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Research Article

A Study on Arithmetic Integer Additive Set-Indexers of Graphs

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Abstract. Let $\mathbb{N}_0$ be the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An integer additive set-indexer (IASI) of a graph G is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. A graph G which admits an IASI is called an IASI-graph. An IASI f is said to be a *weak IASI* if $|f^+(uv)| = \max(|f(u)|, |f(v)|)$ and an IASI f is said to be a *strong IASI* if $|f^+(uv)| = |f(u)|\,|f(v)|$ for all $uv \in E(G)$. In this paper, we introduce the notion of arithmetic integer additive set-indexers of a given graph G as an IASI with respect to which all elements of G have arithmetic progressions as their set-labels and study the characteristics of this type of IASIs.

Keywords. Integer additive set-indexers; Set-indexing number; Arithmetic integer additive set-indexers; Deterministic index; Deterministic ratio

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1. Preliminaries

For all terms and definitions, not defined in this paper, we refer to [5, 6, 11, 19] and for more about graph labeling, we refer to [8]. For the terms and results in number theory, please see to [3, 7, 14]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The *sumset* of two non-empty sets A and B, denoted by $A+B$, is defined by $A+B=\{a+b: a\in A, b\in B\}$ (see [14]). The sumset of two sets A and B is a countably infinite set if either A or B is countably infinite. Hence, all sets mentioned in this paper are finite sets of non-negative integers. We denote the cardinality of a set A by $|A|$.

Let us now recall the following theorem on the cardinality of the sumset of two non-empty finite set of integers.

Theorem 1.1 ([14])**.** *If A and B are two non-empty sets of integers, then $|A|+|B|-1\le|A+B|\le |A||B|$.*

Using the terminology of sumsets, the notion of an integer additive set-indexer of a given graph G have been introduced in [9] as follows.

Definition 1.2. Let $\mathbb{N}_0$ be the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An *integer additive set-indexer* (IASI, in short) of a graph G is an injective function $f: V(G)\to\mathcal{P}(\mathbb{N}_0)$ such that the induced edge-function $f^+: E(G)\to\mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv)=f(u)+f(v)$ is also injective. A graph G which admits an IASI is called an IASI-graph.

The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element. An IASI is said to be *k-uniform* if $|f^+(e)|=k$ for all $e\in E(G)$. The vertex set V of a graph G is defined to be *l-uniformly set-indexed*, if all the vertices of G have the set-indexing number l.

A *weak IASI* is defined in [10] as an IASI f if for every $uv\in E(G)$, $|f^+(uv)|=\max(|f(u)|,|f(v)|)$. A graph which admits a weak IASI may be called a *weak IASI-graph*. A weak IASI is said to be a *weakly uniform IASI* if $|f^+(uv)|=k$, for all $u,v\in V(G)$ and for some positive integer k.

A necessary and sufficient condition for a graph to admit a weak IASI is given below.

Lemma 1.3 ([15])**.** *An IASI f of a given graph G is a weak IASI of G if and only if at least one end vertex of every edge of G has a singleton set-label with respect to f.*

A *strong IASI* is defined as an IASI f if $|f^+(uv)|=|f(u)||f(v)|$ for all $uv\in E(G)$ (see [16]). A graph which admits a strong IASI may be called a *strong IASI-graph*. A strong IASI is said to be *strongly uniform IASI* if $|f^+(uv)|=k$, for all $u,v\in V(G)$ and for some positive integer k.

The following theorem establishes a necessary and sufficient condition for a graph to admit a strong IASI.

Theorem 1.4. *An IASI f of a given graph G is a strong IASI of G if and only if $D_{f(u)}\cap D_{f(v)}=\emptyset$, where $D_{f(u)}$ and $D_{f(v)}$ are the difference sets of the set-labels of u and v respectively, defined by $D_A=\{|a-b|: a,b\in A\}$.*

Some studies on arithmetic graphs have been done in [1], [2] and [13]. Certain studies on the above mentioned types of integer additive set-indexers of graphs have been done in [10], [15], [16] and [17]. Motivated by these studies, in this paper, we initiate a study on a special type of integer additive set-indexers called arithmetic integer additive set-indexer and establish some results on arithmetic integer additive set-indexers.

