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Research Article

# Remarks on Common Fixed Point Results in $C^*$-Algebra-Valued Metric Spaces

Tanusri Senapati and Lakshmi Kanta Dey*

*Department of Mathematics, National Institute of Technology, Durgapur, West Bengal, India*
***Corresponding author:** lakshmikdey@yahoo.co.in

**Abstract.** In this short note, we study the article of Xin *et al.* [*J. Nonlinear Sci. Appl.* 9 (2016)] and unexpectedly notice that the common fixed point results of this article do not produce any new result in literature. In fact the main results of this article coincide with some consequences of previous published results.

**Keywords.** $C^*$-algebra; Common fixed point

**MSC.** 47H10; 54H25

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## 1. Introduction

In 2014, Ma *et al.* [5] introduced the concept of $C^*$-algebra-valued metric space and presented some fixed point results for mappings satisfying contractive or expansive conditions in this space. Many researchers have already done their research in this structure but surprisingly Kadelburg and Radenović [4] and Alsulami *et al.* [2] observed that the all these results in this structure can be directly deduced as consequences of different fixed point results in standard metric and other related structures of metric counterpart. Recently, Xin *et al.* [7] presented common fixed point results on $C^*$-algebra-valued metric spaces. They established the following result.

**Theorem 1.1.** *Let $(X,\mathscr{A},d)$ be a complete $C^*$-algebra-valued metric space. Suppose that two mappings $T,S:X\to X$ satisfy*

$$d(Tx,Sy)\preceq a^*d(x,y)a$$

*for any $x,y\in X$ and $a\in\mathscr{A}$ with $\|a\|<1$. Then $T$ and $S$ have a unique common fixed point in $X$.*

In this article, we show that actually the mapping $T$ is identical with $S$, i.e., $Tx=Sx$ for all $x\in X$. Hence, the Theorem 1.1 coincides with the result of Ma *et al*. [5]. Also, the authors of [7] proved the following theorem as a corollary.

**Theorem 1.2.** *Let $(X,\mathscr{A},d)$ be a complete $C^*$-algebra-valued metric space. Suppose that the mapping $T:X\to X$ satisfies*

$$d(T^mx,T^ny)\preceq a^*d(x,y)a$$

*for any $x,y\in X$; $a\in\mathscr{A}$ with $\|a\|<1$ and $m,n$ are any positive integers. Then $T$ has a unique fixed point in $X$.*

In next section, we show that every point $x\in X$, $T^nx$ is a fixed point of $T$ whenever $m>n$.

In 2007, Huang and Zhang [3] introduced the concept of cone metric spaces. Later on, Radenović and Kadelburg [6] showed that every cone metric space $(X,d)$ with a normal solid cone and normal constant $K=1$ is identical with standard metric space. Hence, the common fixed point results in cone metric spaces presented by Abbas and Jungck [1] also hold if we consider the underlying space as standard metric space. Here, we consider the common fixed point results of Abbas and Jungck [1] in the context of standard metric spaces.

**Theorem 1.3.** *Let $(X,d)$ be a standard metric space. Suppose that the mappings $T,S:X\to X$ satisfy either of the following conditions:*

(C1) $d(Tx,Ty)\le kd(Sx,Sy)$,

(C2) $d(Tx,Ty)\le k[d(Tx,Sx)+d(Ty,Sy)]$,

(C3) $d(Tx,Ty)\le k[d(Tx,Sy)+d(Ty,Sx)]$

*for any $x,y\in X$ and $k\in[0,1)$ for (C1) contraction and $k\in\left[0,\frac{1}{2}\right)$ for rest of the contractions. If $R(T)\subset R(S)$ and $R(S)$ is complete in $X$, then $S$ and $T$ have unique point of coincidence in $X$. Also, if $S$ and $T$ are weakly compatible, then there exists a unique common fixed point of $S$ and $T$ in $X$.*

## 2. Main Result

In this section, we show that the results presented by Xin *et al*. [7] do not produce any new idea in literature.

