 339, Theorem 1 and p. 347, Theorem 2], both adapted considerably to fit the objectives of this paper. The hypotheses of both of these theorems in [5] have been simplified. Furthermore, the density assumptions in both theorems mentioned above are redundant since Lemma 2.1 of the present paper guarantees all that is needed about density for our results in this section.

Theorem 3.1. *Let X be a locally compact Hausdorff space and $\alpha : \mathscr{K}_\delta \to \mathbb{R}$ a finite set function. If α is* (1) *monotone;* (2) *finitely subadditive;* (3) *finitely additive;* (4) *$\mathscr{U}$-regular on $\mathscr{K}_\delta$, then*

(a) *α is countably additive on $\mathscr{K}_\delta$;*
(b) *α can be extended uniquely to a $\mathscr{U}$-regular measure μ on the δ-ring $\mathscr{B}_0$;*
(c) *for every μ-measurable set $A \in \mathscr{M}(\mu)$, we have*

$$\mu^*(A) = \sup\{\mu(K) : K \subseteq A \text{ and } K \in \mathscr{K}_\delta\}$$

and for every set $A \in \mathscr{B}_0$, we have

$$\mu(A) = \inf\{\mu^*(G) : A \subseteq G \text{ and } G \in \mathscr{G}^*\}.$$

Proof. The uniqueness of the extension follows from [5, p. 24, Proposition 6]. By the finite subadditivity of α it follows that if $K \in \mathscr{K}_\delta$, then $\alpha(\emptyset \cup K) = \alpha(\emptyset) + \alpha(K)$, and we deduce that $\alpha(\emptyset) = 0$. If $K, K' \in \mathscr{K}_\delta$ then it follows from conditions (1) and (2) that $\alpha(K') \le \alpha(K \cup K') \le \alpha(K) + \alpha(K')$, so that $\alpha(K) \ge 0$ for every $K \in \mathscr{K}_\delta$. The rest of the proof is subdivided into eleven parts.

(i) Denote the variation of α on $\mathscr{P}$ by $\bar{\alpha}$. Since $\mathscr{K}_\delta$ is a lattice on which α is positive and finitely additive, it follows from [5, p. 38, Proposition 6] that for every set $A \in \mathscr{P}$

$$\bar{\alpha}(A) = \sup\{\alpha(K) : K \subseteq A \text{ and } K \in \mathscr{K}_\delta\}.$$

Since α is increasing it follows that

$$\bar{\alpha}(K) = \alpha(K) \text{ for every set } K \in \mathscr{K}_\delta,$$

therefore, $\bar{\alpha}$ is an extension of α from $\mathscr{K}_\delta$ to $\mathscr{P}$. It follows from [5, §3] that $\bar{\alpha}$ is positive, increasing and superadditive.

We want to show that $\bar{\alpha}$ is countably additive and $\mathscr{U}$-regular on $\mathscr{B}_0$, so that the restriction $\mu = \bar{\alpha} \mid \mathscr{B}_0$ is the measure we need.

(ii) For every set $K \in \mathscr{K}_\delta$ we claim that

$$\alpha(K) = \inf\{\bar{\alpha}(G) : K \subseteq G \text{ and } G \in \mathscr{G}^*\}.$$

To establish this equality, let $K \in \mathscr{K}_\delta$ and let $\varepsilon > 0$. By condition (4), there exists a set $G \in \mathscr{G}^*$ with $K \subseteq G$ such that for every set $K' \in \mathscr{K}_\delta$ with $K \subseteq K' \subseteq G$, we have $\alpha(K') - \alpha(K) < \varepsilon$, that is, $\alpha(K') < \alpha(K) + \varepsilon$. Let K''

be any set in $\mathscr{K}_\delta$ with $K'' \subseteq G$. Then, if $K' = K \cup K''$, then $K \subseteq K' \subseteq G$, and $\alpha(K'') < \alpha(K) + \varepsilon$, therefore

$$\begin{aligned}\bar{\alpha}(G) &= \sup\{\alpha(B) : B \subseteq G \text{ and } B \in \mathscr{K}_\delta\} \\ &\leq \alpha(K) + \varepsilon,\end{aligned}$$

whence

$$\alpha(K) = \inf\{\bar{\alpha}(G) : K \subseteq G \text{ and } G \in \mathscr{G}^*\}.$$

(iii) To establish the countable subadditivity of $\bar{\alpha}$ on $\mathscr{G}^*$, we consider first a finite subclass $\{G_1, G_2, \ldots, G_n\}$ of $\mathscr{G}^*$ and their union G; then $G \in \mathscr{G}^*$. Let $K \subseteq G$, with $K \in \mathscr{K}_\delta$. Let x be any element of K; then $\{x\} \in \mathscr{K}$, and $\{x\}$ is a subset of one of the sets $G_i, 1 \leq i \leq n$. By Lemma 2.1, there exists a set $K_0^x \in \mathscr{K}_\delta$ and a set $G_0^x \in \mathscr{G}_\sigma \cap \mathscr{B}_0$ such that