2. Arithmetic Integer Additive Set-Indexers

Studies about the graphs, the set-labels of whose elements have specific properties arouse much interest. In this paper, we study the graphs, the set-labels whose elements are arithmetic progressions. Since the elements in the set-labels of all elements of G are in arithmetic progression, all these set-labels must contain at least three elements.

By the term, an AP-set, we mean a set whose elements are in arithmetic progression. Then, we introduce the following notion.

Definition 2.1. The common difference of an AP-set, which is the set-label of an element of the given graph G, is called the *deterministic index* of that element of G. The *deterministic ratio* of an edge of G is the ratio, $k \ge 1$ between the deterministic indices of its end vertices.

First consider the graphs, all whose vertices have AP-sets as their set-labels. Hence, we define

Definition 2.2. Let $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ be an IASI on G. For any vertex v of G, if $f(v)$ is an AP-set, then f is called a *vertex arithmetic IASI* of G. A graph that admits a vertex-arithmetic IASI is called a *vertex arithmetic IASI-graph*.

In a similar way we can define an edge arithmetic IASI of a graph also as follows.

Definition 2.3. For an IASI f of G, if $f^+(e)$ is an AP-set, for all $e \in E(G)$, then f is called an *edge-arithmetic IASI* of G. A graph that admits an edge-arithmetic IASI is called an *edge-arithmetic IASI-graph*.

The difference set of a non-empty set A, denoted by D_A, is the set defined by $D_A = \{|a-b| : a,b \in A\}$. It is to be noted that if A is an AP-set, then its difference set D_A is also an AP-set and vice versa. The following result is a necessary and sufficient condition for a graph G to be edge-arithmetic IASI-graph in terms of the difference sets the set-labels of vertices of G.

Theorem 2.4. *Let f be an IASI defined on a graph G. If the set-label of an edge of G is an AP-set if and only if the sumset of the difference sets of set-labels of its end vertices is an AP-set.*

Proof. Let $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ be an IASI defined on G. Let a_i, a_j be two arbitrary elements in $f(u)$ and let b_r, b_s be two elements in $f(v)$. Then, $|a_i - a_j| \in D_{f(u)}$ and $|a_i - a_j| \in D_{f(u)}$. That is, $D_{f(u)} = \{|a_i - a_j| : a_i, a_j \in f(u)\}$ and $D_{f(v)} = \{|b_r - b_s| : b_r, b_s \in f(v)\}$.

Now, assume that $f^+(e) = f^+(uv)$ is an AP-set for an edge $e = uv \in E(G)$. That is, $A = f(u) + f(v)$ is an AP-set. Then, the difference set $D_A = \{|a-b| : a,b \in A = f(u)+f(v)\}$ is also an AP-set. Since $a,b \in A$, we have $a = a_i + b_r$ and $b = a_j + b_s$, where $a_i, a_j \in f(u)$ and $b_r, b_s \in f(v)$. Then,

$$\begin{aligned} D_A &= \{|a-b| : a,b \in A\} \\ &= \{|a_i + b_r - (a_j + b_s)| : a_i, a_j \in f(u), b_r, b_s \in f(v)\} \\ &= \{|a_i - a_j| + |b_r - b_s| : a_i, a_j \in f(u), b_r, b_s \in f(v)\} \end{aligned}$$

$$= \{|a_i - a_j| : a_i, a_j \in f(u)\} + \{|b_r - b_s| : b_r, b_s \in f(v)\}$$
$$= D_{f(u)} + D_{f(v)}.$$

Hence, $D_{f(u)} + D_{f(v)}$ is an AP-set.

Conversely, assume that $D_{f(u)} + D_{f(v)}$ is an AP-set. Then, by previous step, we have $D_{f(u)} + D_{f(v)} = D_A$, where $A = f(u) + f(v)$. Then, we have D_A is an AP-set. Since the difference set D_A is an AP-set, then by the above remark, we have $A = f(u) + f(v) = f^+(uv)$ is also an AP-set. Hence, the edge $e = uv$ has an AP-set as its set-label. □

In view of the above definitions, we note that there are some graphs, all whose elements have AP-sets as their set-labels and there are some graphs, the set-labels of whose edges are not AP-sets. Keeping this in mind, we define the following notion.

