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Research Article

Family of Frames

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Abstract. We introduce here the concept of Family of Frames, a new type of generator systems for Hilbert separable spaces. These families, similar to frames and bases, allow the representation of all vectors of the space through a process of analysis, and reconstruction of them by a process of synthesis. These families are defined using the concept of outer frame of a closed subspace. These new systems are even less restrictive than the known frames. In addition, this new concept has a particular characteristic for some *multiresolution analysis* (MRA), which is the ability to represent any vector of the space from the generators of the subspaces of MRA, without needing to build wavelets frames associated to the MRA for this end.

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1. Introduction

The importance of generator systems lies in the fact that each element of the space can be described in terms of simpler elements. Schauder bases were the first generator systems for Banach spaces. These bases provide a unique description for each vector space, as a linear combination (infinite series) of its elements. Later other conditions were required to bases, such as the unconditional convergence of the series involved in the reconstruction. This means that no matter the sequence in which the elements of the base are ordered. Examples of such generator systems are orthonormal bases or Riesz bases of Hilbert spaces.

The theory of frames in Hilbert spaces originates in the work of R.J. Duffin and A.C. Schaeffer (1952) ([4]). They continue the study of non-harmonic Fourier series initiated by Paley and Wiener two decades earlier. Duffin and Schaeffer show that certain density conditions on the coefficients (real or complex) $\{\lambda_n\}_{n\in\mathbb{Z}}$, imply that the sequence $\{e^{i\lambda_n \cdot t}\}_{n\in\mathbb{Z}}$ not only generates the space $L^2(-T,T)$, with $0 \le T \le \pi$, but also constitutes what they call a *frame*, because it satisfies the following inequality:

$$m \le \frac{\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} |\int_{-T}^{T} f(t)e^{i\lambda_n \cdot t}dt|^2}{\int_{-T}^{T}|f(t)|^2} \le M \tag{1}$$

for some pair of constants m and M greater than 0, and all $f \in L^2(-T,T)$. For an abstract Hilbert space H, this equation becomes

$$m\|f\|^2 \le \sum_{i\in I} |\langle f_i, f\rangle|^2 \le M\|f\|^2 \quad \forall\ f \in H \tag{2}$$

for $\{f_i\}_{i\in I}$ sequence in H, which is called frame. In particular, the lower inequality in (2) determines that a frame be a generator sequence for H.

The rapid development of the theory of wavelets, introduce frames as a very useful tool in analyzing time-frequency signals and images processing.

The frame is an overdimensioned system (the representation of each element of space is not unique). This causes lack of unicity in the reconstruction (property that the bases do have). This seemed at first a problem, however this lack of unicity proved to be interesting in applications years later. This oversizing of the frame makes that a vector can be reconstructed with more than one sequence of coefficients, giving certain flexibility to the reconstruction of a vector using a frame, in contrast to the stiffness which bases presents.

In this article we define and study the structure and properties of *Family of Frames*, a new type of generator systems for a Hilbert space, even less restrictive than the frames. First, we define this new concept, from the notion of orthogonal projection on a subspace. An analysis process for each vector of space is produced from this structure. Subsequently, we show two methods of reconstruction or synthesis for this families.

Moreover, we extend the concepts of *Multiresolution Analysis* (MRA) and associated Wavelets, using the notion of outer frame. This allows us to prove an interesting theorem: under certain hypotheses for the bounds of the outer frames of subspaces of the MRA, we establish a family of frames and a frame, both associated with it, determined by the generating functions of the subspaces of the MRA. Finally, we show as an example, a family of frames associated with the Haar MRA, generated from its scaling function.

2. Preliminaries

Definition 2.1 (Frames in Hilbert spaces)**.** Let H be a Hilbert space, and I a numerable set of indexes, we say that $\{f_i\}_{i\in I} \subset H$ is a frame for H, if $\exists\ A,B : 0 < A \le B < \infty$ such that $\forall\ f \in H$

$$A\|f\|^2 \le \sum_{j\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2$$

$\{\langle f, f_i\rangle\}_{i\in I}$ are called *coefficients* of f in the frame $\{f_i\}_{i\in I}$.