$$\{x\} \subseteq G_0^x \subseteq K_0^x \subseteq G_i$$

for some i, $1 \leq i \leq n$. Then K can be covered by a finite subclass $\{K_0^{x_j} : j = 1, 2, \ldots, m\}$ of the class $\{K_0^x : x \in K\}$. Let K_i be the union of all those sets $K_0^{x_j}$ which are contained in G_i. Then $K_i \in \mathscr{K}_\delta$, $i = 1, 2, \ldots, n$, and also, $\cup_{i=1}^n K_i \in \mathscr{K}_\delta$ because $\mathscr{K}_\delta$ is a lattice and

$$K \subseteq \bigcup_{i=1}^n K_i \subseteq \bigcup_{i=1}^n G_i = G,$$

and

$$\alpha(K) \leq \alpha\left(\bigcup_{i=1}^n K_i\right) \leq \sum_{i=1}^n \alpha(K_i) \leq \sum_{i=1}^n \bar{\alpha}(G_i).$$

Then

$$\bar{\alpha}(G) = \sup\{\alpha(K) : K \subseteq G \text{ and } K \in \mathscr{K}_\delta\} \leq \sum_{i=1}^n \bar{\alpha}(G_i).$$

Let now $\{G_n : n \in \mathbb{N}\}$ be a subclass of $\mathscr{G}^*$ and put $G = \cup_{i=1}^\infty G_i$. Then $G \in \mathscr{G}^*$ by Lemma 2.2. Let $K \in \mathscr{K}_\delta$ such that $K \subseteq G$. Then K is covered by a finite subclass, $\{G_1, G_2, ..., G_r\}$, say, of the given class $\{G_n : n \in \mathbb{N}\}$. Then

$$\alpha(K) \leq \bar{\alpha}\left(\bigcup_{i=1}^r G_i\right) \leq \sum_{i=1}^r \bar{\alpha}(G_i) \leq \sum_{i=1}^\infty \bar{\alpha}(G_i)$$

by what have been established above, whence,

$$\bar{\alpha}(G) \leq \sum_{i=1}^\infty \bar{\alpha}(G_i),$$

showing that $\bar{\alpha}$ is countably subadditive on $\mathscr{G}^*$. Since $\bar{\alpha}$ is also countably superadditive on $\mathscr{G}^*$, we deduce that $\bar{\alpha}$ is countably additive on $\mathscr{G}^*$.

(iv) We now show that for every set $K \in \mathscr{K}_\delta$ and for every set $G \in \mathscr{G}^*$ for which $K \subseteq G$, we have

$$\bar{\alpha}(G \setminus K) = \bar{\alpha}(G) - \alpha(K).$$

For this purpose, let $\varepsilon > 0$. By part (ii), there exists a set $G_\varepsilon \in \mathscr{G}^*$, such that $K \subseteq G_\varepsilon$ and

$$\alpha(K) \leq \bar{\alpha}(G_\varepsilon) < \alpha(K) + \varepsilon.$$

Now, $G \setminus K = G \cap (X \setminus K) \in \mathscr{G}^*$, because $\mathscr{K}_\delta \subseteq \mathscr{F}^*$. Also, $G = (G \setminus K) \cup K \subseteq (G \setminus K) \cup G_\varepsilon$. Since $\bar{\alpha}$ is increasing and subadditive on the class $\mathscr{G}^*$, we have

$$\bar{\alpha}(G) \leq \bar{\alpha}(G \setminus K) + \bar{\alpha}(G_\varepsilon) < \bar{\alpha}(G \setminus K) + \alpha(K) + \varepsilon.$$

Then

$$\bar{\alpha}(G) \leq \bar{\alpha}(G \setminus K) + \alpha(K),$$

and from the superadditivity of $\bar{\alpha}$,

$$\bar{\alpha}(G) = \bar{\alpha}(G \setminus K) + \alpha(K).$$

Therefore,

$$\bar{\alpha}(G \setminus K) = \bar{\alpha}(G) - \alpha(K).$$

(v) Let

$$\Phi = \{A \subseteq T : \bar{\alpha}(A) < \infty \text{ and } \bar{\alpha}(A) = \inf\{\bar{\alpha}(G) : A \subseteq G \text{ and } G \in \mathscr{G}^*\}\}.$$

Since $\bar{\alpha}$ is increasing,

$$\bar{\alpha}(A) = \inf\{\bar{\alpha}(G) : A \subseteq G \text{ and } G \in \mathscr{G}^*\}$$

for every set $A \in \mathscr{G}^*$. Then a set $A \in \mathscr{G}^*$ belongs to Φ if and only if $\bar{\alpha}(A) < \infty$. From parts (i) and (ii), $\mathscr{K}_\delta \subseteq \Phi$. By part (i), for every set $A \in \Phi$,

$$\bar{\alpha}(A) = \sup\{\alpha(K) : K \subseteq A \text{ and } K \in \mathscr{K}_\delta\}.$$

Therefore, a set A belongs to Φ if and only if for every $\varepsilon > 0$, there exists a set $K_\varepsilon \in \mathscr{K}_\delta$ and a set $G_\varepsilon \in \mathscr{G}^*$ such that $K_\varepsilon \subseteq A \subseteq G_\varepsilon$ and $\bar{\alpha}(G_\varepsilon) - \bar{\alpha}(K_\varepsilon) < \varepsilon$. By part (iv), $\bar{\alpha}(G_\varepsilon \setminus K_\varepsilon) < \varepsilon$. This shows that $\bar{\alpha}$ is $\mathscr{U}$-regular on Φ.