Definition 2.5. An *arithmetic integer additive set-indexer* of a graph G is an integer additive set-indexer f of G, with respect to which the set-labels of all vertices and edges of G are AP-sets. A graph that admits an arithmetic IASI is called an *arithmetic IASI-graph*.

Invoking the fact that the set-labels assigned to every element of G must have at least three elements, we have the following theorem as an immediate consequence of Lemma 1.3.

Proposition 2.6. *An arithmetic integer additive set-indexer of a graph G is not a weak integer additive set-indexer of G and vice versa.*

Let us now proceed to verify the conditions required for a vertex arithmetic IASI of a given graph G to be an arithmetic IASI of G. In the following proposition, we check the case when both end vertices of an edge of a vertex arithmetic IASI-graph G have the same deterministic index.

Proposition 2.7. *If the set-labels of two adjacent vertices are AP-sets with the same common difference, say d, then the set-label of the corresponding edge is also an AP-set with the same common difference d.*

Proof. Let u and v be two adjacent vertices in G. Let $f(u) = \{a + id : 0 \le i \le m\}$ and $f(v) = \{b + jd : 0 \le j \le n\}$, where a, b, m, n are non-negative integers. Then, $f^+(uv) = \{(a+b) + (i+j)d : 0 \le i \le m, 0 \le j \le n\}$, which is also an AP-set, with the same common difference d. □

Invoking Proposition 2.7, we have the following theorem.

Lemma 2.8. *Let f be a vertex arithmetic IASI defined on a given graph G. If the deterministic indices of any two adjacent vertices of G are the same, then f is an arithmetic IASI of G.*

The following proposition establishes the condition for an edge to have an AP-set as its set-label if the deterministic index of one end vertex of it is a multiple of the deterministic index of the other.

Proposition 2.9. *Let f be a vertex arithmetic IASI of G and let u and v be two adjacent vertices in G with deterministic indices d_u and d_v respectively such that $d_u \le d_v$. Then, f is an arithmetic IASI if the deterministic ratio of the edge uv is a positive integer less than or equal to $|f(u)|$.*

Proof. Let $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ be a vertex arithmetic IASI on G such that $f(u)$ and $f(v)$ has the deterministic indices d_u and d_v respectively, where $d_u \le d_v$. Assume that $f(u) = \{a_r = a + rd_u : 0 \le r < m\}$ and $f(v) = \{b_s = b + sd_v : 0 \le s < n\}$. Then, $|f(u)| = m$ and $|f(v)| = n$. Now, arrange the terms of $f(u)+f(v) = f^+(uv)$ in rows and columns as follows. For any $b_s \in f(v), 0 \le s < n$, arrange the terms of $A + b_s$ in $(s+1)$-th row in such a way that equal terms of different rows come in the same column of this arrangement.

Assume, without loss of generality, that $d_v = kd_u$ and $k \le m$. If $k < m$, then for any $a \in f(u)$ and $b \in f(v)$ we have $a + (b + d_v) = a + b + kd_u < a + b + md_i$. That is, a few final elements of each row of the above arrangement occur as the initial elements of the succeeding row (or rows) and the difference between two successive elements in each row is d_u itself. If $k = m$, then the difference between the final element of each row and the first element of the next row is d_u and the difference between two consecutive elements in each row is d_u. Hence, if $k \le m$, then $f^+(uv)$ is an AP-set with common difference d_u. This completes the proof. □

Invoking the above proposition, we have the following lemma.

Lemma 2.10. *Let f be a vertex arithmetic IASI of the graph G. Then f is an arithmetic IASI if for any pair of adjacent vertices in G the deterministic index of one of them is a positive integral multiple of the deterministic index of the other where this positive integer is less than or equal to the set-indexing number of the vertex having smaller deterministic index.*

A necessary and sufficient condition for a graph to admit an arithmetic IASI is established in the following theorem.

Theorem 2.11. *A graph G admits an arithmetic IASI f if and only if for any two adjacent vertices in G, the deterministic ratio of every edge of G is a positive integer, which is less than or equal to the set-indexing number of its end vertex having smaller deterministic index.*

Proof. The necessary part follows immediately from Lemmas 2.8 and 2.10.

We prove the converse part by contradiction method. For this, assume that f is an arithmetic IASI of G. Let us proceed by considering the following two cases.