A and B are known as *lower bound* and *upper bound* of the frame $\{f_i\}_{i\in I}$ respectively. Bounds of a frame are not unique.

A frame will be said *tight* if $A = B$, and it will be of *Parseval* if $A = B = 1$.

2.1 Reconstruction (Synthesis) for Frames: Dual Frames

The process to obtain a set of frame coefficients, of each vector of space in a given frame is called *analysis*. In theory of frames the process of *synthesis* is essential, because it allows recovering each vector of space, from their frame coefficients using a dual frame, that used during the analysis.

Definition 2.2 (Dual Frames)**.** Let $\{f_j\}_{j\in I}$ be a frame for H, we say that the frame $\{g_j\}_{j\in I}$ is a dual frame for $\{f_j\}_{j\in I}$ if:

$$f = \sum_{k\in I} \langle f, f_k\rangle g_k \quad \forall\, f \in H. \tag{3}$$

The frame $\{f_j\}_{j\in I}$ carries out the analysis to obtain $\{\langle f, f_k\rangle\}_{k\in I}$, and by (3) the frame $\{g_j\}_{j\in I}$ carries out the synthesis.

Remark 2.3. The reconstruction or synthesis of space vectors from a tight frame is very easy, since $\{A^{-1} f_i\}_{i\in I}$ is dual of the tight frame $\{f_i\}_{i\in I}$ of bound A; from this results that a Parseval frame ($A = B = 1$) is dual of itself.

Below we recall the definition of *outer frame*, strongly used in demonstrations of results of this work:

Definition 2.4 (Outer Frame)**.** Let H be a Hilbert space, the collection of vectors $\{f_j\}_{j\in\mathbb{Z}} \subset H$ is an outer frame for a closed subspace F of H, if $\{P_F(f_j)\}_{j\in\mathbb{Z}}$ is frame for F, where we call P_F to the orthogonal projection over the subspace F. This is equivalent to the existence of constants $0 < m \le M < \infty$ such that:

$$m\|f\|^2 \le \sum_{j\in\mathbb{Z}} |\langle f, f_j\rangle|^2 \le M\|f\|^2 \quad \forall\, f \in F.$$

Remark 2.5. If $\{f_j\}_{j\in\mathbb{Z}} \subset H$ is a frame of the Hilbert space H, then $\{f_j\}_{j\in\mathbb{Z}}$ is outer frame of all closed subspace F of H.

3. Family of Frames

Definition 3.1. Let H be a Hilbert space. $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}} \subset H$ is a $f$amily of frames for H, if there is at least one collection of closed subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$ of H such that:

(a) $\bigoplus^{\perp}_{j\in\mathbb{Z}} \mathscr{W}_j = H$,

(b) There are $0 < m \le M < \infty$ such that $\forall\ sf \in H$:

$$m\|f\|^2 \le \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} |\langle f_j, h_{j,k}\rangle|^2 \le M\|f\|^2 \tag{4}$$

being f_j the orthogonal projection of f over the $\mathscr{W}_j$ subspace.

- m and M are the *upper and lower bounds* of the family of frames, respectively.
- $\{\langle f_j, h_{jk}\rangle\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ are coefficients of the function f with respect the family.
- If $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$ is a collection of subspaces for which are verified (a) and (b), we say that the family $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ *is linked* to the collection of subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$.

Remark 3.2. If $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a family of frames for $L^2(\mathbb{R}^d)$ linked to the collection of subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$, with bounds m and M, then $\forall\ j\in\mathbb{Z}$, $\{h_{j,k}\}_{k\in\mathbb{Z}}$ is outer frame of $\mathscr{W}_j$ with bounds m and M.

Definition 3.3. A family of frames will be *tight*, with bound M, if $m=M$; and the family of frames will be a **Parseval family of frames**, if $m=M=1$.