(vi) We show that Φ is a δ-ring containing $\mathscr{B}_0$. Let $A_1, A_2 \in \Phi$ and let $\varepsilon > 0$ be arbitrary. Then $\bar{\alpha}(A_1) < \infty$, $\bar{\alpha}(A_2) < \infty$, and there are sets $K_1, K_2 \in \mathscr{K}_\delta$ and $G_1, G_2 \in \mathscr{G}^*$ such that $K_i \subseteq A_i \subseteq G_i$, $\bar{\alpha}(G_i) < \infty$ and $\bar{\alpha}(G_i \setminus K_i) < \frac{\varepsilon}{2}$ for $i = 1, 2$. Then $K = K_1 \setminus G_2 = K_1 \cap (X \setminus G_2) \in \mathscr{K}_\delta$, because $\mathscr{K}_\delta \subseteq \mathscr{F}^* \subseteq \mathscr{F}_\delta$, see section 2; also, if $G = G_1 \setminus K_2$, then $G \in \mathscr{G}^*$, and $\bar{\alpha}(G) \leq \bar{\alpha}(G_1) < \infty$. Also, $K = K_1 \setminus G_2 = K_1 \cap (X \setminus G_2) \subseteq A_1 \cap (X \setminus G_2) \subseteq A_1 \cap (X \setminus A_2) = A_1 \setminus A_2 \subseteq G_1 \cap (X \setminus A_2) \subseteq G_1 \cap (X \setminus K_2) = G_1 \setminus K_2 = G$, and $G \setminus K \subseteq (G_1 \setminus K_1) \cup (G_2 \setminus K_2)$. Since $G_1 \setminus K_1, G_2 \setminus K_2 \in \mathscr{G}^*$, we deduce that $\bar{\alpha}(G \setminus K) \leq \bar{\alpha}(G_1 \setminus K_1) + \bar{\alpha}(G_2 \setminus K_2) < \varepsilon$. By part (v), $A_1 \setminus A_2 \in \Phi$. Furthermore, $K' = K_1 \cup K_2 \in \mathscr{K}_\delta$, $G' = G_1 \cup G_2 \in \mathscr{G}^*$ and $K' \subseteq A_1 \cup A_2 \subseteq G'$, and $G' \setminus K' \subseteq (G_1 \setminus K_1) \cup (G_2 \setminus K_2)$, so that $\bar{\alpha}(G' \setminus K') < \varepsilon$.

This means that $A_1 \cup A_2 \in \Phi$. We now invoke [5, p. 5, Proposition 8] to show that $\mathscr{K}_\delta$ is closed under countable intersections. Firstly, let $(A_i : i \in \mathbb{N})$ be a sequence in Φ contained in a set $B \in \Phi$, and let $A = \cup_{i=1}^{\infty} A_i$. We show that $A \in \Phi$. Put $B_1 = A_1, B_2 = A_2 \setminus B_1, \ldots, B_i = A_i \setminus \cup_{j=1}^{i-1} B_j, \ldots,\ i \geq 3$. Then $B_i \in \Phi$, $B_i \cap B_j = \emptyset$ if $i \neq j$ and $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i = A$. Let $U \in \mathscr{G}^*$ with $\bar{\alpha}(U) < \infty$ such that $B \subseteq U$. Then $B_i \subseteq U$ for all $i \in \mathbb{N}$. Let $\varepsilon > 0$. For every $i \in \mathbb{N}$, there exist a set $K_i \in \mathscr{K}_\delta$ and a set $G_i \in \mathscr{G}^*$ such that

$$K_i \subseteq B_i \subseteq G_i,\ \bar{\alpha}(G_i) < \infty \text{ and } \bar{\alpha}(G_i \setminus K_i) < \frac{\varepsilon}{2^{i+1}},$$

by part (v). We can choose the sets G_i to be contained in U. Then $G = \cup_{i=1}^{\infty} G_i \subseteq U$ and therefore, $\bar{\alpha}(G) \leq \bar{\alpha}(U) < \infty$. The sets K_i are disjoint, so by the countable superadditivity and monotonicity of $\bar{\alpha}$ on $\mathscr{P}$,

$$\sum_{i=1}^{\infty} \bar{\alpha}(K_i) \leq \bar{\alpha}\left(\bigcup_{i=1}^{\infty} K_i\right) \leq \bar{\alpha}(G) < \infty,$$

whence, because $\bar{\alpha}(G_i) - \bar{\alpha}(K_i) = \bar{\alpha}(G_i \setminus K_i)$, it follows that

$$\sum_{i=1}^{\infty} \bar{\alpha}(G_i) = \sum_{i=1}^{\infty} \bar{\alpha}(G_i \setminus K_i) + \sum_{i=1}^{\infty} \bar{\alpha}(K_i) \leq \varepsilon + \sum_{i=1}^{\infty} \bar{\alpha}(K_i) < \infty.$$