***Case* 1:** Assume that d_j is not a multiple of d_i (or d_i is not a multiple of d_j). Without loss generality, let $d_i < d_j$. Then, by division algorithm, $d_j = pd_i + q, 0 < q < d_i$. Then, the difference between any two consecutive terms in $f^+(v_iv_j)$ are not equal. Hence, in this case also f is not an arithmetic IASI, contradiction to the hypothesis. Therefore, $d_i | d_j$.

***Case* 2:** Let $d_j = kd_i$ where $k > m$. Then, the difference between two successive elements in each row is d_i, but the the difference between the final element of each row and the first element of the next row is td_i, where $t = k - m + 1 \ne 1$. Hence, f is not an arithmetic IASI, a contradiction to the hypothesis. Hence, we have $d_j = k\,d_i; k \le m$.

Therefore, from the above cases it can be noted that if a vertex arithmetic IASI of G is an arithmetic IASI of G, then the deterministic ratio of every edge of G is a positive integer, which is greater than or equal to the set-indexing number of its end vertex having smaller deterministic index. This completes the proof. □

In the following theorem, we establish a relation between the deterministic indices of the elements of an arithmetic IASI-graph G.

Theorem 2.12. *If G is an arithmetic IASI-graph, the greatest common divisor of the deterministic indices of vertices of G and the greatest common divisor of the deterministic indices of the edges of G are equal to the smallest among the deterministic indices of the vertices of G.*

Proof. Let f be an arithmetic IASI of G. Then, by Theorem 2.11, for any two adjacent vertices v_i and v_j of G with deterministic indices d_i and d_j respectively, either $d_i = d_j$, or if $d_j > d_i, d_j = k\,d_j$, where k is a positive integer such that $1 < k \le |f(v_i)|$.

If the deterministic indices of the elements of G are the same, the result is obvious. Hence, assume that for any two adjacent vertices v_i and v_j of G, $d_j = k\,d_j, k \le |f(v_i)|$, where d_i is the smallest among the deterministic indices of the vertices of G. If v_r is another vertex that is adjacent to v_j, then it has the deterministic index d_r which is equal to either d_i or d_j or $l\,d_j$. In all the three cases, d_r is a multiple of d_i. Hence, the g.c.d of d_i, d_j, d_r is d_i. Proceeding like this, we have the g.c.d of the deterministic indices of the vertices of G is d_i.

Also, by Lemma 2.10, the edge u_iv_j has the deterministic index d_i. The edge v_jv_k has the deterministic index d_i, if $d_k = d_i$, or d_j in the other two cases. Proceeding like this, we observe that the g.c.d of the deterministic indices of the edges of G is also d_i. This completes the proof. □

The study on the set-indexing number of edges of an arithmetic IASI-graphs arouses much interest. The set-indexing number of an edge of an arithmetic IASI-graph G, in terms of the set-indexing numbers of its end vertices, is determined in the following theorem.

Theorem 2.13. *Let G be a graph which admits an arithmetic IASI, say f and let d_i and d_j be the deterministic indices of two adjacent vertices v_i and v_j in G. If $|f(v_i)| \ge |f(v_j)|$, then for some positive integer $1 \le k \le |f(v_i)|$, the edge v_iv_j has the set-indexing number $|f(v_i)| + k(|f(v_j)| - 1)$.*

Proof. Let f be an arithmetic IASI defined on G. For any two vertices v_i and v_j of G, let $f(v_i) = \{a_i, a_i + d_i, a_i + 2d_i, a_i + 3d_i, \ldots, a_i + (m-1)d_i\}$ and let $f(v_j) = \{a_j, a_j + d_j, a_j + 2d_j, a_j + 3d_j, \ldots, a_j + (n-1)d_j\}$. Here $|f(v_i)| = m$ and $|f(v_j)| = n$.