Remark 3.4. It is important to note that:

(a) As $\langle f_j, h_{j,k}\rangle = \langle P_j(f), h_{j,k}\rangle = \langle f, P_j(h_{j,k})\rangle$, according to (4) the set $\{P_j(h_{j,k})\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is frame for H with bounds m and M:

$$m\|f\|^2 \le \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} |\langle f, P_j(h_{j,k})\rangle|^2 \le M\|f\|^2. \tag{5}$$

(b) If $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a family of frames linked to subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$, and $\{h_{j,k}\}_{k\in\mathbb{Z}}$ is a frame of $\mathscr{W}_j\ \forall\ j$, then the family of frames $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a **frame** of all space with the same bounds.

(c) For a family of frames $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ of a Hilbert space H, there may be several decompositions of H as direct sum of mutually orthogonal subspaces, so that this family is linked to these decompositions.

Theorem 3.5. *Let $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ be a set of functions of H.*

(a) *If $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a family of frames for H linked to the set of subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$, then $\sup_j M_j$ and $\inf_j m_j$ are also upper and lower bounds of the family (para M_j the least, and m_j supreme of upper and lower bounds respectively of the outer frames $\{h_{j,k}\}_{k\in\mathbb{Z}}$ of $\mathscr{W}_j$).*

(b) *They are equivalent:*

(1) *$\{h_{jk}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a family of frames for H linked to the collection of subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$.*

(2)
- *$\forall\ j\in\mathbb{Z}$ the set $\{h_{jk}\}_{k\in\mathbb{Z}}$ is outer frame of the subspace $\mathscr{W}_j$,*
- *$H = \overset{\perp}{\bigoplus_{j\in\mathbb{Z}}} \mathscr{W}_j$*
- *$M := \sup_j M_j < \infty$ and $m := \inf_j m_j > 0$, for M_j and m_j upper and lower bounds of the frame of $\mathscr{W}_j$.*

Proof. (a) Let m_j and M_j be optimal bounds of the outer frame $\{h_{j,k}\}_{k\in\mathbb{Z}}$ of $\mathscr{W}_j$. From remark 3.2, $m \le m_j$ and $M_j \le M\ \forall\ j\in\mathbb{Z}$, being m and M bounds of the family.

Therefore, there is $\widetilde{m} := \inf_j m_j$ and $\widetilde{M} := \sup_j M_j$. Then $m \le \widetilde{m}$ and $\widetilde{M} \le M$. From Definition 3.1, (b): Let $f \in H$, then $f = \sum_{j\in\mathbb{Z}} f_j$, with f_j the orthogonal projection of f over $\mathscr{W}_j$, besides:

- Being m_j and M_j optimal bounds for the outer frame $\{h_{j,k}\}_{k\in\mathbb{Z}}$ of $\mathscr{W}_j$ we have:
$$m_j\|f_j\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f_j, h_{j,k}\rangle|^2 \le M_j\|f_j\|^2. \tag{6}$$
- Adding in the previous expression over j, and considering definitions of $\widetilde{m}$ and $\widetilde{M}$, we obtain:
$$\widetilde{m}\|f\|^2 \le \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} |\langle f_j, h_{j,k}\rangle|^2 \le \widetilde{M}\|f\|^2. \tag{7}$$

The expression (7) states that $\widetilde{m}$ and $\widetilde{M}$ are bounds for the family of frames $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$.

(b) (1)$\Longrightarrow$(2): It follows from the definition of family of frames (see Remark 3.2), and from the demonstration in the first part of the previous section.

(2)$\Longrightarrow$(1): Let $f \in H$, then $f = \sum_{j\in\mathbb{Z}} f_j$ (being f_j the orthogonal projection of f over $\mathscr{W}_j$). As $\{h_{j,k}\}_{k\in\mathbb{Z}}$ is an outer frame of $\mathscr{W}_j$ with bounds m_j and M_j then:
$$m_j\|f_j\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f_j, h_{jk}\rangle|^2 \le M_j\|f_j\|^2. \tag{8}$$
Then by hypothesis:
$$m\|f_j\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f_j, h_{jk}\rangle|^2 \le M\|f_j\|^2. \tag{9}$$
Adding now over $j \in \mathbb{Z}$ in (9), and considering that $\|f\|^2 = \sum_{j\in\mathbb{Z}} \|f_j\|^2$ we obtain:
$$m\|f\|^2 \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f_j, h_{jk}\rangle|^2 \le M\|f\|^2. \tag{10}$$
The expression (10) states that the collection $\{h_{t,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a family of frames for H linked to subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$ with bounds m and M. □