Let $n \in \mathbb{N}$ be such that $\sum_{i>n} \bar{\alpha}(G_i) < \frac{\varepsilon}{2}$. Then $K = \cup_{i=1}^{n} K_i \in \mathscr{K}_\delta$, $K \subseteq A \subseteq G$ and

$$\begin{aligned} G \setminus K &\subseteq \left(\bigcup_{i=1}^{\infty} G_i\right) \Bigg\backslash \left(\bigcup_{i=1}^{n} K_i\right) = \left(\bigcup_{i=1}^{n} G_i\right) \Bigg\backslash \left(\bigcup_{i=1}^{n} K_i\right) \bigcup \left(\bigcup_{i>n} G_i\right) \\ &= \left(\bigcup_{i=1}^{n} (G_i \setminus K_i)\right) \bigcup \left(\bigcup_{i>n} G_i\right), \end{aligned}$$

thus

$$\begin{aligned} \bar{\alpha}(G \setminus K) &\leq \bar{\alpha}\left[\left(\bigcup_{i=1}^{n} (G_i \setminus K_i)\right) \bigcup \left(\bigcup_{i>n} G_i\right)\right] \\ &\leq \sum_{i=1}^{n} \bar{\alpha}(G_i \setminus K_i) + \sum_{i>n} \bar{\alpha}(G_i) \\ &\leq \sum_{i=1}^{\infty} \bar{\alpha}(G_i \setminus K_i) + \sum_{i>n} \bar{\alpha}(G_i) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Because $G \in \mathscr{G}^*$ by Lemma 2.2, it follows from part (v) that $A \in \Phi$. Consider now, in the second place, an arbitrary sequence $(A_i : i \in \mathbb{N})$ of sets in Φ. Then $A_1 \setminus A_i \in \Phi$ for every $i \in \mathbb{N}$, and also $A_1 \setminus A_i \subseteq A_1$ for every $i \in \mathbb{N}$. Then

$\cup_{i=1}^{\infty}(A_1 \setminus A_i) \in \Phi$ from what has been established above. Then

$$\bigcap_{i=1}^{\infty} A_1 = A_1 \Bigg\backslash \left(\bigcup_{i=1}^{\infty} (A_1 \setminus A_i) \right) \in \Phi$$

and we conclude that Φ is a δ-ring. Obviously, $\mathscr{B}_0 \subseteq \Phi$.

(vii) To prove that $\bar{\alpha}$ is countably additive on Φ, let $(A_i : i \in \mathbb{N})$ be a sequence of disjoint sets in Φ and let $\cup_{i=1}^{\infty} A_i = A \in \Phi$. Then $\bar{\alpha}(\cup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} \bar{\alpha}(A_i)$, since $\bar{\alpha}$ is superadditive on Φ. To prove the converse, let $\varepsilon > 0$. For each $i \in \mathbb{N}$, there exists a set $G_i \in \mathscr{G}^*$ such that $A_i \subseteq G_i$ and

$$\bar{\alpha}(G_i) < \bar{\alpha}(A_i) + \frac{\varepsilon}{2^i}.$$

Since $A \in \Phi$, $\cup_{i=1}^{\infty} G_i \in \mathscr{G}^*$ and $\bar{\alpha}$ is countably subadditive on $\mathscr{G}^*$ by (iii), we have

$$\bar{\alpha}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \bar{\alpha}\left(\bigcup_{i=1}^{\infty} G_i\right) \leq \sum_{i=1}^{\infty} \bar{\alpha}(G_i) \leq \sum_{i=1}^{\infty} \bar{\alpha}(A_i) + \varepsilon,$$

whence, ε being arbitrary,

$$\bar{\alpha}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \bar{\alpha}(A_i).$$

The result follows.

(viii) Since $\mathscr{K}_\delta \subseteq \Phi$, from part (v) and $\bar{\alpha} = \alpha$ on $\mathscr{K}_\delta$, α is countably additive on $\mathscr{K}_\delta$, proving (a).

(ix) Claim: $\bar{\alpha}$ is complete on Φ. To prove this, let $A \in \Phi$ with $\bar{\alpha}(A) = 0$ and let $B \subseteq A$. For every $\varepsilon > 0$, there exists a set $G_\varepsilon \in \mathscr{G}^*$ with $A \subseteq G_\varepsilon$ and $\bar{\alpha}(G_\varepsilon) < \varepsilon$. Since $B \subseteq G_\varepsilon$ and $\bar{\alpha}$ is increasing on $\mathscr{P}$, we deduce that $B \in \Phi$ and $\bar{\alpha}(B) = 0$.

(x) Let $\mu = \bar{\alpha}|\mathscr{B}_0$. Since $\mathscr{K}_\delta \subseteq \mathscr{B}_0 \subseteq \Phi$, we deduce that μ is a finite measure on $\mathscr{B}_0$ and is an extension of α from $\mathscr{K}_\delta$ to $\mathscr{B}_0$, by part (i). We also deduce from part (v) that μ is $\mathscr{U}$-regular on $\mathscr{B}_0$, whereby (b) is finally established.