Let d_i and d_j be the deterministic indices of the vertices v_i and v_j respectively, such that $d_i < d_j$. Since f is an arithmetic IASI on G, by Theorem 2.11, there exists a positive integer k such that $d_j = k.d_i$, where $1 \le k \le |f(v_i)|$. Then, $f(v_j) = \{a_j, a_j + kd_i, a_j + 2kd_i, a_j + 3kd_i, \ldots, a_j + (n-1)kd_i\}$. Therefore, $f^+(v_iv_j) = \{a_i + a_j, a_i + a_j + d_i, a_i + a_j + 2d_i, \ldots, a_i + a_j + [(m-1) + k(n-1)]d_i\}$. That is, the set-indexing number of the edge v_iv_j is $m + k(n-1)$. □

Can we define an arithmetic IASI every given graph admit? The following theorem establishes the existence of an arithmetic IASI for a given graph.

Theorem 2.14. *Every graph G admits an arithmetic integer additive set-indexer.*

Proof. Let f be an IASI defined on a given graph G under which all the vertices of G are labeled by AP-sets in such a way that the common difference $f(v_i)$ is d_i, where d_i is a positive integer. Let v_1 be an arbitrary vertex of G and let d_1 be any positive integer. Label v_1 by an AP-set with common difference d_1. Let v_2 be a vertex of G adjacent to v_1. Label this vertex by an AP-set with common difference $d_2 = k_1 d_1, 1 \le k_1 \le |f(v_1)|$. Let v_3 be a vertex of G adjacent to v_2. Label this vertex by an AP-set with common difference $d_3 = k_2 d_2, 1 \le k_1 \le |f(v_1)|$. If v_3 is adjacent to v_1, then $d_3 = k_3 d_1, 1 \le k_3 \le \min(|f(v_1)|, |f(v_2)|)$. Label all vertices of G in this manner. Then, by Theorem, the set-label of every edge of G is also an AP-set. Hence, f is arithmetic IASI of G. □

We observe that a vertex arithmetic IASI of a given graph need not be an arithmetic IASI of G. Then we have the following notion.

Definition 2.15. If all the set-labels of all vertices of a graph G are AP-sets and the set-labels of edges are not AP-sets, then the corresponding IASI is called *semi-arithmetic IASI*. In other words, a semi-arithmetic IASI of a graph G is a vertex arithmetic IASI of G which is not an arithmetic IASI.

Invoking Theorem 2.11, a vertex arithmetic IASI f of a given graph G is a semi-arithmetic IASI of G if for any two adjacent vertices u and v with deterministic indices d_u and d_v respectively such that $d_u \le d_v$, either $\frac{d_v}{d_u}$ is not a positive integer or $\frac{d_v}{d_u} > |f(u)|$.

3. Arithmetic IASIs on Graph Classes

In this section, we discuss the admissibility of arithmetic IASIs by certain graph classes. The following theorem establishes the necessary condition for a complete graph to admit an arithmetic IASI.

Theorem 3.1. *If a complete graph K_n admits an arithmetic IASI if and only if the deterministic indices of any vertex of K_n is either an integral multiple or divisor of the deterministic indices of all other vertex of K_n.*

Proof. If all the vertices of G have the same deterministic index, then by Proposition 2.7, K_n admits an arithmetic IASI. Hence, assume that not all vertices of K_n have the same deterministic index and K_n admits an arithmetic IASI, say f. Let v_i, v_j and v_r be any three vertices of K_n and d_i, d_j, d_r be the deterministic indices of these vertices, respectively. Let $d_i \le d_j$. Then $d_j = k d_i$, where $k \le |f(v_i)|$ is a positive integer. Similarly, if $d_j \le d_r$, then $d_r = l d_j$, where $l \le |f(v_j)|$ is a positive integer. Then, obviously d_i divides d_r also. The similar argument can be made if $d_i \ge d_j$ also. Therefore, the deterministic index of any vertex of K_n is either a multiple or a divisor of the deterministic index of every other vertex of K_n. □

The converse of the above theorem is also valid subject to the existence of the inequality concerned between the deterministic ratio of every edge of K_n and the set-indexing numbers of its end vertices as mentioned in Theorem 2.11.

Theorem 3.2. *A complete graph admits an arithmetic IASI if and only if the deterministic indices of any vertex of K_n is either an integral multiple or divisor of the deterministic indices of all other vertex of K_n such that the deterministic ratio of every edge of K_n is less than or equal to the set-indexing number of its end vertex having smaller deterministic index.*

The hereditary nature of the existence of arithmetic IASI for a given graph G is established in the following theorem.