**Theorem 2.1.** *Let $(X,\mathscr{A},d)$ be a $C^*$-algebra-valued metric space. Suppose that two mappings $T,S:X\to X$ satisfy*

$$d(Tx,Sy)\preceq a^*d(x,y)a$$

*for any $x,y\in X$ and $a\in\mathscr{A}$ with $\|a\|<1$. Then $Tx=Sx$ for all $x\in X$.*

*Proof.* Let $x \in X$. Then from the hypothesis of the theorem we have

$$\begin{aligned} & d(Tx,Sx) \preceq a^* d(x,x)a \\ \Rightarrow \quad & d(Tx,Sx) \preceq \theta. \end{aligned}$$

Thus we have that for all $x \in X$, $Tx = Sx$. Hence, $S$ and $T$ are identical. □

**Remark 2.2.** From the above theorem, we observe that Theorem 1.1 does not give anything new and it coincides with the fixed point result in $C^*$-algebra-valued metric space [5, Theorem 2.1, p. 4]. On the other hand, Kadelburg and Radenović [4] and Alsulami *et al*. [2] independently proved that fixed point results in this space are the direct consequences of metric fixed point results. Hence this result contributes nothing new.

**Theorem 2.3.** *Let $(X,\mathscr{A},d)$ be a complete $C^*$-algebra-valued metric space. Suppose that the mapping $T : X \to X$ satisfies*

$$d(T^m x, T^n y) \preceq a^* d(x,y)a$$

*for any $x,y \in X$; $a \in \mathscr{A}$ with $\|a\| < 1$ and $m,n$ are any positive integers. Then every point $x \in X$, $T^n x$ is a fixed point of $T$ whenever $m > n$.*

*Proof.* As previous case, for all $x \in X$ and for $m > n$, we obtain

$$\begin{aligned} & d(T^m x, T^n x) \preceq a^* d(x,x)a \\ \Rightarrow \quad & d(T^m x, T^n x) \preceq \theta \\ \Rightarrow \quad & T^m x = T^n x \\ \Rightarrow \quad & T^{m-n}(T^n x) = T^n x, \end{aligned}$$

which implies that $T^n x$ is fixed point of $T$ for every $x \in X$ and for all $n \in \mathbb{N}$. □

Next, we pick up the following result from [7].

**Theorem 2.4.** *Let $(X,\mathscr{A},d)$ be a complete $C^*$-algebra-valued metric space. Suppose that two mappings $T,S : X \to X$ satisfy*

$$d(Tx,Ty) \preceq a^* d(Sx,Sy)a$$

*for any $x,y \in X$ and $a \in \mathscr{A}$ with $\|a\| < 1$. If $R(T) \subset R(S)$ and $R(S)$ is complete in $X$, then $S$ and $T$ have unique point of coincidence in $X$. Also, if $S$ and $T$ are weakly compatible, then there exists a unique common fixed point of $S$ and $T$ in $X$.*

Now, we prove the following result.

**Theorem 2.5.** *Theorem* 2.4 *is equivalent with Theorem* 1.3.

*Proof.* In Theorem 2.4, if we consider $\mathscr{A} = \mathbb{R}$, absolute value as norm and $a^* = a$ for involution then we obtain

$$d(Tx,Ty) \le a^2 d(Sx,Sy)$$

for all $x, y \in X$ and $a^2 \in [0,1)$. Then clearly the contraction principle of the Theorem 2.4 coincides with the contraction (C1) in Theorem 1.3.

On the other hand, we consider the hypotheses of Theorem 2.4. Then by choosing $\widetilde{d}(x,y) = \|d(x,y)\|$, obviously, one can get

$$\widetilde{d}(Tx,Ty) = \|d(Tx,Ty)\| \le \|a^* d(Sx,Sy)a\| \le \|a\|^2 \widetilde{d}(x,y),$$

where $\|a\|^2 \in [0,1)$. Hence by Theorem 1.3 with (C1) contraction principle, $S$ and $T$ have unique coincidence point. Also, if the mappings are weakly compatible then they have unique common fixed point in $X$. □

**Remark 2.6.** In a similar fashion, one can see that the Theorem 2.9 and Theorem 2.10 in Xin *et al*. [7] are equivalent to Theorem 1.3 with (C2) and (C3) contraction principles, respectively.

## 3. Conclusion

We conclude that the common fixed point results of Xin *et al*. [7] are not new in literature. Infact, many fixed point results as well as common fixed point results in $C^*$-algebra-valued metric spaces can be easily deduced from their metric counterparts.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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