(xi) We now proof (c). We show that $\mathscr{G}^* \subseteq \mathscr{M}(\mu)$. For this purpose, let $\mathscr{R}$ be the ring generated by $\mathscr{K}_\delta$ and let

$$\mathscr{L}(\mathscr{R}) = \{E \in \mathscr{P} : E \cap A \in \mathscr{R} \text{ for every set } A \in \mathscr{R}\}.$$

Then

$$\mathscr{R} \subseteq \mathscr{B}_0 \subseteq \mathscr{T}(\mu) \subseteq \mathscr{M}(\mu).$$

Now, by invoking [5, p. 70, Corollary 2],

$$\begin{aligned} E \in \mathscr{L}(\mathscr{R}) &\Rightarrow E \cap A \in \mathscr{R} \text{ for every set } A \in \mathscr{R} \\ &\Rightarrow E \cap A \in \mathscr{B}_0 \text{ for every set } A \in \mathscr{R} \\ &= E \cap A \in \mathscr{T}(\mu) \text{ for every set } A \in \mathscr{R} \\ &\Rightarrow E \in \mathscr{M}(\mu), \end{aligned}$$

showing that $\mathscr{L}(\mathscr{R}) \subseteq \mathscr{M}(\mu)$. Let now $G \in \mathscr{G}^*$, and let $K \in \mathscr{K}_\delta$. Then $K \setminus G = K \cap (X \setminus G) \in \mathscr{K}_\delta$ because $X \setminus G \in \mathscr{F}^* \subseteq F_\delta$. So, $G \cap K = K \setminus (K \setminus G) \in \mathscr{R}$. If $A \in \mathscr{R}$ is arbitrary, then by [5, p. 9, Corollary],

$$A = \bigcup_{i=1}^{n} (K_i \setminus K_i')$$

where $K_i, K_i' \in \mathscr{K}_\delta$. Then

$$G \cap A = \bigcup_{i=1}^{n} \big((K_i \cap G) \setminus K_i'\big) \in \mathscr{R}.$$

This shows that $G \in \mathscr{L}(\mathscr{R})$, from which we deduce that $\mathscr{G}^* \subseteq \mathscr{M}(\mu)$. For every set $A \in \mathscr{B}_0$,

$$(1) \qquad \mu(A) = \inf\{\bar{\alpha}(G) : A \subseteq G \text{ and } G \in \mathscr{G}^*\} < \infty$$

by parts (v), (vi) and (x). By [5, p. 73, Proposition 7], and parts (v) and (i) of this paper, for any set $A \in \mathscr{M}(\mu)$, and since $\mathscr{B}_0$ is the δ-ring generated by $\mathscr{B}_0$,

$$\begin{aligned}(2) \qquad \mu^*(A) &= \sup\{\mu(B) \ : \ B \subseteq A \text{ and } B \in \mathscr{B}_0\} \\ &= \sup\{\sup\{\alpha(K) : K \subseteq B \text{ and } K \in \mathscr{K}_\delta\} : B \subseteq A \text{ and } B \in \mathscr{B}_0\} \\ &= \sup\{\alpha(K) \ : \ K \subseteq A \text{ and } K \in \mathscr{K}_\delta\} \\ &= \bar{\alpha}(A).\end{aligned}$$

Since $\mathscr{K}_\delta \subseteq \mathscr{B}_0$,

$$\mu^*(A) = \sup\{\mu(K) \quad K \subseteq A \text{ and } K \in \mathscr{K}_\delta\}$$

for any set $A \in \mathscr{M}(\mu)$, and from (1) and (2) above,

$$\mu(A) = \inf\{\mu^*(G) : A \subseteq G \text{ and } G \in \mathscr{G}^*\}$$

for any set $A \in \mathscr{B}_0$. This completes the proof. □

The finite $\mathscr{U}$-regular measure $\mu : \mathscr{B}_0 \to \mathbb{R}$ obtained in Theorem 3.1 will be called a *Baire measure*. In Theorem 3.2 below we extend this Baire measure μ to a $\mathscr{V}$-regular measure $\mu_1 : \mathscr{B}_1 \to \mathbb{R}$; μ_1 will be referred to as a *Borel measure*.