Proposition 3.3. *If a graph G admits an arithmetic integer additive set-indexer, then any non-trivial subgraph of G also admits an arithmetic integer additive set-indexer.*

Proof. let f be an arithmetic IASI on G and let $H \subset G$. The proof follows from the fact that the restriction $f|_H$ of f to the subgraph H is an arithmetic IASI on H. □

The following theorems establishes the admissibility of an arithmetic IASI by some graphs associated to a given arithmetic IASI-graphs. First recall the definition of an edge contraction of a graph.

Definition 3.4 ([11])**.** By *edge contraction operation* in G, we mean an edge, say e, is removed and its two incident vertices, u and v, are merged into a new vertex w, where the edges incident to w each correspond to an edge incident to either u or v.

The following theorem discusses the admissibility of arithmetic IASIs by the graph obtained by contracting an edge of a given arithmetic IASI-graph G.

Theorem 3.5. *Let G be an arithmetic IASI-graph and let e be an edge of G. Then, $G \circ e$ admits an induced arithmetic IASI.*

Proof. Let G admits a weak IASI. Let e be an edge in $E(G)$. Let $e = uv$ be an arbitrary edge of G. Let d_i and d_j be the deterministic number of u and v respectively. Then, by Theorem 2.11, either $d_i = d_j$ or if $d_j > d_j$, $d_j = k\,d_j, 1 \le k \le |f(u)|$. $G \circ e$ is the graph obtained from G by deleting e of G and identifying u and v to get anew vertex, say w. Label the w, by the set-label of the deleted edge e. Then, w has the deterministic number d_i and all elements in $G \circ e$ are AP-sets. Hence, $G \circ e$ is a isoarithmetic IASI-graph. □

Now consider the following definition of homeomorphic graphs.

Definition 3.6 ([12])**.** Let G be a connected graph and let v be a vertex of G with $d(v) = 2$. Then, v is adjacent to two vertices, say u and w, in G. If u and v are non-adjacent vertices in G, then delete v from G and add the edge uw to $G - \{v\}$. This operation is known as an *elementary topological reduction* on G. If H is a graph obtained from a given graph G by applying a finite number of elementary topological reductions, then we say that G and H are *homeomorphic graphs*.

The admissibility of an induced arithmetic IASI by a graph that is homeomorphic to a given arithmetic IASI-graph is established in the following theorem.

Theorem 3.7. *Let G be a graph which admits an arithmetic IASI. Then any graph G', obtained by applying finite number of elementary topological reductions on G, admits also admits an arithmetic IASI.*

Proof. Let G be a graph which admits an arithmetic IASI, say f. Let v be a vertex of G with $d(v) = 2$ and deterministic index d_i. Since $d(v) = 2$, v must be adjacent two vertices u and w in G. Let these vertices u and w re on-adjacent. Now remove the vertex v from G and introduce the edge uw to $G - v$. Let $G' = (G - v) \cup \{uw\}$. Let $f' : V(G') \to \mathcal{P}(\mathbb{N}_0)$ such that $f'(v) = f(v)\ \forall\ v \in V(G')$ and the associated function $f'^{+} : E(G') \to \mathcal{P}(\mathbb{N}_0)$ and defined by

$$f'^{+}(e) = \begin{cases} f^{+}(e) & \text{if } e \neq uw \\ f(u) + f(w) & \text{if } e = uw. \end{cases}$$

Hence, f' is an arithmetic IASI of G'. □

Next, recall the definition of the subdivision of a graph.

Definition 3.8 ([18])**.** A *subdivision* of a graph G is the graph obtained by adding vertices of degree two into its edges.

The following theorem establishes the admissibility of an induced arithmetic IASI by the subdivision of an arithmetic IASI-graph.

Theorem 3.9. *A graph subdivision G^* of a given arithmetic IASI-graph G also admits an induced arithmetic IASI.*

Proof. Let u and v be two adjacent vertices in G. Since G admits an arithmetic IASI, the deterministic indices d_i and d_j of u and v respectively are either equal or, if $d_j > d_i$, $d_j = k d_i, 1 \le k \le |f(u)|$. Introduce a new vertex w to the edge uv. Now, we have two new edges uw and vw in place of uv. Extend the set-labeling of G by labeling the vertex w by the same set-label of the edge uv. Then, the vertices u and w have the same deterministic indices d and the deterministic index of v is a positive integral multiple of the deterministic index of w, where this positive integer is clearly less than the cardinality of the labeling set of w. Hence, G^* admits an arithmetic IASI. □

Another important graph associated with a graph G is its line graph which is defined as given below.

Definition 3.10 ([19])**.** For a given graph G, its *line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G incident on a common vertex in G.

The admissibility of an induced arithmetic IASI by the line graph of a given arithmetic IASI-graph G is examined in the following theorem.

Theorem 3.11. *The line graph $L(G)$ of an arithmetic IASI-graph G admits an induced arithmetic IASI. Moreover, the function f^+ associated to f in G is an arithmetic IASI on $L(G)$.*

Proof. Consider two adjacent edges $e_1 = v_1v_2$ and $e_2 = v_2v_3$ in G. Let d_i be the deterministic index of the vertex v_i in G. Since G is arithmetic graph, $d_1 = d_2$ or $d_2 = k_1d_1, 1 \le k \le |f(v_1)|$. In both cases, deterministic number of the edge v_1v_2 is d_1 itself. Similarly, since v_2 is adjacent to v_3, either $d_3 = d_1$ or $d_3 = d_2$ or $d_3 = k_2d_2$. In all these three cases, the deterministic number of the edge e_2 is d_1 or a positive integer multiple of d_1. Proceed like this until all the elements of G are set-labeled. Hence, we have a set-labeling in which for every pair of adjacent edges in G, the deterministic indices are either equal or the deterministic index of one of these vertices is a positive integral multiple of the other. Hence, the deterministic indices of corresponding vertices in $L(G)$ are either equal or the deterministic index of one of these vertices is a positive integral multiple of the other. That is, f^+ defined in G is an arithmetic IASI on $L(G)$. □

Next, consider the definition of another type graph associated with a graph G, called the total graph, as given below.

Definition 3.12 ([4])**.** The *total graph* of a graph G is the graph, denoted by $T(G)$, is the graph having the property that a one-to one correspondence can be defined between its points and the elements (vertices and edges) of G such that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (either if both elements are edges or if both elements are vertices) or they are incident (if one element is an edge and the other is a vertex).

The existence of an induced arithmetic IASI for the total graph of an arithmetic IASI-graph is discussed in the theorem given below.

Theorem 3.13. *The total graph $T(G)$ of an arithmetic IASI-graph G admits an induced arithmetic IASI.*

Proof. Define a function $g : V(T(G)) \to \mathcal{P}(\mathbb{N}_0)$ as follows.

$$g(v') = \begin{cases} ccf(v) & \text{if } v' \in V(L(G)) \text{ corresponds to } v \in V(G) \\ f^+(e) & \text{if } v' \in V(L(G)) \text{ corresponds to } e \in E(G) \end{cases}$$

Since the vertices in $T(G)$ corresponding to the vertices of G preserve the same set-labelings of the corresponding vertices of G and the vertices of $T(G)$ corresponding to the edges of G preserve the same set-labeling of the corresponding edges in G, these vertices in $T(G)$ satisfy the conditions required for admitting an arithmetic IASI. Hence g is arithmetic IASI on $T(G)$. □

4. Conclusion

In this paper, we have discussed some characteristics of graphs which admit a certain type of IASI called arithmetic IASI. We have formulated some conditions for some graph classes to admit arithmetic IASIs. Problems related to the characterisation of different biarithmetic IASI-graphs are still open.

The following problems on biarithmetic and semi-arithmetic IASI-graphs, analogous to the results proved for isoarithmetic IASI-graphs, are to be investigated.

Problem 1. Discuss the admissibility of certain binary operations and products graphs which admit different types arithmetic IASI-graphs.

Problem 2. Characterise the graphs which admit different arithmetic and semi-arithmetic IASIs.

Problem 3. Discuss the existence and cardinality of saturated classes in the set-labels of the elements of given graphs which admit arithmetic and semi-arithmetic IASIs.

Problem 4. Characterise the graphs which admit uniform arithmetic and semi-arithmetic IASIs.

The IASIs under which the vertices of a given graph are labeled by different standard sequences of non negative integers, are also worth studying. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain IASIs still remain unsettled. All these facts highlight a wide scope for further studies in this area.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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