Theorem 3.2. *The Baire measure $\mu : \mathscr{B}_0 \to \mathbb{R}$ obtained in Theorem 3.1 can be uniquely extended to a finite positive $\mathscr{V}$-regular Borel measure $\mu_1 : \mathscr{B}_1 \to \mathbb{R}$. Furthermore,*

(a) *for every set $K \in \mathscr{K}$,*

$$\mu_1(K) = \inf\{\mu(A) : K \subseteq A \textit{ and } A \in \mathscr{B}_0\};$$

(b) *for every set $G \in \mathscr{G}$,*

$$\mu_1^*(G) = \sup\{\mu(A) : A \subseteq G \textit{ and } A \in \mathscr{B}_0\};$$

(c) *for every set* $A \in \mathscr{B}_0$,

$$\mu(A) = \sup\{\mu_1(K) : K \subseteq A \text{ and } K \in \mathscr{K}\}$$
$$= \inf\{\mu_1^*(G) : A \subseteq G \text{ and } G \in \mathscr{G}\}.$$

Proof. Let $K \in \mathscr{K}$. By Lemma 2.1 there are sets $A \in \mathscr{B}_0$ such that $K \subseteq A$. Put

$$\beta(K) = \inf\{\mu(A) : K \subseteq A \text{ and } A \in \mathscr{B}_0\}.$$

We show that β has the following four properties on $\mathscr{K}$:

(1) β *is increasing:* $\beta(K_1) \leq \beta(K_2)$ if $K_1 \subseteq K_2$.
This follows from the definition of β.

(2) β *is finitely subadditive:* $\beta(K_1 \cup K_2) \leq \beta(K_1) + \beta(K_2)$.
Let $\varepsilon > 0$. There are sets $A_1, A_2 \in \mathscr{B}_0$ such that

$$\mu(A_1) < \beta(K_1) + \frac{\varepsilon}{2} \text{ and } \mu(A_2) < \beta(K_2) + \frac{\varepsilon}{2}.$$

Then $K_1 \cup K_2 \subseteq A_1 \cup A_2 \in \mathscr{B}_0$ and

$$\beta(K_1 \cup K_2) \leq \mu(A_1 \cup A_2) \leq \mu(A_1) + \mu(A_2) \leq \beta(K_1) + \beta(K_2) + \varepsilon.$$

Because ε is arbitrary, $\beta(K_1 \cup K_2) \leq \beta(K_1) + \beta(K_2)$.

(3) β *is finitely additive:* $\beta(K_1 \cup K_2) = \beta(K_1) + \beta(K_2)$ if $K_1 \cap K_2 = \emptyset$.
Let $K_1, K_2 \in \mathscr{K}$, with $K_1 \cap K_2 = \emptyset$. Then because X is a locally compact Hausdorff space there are disjoint sets $U_1, U_2 \in \mathscr{G}$ and also disjoint relatively compact sets $G_1, G_2 \in \mathscr{G}$ such that

$$K_1 \subseteq G_1 \subseteq \overline{G}_1 \subseteq U_1 \text{ and } K_2 \subseteq G_2 \subseteq \overline{G}_2 \subseteq U_2.$$

Then, $G_1, G_2 \in \mathscr{B}_1$. By Lemma 2.1, we can find two sets $B_1, B_2 \in \mathscr{K}_\delta$ such that

$$K_1 \subseteq B_1 \subseteq G_1 \text{ and } K_2 \subseteq B_2 \subseteq G_2.$$

Let $A \in \mathscr{B}_0$ be such that $K_1 \cup K_2 \subseteq A$. Put $A_1 = A \cap B_1 \in \mathscr{B}_0$ and $A_2 = A \cap B_2 \in \mathscr{B}_0$. Then $K_1 \subseteq A_1$, $K_2 \subseteq A_2$, $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 \subseteq A$. Then

$$\beta(K_1) + \beta(K_2) \leq \mu(A_1) + \mu(A_2) = \mu(A_1 \cup A_2) \leq \mu(A),$$

and so $\beta(K_1) + \beta(K_2) \leq \beta(K_1 \cup K_2)$. The converse inequality follows from (2) above.

(4) β *is* $\mathscr{V}$*-regular on* $\mathscr{K}$*:*
Let $K \in \mathscr{K}$ and let $\varepsilon > 0$. There exists a set $A \in \mathscr{B}_0$ such that $K \subseteq A$ and

$$\mu(A) < \beta(K) + \frac{\varepsilon}{2}.$$

Because μ is $\mathscr{U}$-regular on $\mathscr{B}_0$, we can find sets $C \in \mathscr{K}_\delta$ and $G \in \mathscr{G}^*$, with $C \subseteq A \subseteq G$, such that if $A' \in \mathscr{B}_0$ and $C \subseteq A' \subseteq G$, then $\mid \mu(A) - \mu(A') \mid < \frac{\varepsilon}{2}$, therefore

$$\mu(A') < \mu(A) + \frac{\varepsilon}{2}.$$

Let $K' \in \mathscr{K}$ such that $K \subseteq K' \subseteq G$. Choose a set $A' \in \mathscr{B}_0$ such that $C \cup K' \subseteq A' \subseteq G$; this is possible by Lemma 2.1. Then

$$\beta(K') \leq \beta(C \cup K') \leq \mu(A') \leq \mu(A) + \frac{\varepsilon}{2} \leq \beta(K) + \varepsilon,$$

therefore,

$$\beta(K') - \beta(K) < \varepsilon,$$

hence β is $\mathscr{V}$-regular on the class $\mathscr{K}$ of compact sets.

Now apply [5, p. 339, Theorem 1], to β on the lattice $\mathscr{K}$ to deduce the existence of a unique finite positive $\mathscr{V}$-regular Borel measure $\mu_1 : \mathscr{B}_1 \to \mathbb{R}$ such that $\mu_1(K) = \beta(K)$ for every set $K \in \mathscr{K}$. Now, (a) follows from the definition of β. We now show that $\mu_1 = \mu$ on $\mathscr{B}_0$. Denote the variation of μ_1 on $\mathscr{P}$ by $\bar{\mu}_1$. By [5, p. 320, Proposition 26], for any $A \in \mathscr{P}$,

$$(*) \qquad \bar{\mu}_1(A) = \sup\{\mu_1(K) : K \subseteq A \text{ and } K \in \mathscr{K}\}.$$

Let now $A \in \mathscr{B}_0$. Then, $\mu_1(K) = \beta(K) \leq \mu(A)$ if $K \subseteq A$ and $K \in \mathscr{K}$, thus, $\bar{\mu}_1(A) \leq \mu(A)$. Let now $\varepsilon > 0$. Since μ is $\mathscr{U}$-regular on $\mathscr{B}_0$, we can repeat the arguments in (4) above to find a set $C \in \mathscr{K}_\delta$ and a set $G \in \mathscr{G}^*$, where $C \subseteq A \subseteq G$, such that if $A' \in \mathscr{B}_0$ and $C \subseteq A' \subseteq G$, then

$$\mu(A) < \mu(A') + \frac{\varepsilon}{2}.$$

Choose a set $A_0 \in \mathscr{B}_0$ such that $C \subseteq A_0 \subseteq G$, by Lemma 2.1 again. There exists a set $A_1 \in \mathscr{B}_0$, $C \subseteq A_1$, such that

$$\mu(A_1) < \beta(C) + \frac{\varepsilon}{2}.$$

Then $C \subseteq A_0 \cap A_1 \subseteq G$ and $A_0 \cap A_1 \in \mathscr{B}_0$, consequently,

$$\begin{aligned}\mu(A) &\leq \mu(A_0 \cap A_1) + \frac{\varepsilon}{2} \\ &\leq \mu(A_1) + \frac{\varepsilon}{2} \\ &\leq \beta(C) + \varepsilon \\ &= \mu_1(C) + \varepsilon \leq \bar{\mu}_1(A) + \varepsilon.\end{aligned}$$

Since ε is arbitrary, $\mu(A) \leq \bar{\mu}_1(A)$, therefore, $\mu(A) = \bar{\mu}_1(A)$ for $A \in \mathscr{B}_0$. This equality together with (*) proves the first part of (c). Also, since μ_1 is positive on the δ-ring $\mathscr{B}_1$, we have that $\mu_1 = \bar{\mu}_1$ on $\mathscr{B}_1$, consequently, $\mu = \mu_1$ on $\mathscr{B}_0$. Since $\mu = \beta = \mu_1$ on $\mathscr{K}_\delta$ and μ_1 is the unique extension of β to a finite positive $\mathscr{V}$-regular measure on $\mathscr{B}_1$, it follows that μ_1 is the unique extension of μ to a finite positive $\mathscr{V}$-regular measure on $\mathscr{B}_1$.

In order to prove (b) let

$$\mathscr{M}(\mathscr{B}_1) = \{E \in \mathscr{P} : E \cap A \in \mathscr{B}_1 \text{ for every set } A \in \mathscr{B}_1\}.$$

Then $\mathscr{M}(\mathscr{B}_1)$ is a σ-algebra containing $\mathscr{B}_1$, and by [5, p. 291, Corollary 1], $\mathscr{G} \subseteq \mathscr{M}(\mathscr{B}_1)$. Since $\mathscr{B}_1 \subseteq \mathscr{T}(\mu_1) \subseteq \mathscr{H}(\mathscr{B}_1)$, every set $E \in \mathscr{T}(\mu_1)$ can be covered by a sequence of sets from $\mathscr{B}_1$. Since $\mathscr{T}(\mu_1)$ is a σ-ring, it follows from [5, p. 13, Proposition 21], that $\mathscr{M}(\mathscr{B}_1) \subseteq \mathscr{M}(\mu_1)$. This shows that all open sets are μ_1-measurable. Let $G \in \mathscr{G}$. By Lemma 2.1, for every set $K \in \mathscr{K}$, there exists a set $A \in \mathscr{B}_0$ such that $K \subseteq A \subseteq G$. Then $\overline{\mu}_1(K) \leq \overline{\mu}_1(A) = \mu(A) \leq \overline{\mu}_1(G)$. Hence, by (*) above,

$$(**) \qquad \begin{aligned} \overline{\mu}_1(G) &= \sup\{\mu_1(K) : K \subseteq G \text{ and } K \in \mathscr{K}\} \\ &= \sup\{\overline{\mu}_1(K) : K \subseteq G \text{ and } K \in \mathscr{K}\} \\ &\leq \sup\{\mu_1(A) : A \subseteq G \text{ and } A \in \mathscr{B}_0\} \\ &\leq \overline{\mu}_1(G) = \mu_1^*(G). \end{aligned}$$

We now prove the second part of (c). Let $A \in \mathscr{B}_0$ and let $\varepsilon > 0$. Since μ is $\mathscr{U}$-regular on $\mathscr{B}_0$, there exist sets $K \in \mathscr{K}_\delta$ and $G \in \mathscr{G}^*$, with $K \subseteq A \subseteq G$, such that if $A' \in \mathscr{B}_0$ and $K \subseteq A' \subseteq G$, then $|\mu(A) - \mu(A')| < \frac{\varepsilon}{2}$, hence

$$\mu(A') \leq \mu(A) + \frac{\varepsilon}{2}.$$

By $(**)$ above, there exists a set $A_1 \in \mathscr{B}_0$ with $A_1 \subseteq G$ and

$$\mu_1^*(G) < \mu(A_1) + \frac{\varepsilon}{2}.$$

Then $K \subseteq A \cup A_1 \subseteq G$ and $A \cup A_1 \in \mathscr{B}_0$, therefore

$$\mu(A \cup A_1) \leq \mu(A) + \frac{\varepsilon}{2},$$

hence

$$\mu_1^*(G) < \mu(A_1) + \frac{\varepsilon}{2} \leq \mu(A \cup A_1) + \frac{\varepsilon}{2} \leq \mu(A) + \varepsilon.$$

Then,

$$\mu_1^*(A) = \mu(A) = \inf\{\mu_1^*(G) : A \subseteq G \text{ and } G \in \mathscr{G}\}.$$

This completes the proof. □

Remark 3.3. Of the numerous publications on lattice measures and their extensions, we would like to mention three. In [2], Aldaz and Render develop an approach to the measure extension problem that is based on nonstandard analysis. They introduce the class of thick topological spaces, which includes all locally compact and all K-analytic spaces, and show that if $(X, \mathscr{G})$ is regular, Lindel f and thick, $\mathscr{A} \subseteq \mathscr{B}$ is a sub-σ-algebra, and $\nu : \mathscr{A} \to \mathbb{R}$ is a finite measure, inner regular with respect to the closeds sets in $\mathscr{A}$, then ν has a Radon extension. Bachman and Sultan [3] construct a single general abstract measure procedure dealing with some of the extensions of regular measures from lattices of sets to larger classes of sets. They obtain Mař k's result ([9], [10]) as a corollary to one of their more general results. They also show ([3, p. 547]) that if X is a locally compact

Hausdorff space, then every countably additive regular measure $\mu : \mathscr{K}_0 \to \mathbb{R}$ can be extended uniquely to a countably additive regular measure $\nu : \mathscr{K} \to \mathbb{R}$, but in their case, both $\mathscr{K}_0$ and $\mathscr{K}$ contain X as an element. Finally, the unique extension of the finite regular set function $\alpha : \mathscr{K}_\delta \to \mathbb{R}$ to the $\mathscr{V}$-regular measure $\mu_1 : \mathscr{B}_1 \to \mathbb{R}$ in the present paper differs from the method followed by Bourbaki [4, §4, pp. 53–59], mainly due to our two-step method.

References

[1] J.M. Aldaz, Borel extensions of Baire measures, *Fund. Math.* **154** (1997), 275–293.

[2] J.M. Aldaz and H. Render, Borel measure extensions of measures defined on sub-σ-algebras, *Adv. Math.* **150** (2000), 233–263.

[3] G. Bachman and A. Sultan, Extensions of regular lattice measures with topological applications, *J. Math. Anal. Appl.* **57** (1977), 539–559.

[4] N. Bourbaki, *Elements of Mathematics. Integration I*, Chapters 1–6, Springer-Verlag, Heidelberg, 2004.

[5] N. Dinculeanu, *Vector Measures*, Pergamon Press, Oxford, 1967.

[6] R. Engelking, *General Topology*, PWN-Polish Scientic Publishers, Warsaw, 1977.

[7] P. Halmos, *Measure Theory*, Van Nostrand Co., Princeton, 1950.

[8] E. Hewitt, Linear functionals on spaces of continuous functions, *Fund. Math.* **37** (1950), 161–189.

[9] J. Mař k, The Baire and Borel measure, *Časopis Pěst. Mat.* **81** (1956), 431-450 (in Czech).

[10] J. Mař k, The Baire and Borel measure, *Czechoslovak Math. J.* **7**(82) (1957), 248–253.

[11] I. Netuka, Measure and topology: Mař k spaces, *Math. Bohemia* **121** (1996), 357–367.

[12] H. Ohta and K-I. Tamano, Topological spaces whose Baire measure admits a regular Borel extension, *Trans. Amer. Math. Soc.* **317** (1990), 393–415.

[13] K.A. Ross and and K. Stromberg, Baire sets and Baire measures, *Ark. Mat.* **6** (1965), 151–160.

[14] M.E. Rudin, Dowker spaces, in *Handbook of Set-Theoretic Topology*, K. Kunen and J.E. Vaughan (editors), North-Holland, Amsterdam, 1988, 761-780.

[15] R.F. Wheeler, A survey of Baire measures and strict topologies, *Exposition. Math.* **2** (1983), 97–190.

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