3.1 Reconstruction (Synthesis) for Families of Frames

Given a frame in a Hilbet space, there is always a dual frame for it; and any function of the space can be reconstructed from it and the frame coefficients of the given function.

We will see here, in a similar way, two methods for the reconstruction of any vector f in the space H, using its coefficients with respect to a family of frames $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ of H ($\{\langle f_j, h_{j,k}\rangle\}_{j,k}$).

Let $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ be a family of frames for a Hilbert space H linked to subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$. Let $f \in H$ be and P_j the orthogonal projection over the subspace $\mathscr{W}_j$. Then:
$$\langle f_j, h_{jk}\rangle = \langle P_j f, h_{jk}\rangle = \langle P_j f, P_j h_{jk}\rangle. \tag{11}$$

- As $\{h_{j,k}\}_{k\in\mathbb{Z}}$ is an outer frame of $\mathscr{W}_j$ the set $\{P_j h_{j,k}\}_{k\in\mathbb{Z}}$ is frame of $\mathscr{W}_j$, then there exists

$\{\widetilde{h}_{j,k}\}_{k\in\mathbb{Z}}$ which is a dual frame of it and:

$$f_j = \sum_{k\in\mathbb{Z}} \langle f_j, P_j h_{j,k}\rangle \widetilde{h}_{j,k} \quad \forall\, j \in \mathbb{Z}. \tag{12}$$

Adding over j in the previous expression and considering that $f = \sum_{j\in\mathbb{Z}} f_j$ we obtain:

$$f = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} \langle f_j, P_j h_{j,k}\rangle \widetilde{h}_{j,k} = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} \langle f_j, h_{j,k}\rangle \widetilde{h}_{j,k}. \tag{13}$$

- In the remark 3.4, (a) we see that if $\{h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a family of frames for H, then $\{P_j h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a frame of H (where P_j is the orthogonal projection over the j-th subspace of the orthogonal decomposition of H). Let $\{g_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ be a dual frame of $\{P_j h_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$, then:

$$f = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} \langle f, P_j h_{j,k}\rangle g_{j,k} = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} \langle f_j, h_{j,k}\rangle g_{j,k}. \tag{14}$$

Remark 3.6. It follows that:

- Expressions (13) and (14) are in principle different, since $\{\widetilde{h}_{j,k}\}_{k\in\mathbb{Z}}$ is frame of the $\mathscr{W}_j$ subspace for each j, instead $\{g_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is frame of all the space H, which does not necessarily mean that $\{g_{j,k}\}_{k\in\mathbb{Z}}$ be a frame of $\mathscr{W}_j$.
- If m_j and M_j are optimal bounds for the outer frame $\{h_{j,k}\}_{k\in\mathbb{Z}}$ of $\mathscr{W}_j$, then m_j and M_j will also be bounds for the frame $\{P_j h_{j,k}\}_{k\in\mathbb{Z}}$ of $\mathscr{W}_j$. Then, if $\{\widetilde{h}_{j,k}\}_{k\in\mathbb{Z}}$ is its canonical dual, it will have optimal bounds M_j^{-1} and m_j^{-1}, and besides $M^{-1} \le \inf_j M_j^{-1}$ and $\sup_j m_j^{-1} \le m^{-1}$ (for m and M lower and upper bounds of the family, respectively). Therefore, $\{\widetilde{h}_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a family of frames associated to the same collection of subspaces. In addition $\{\widetilde{h}_{j,k}\}_{k\in\mathbb{Z}}$ is a frame of $\mathscr{W}_j$ $\forall\, j \in \mathbb{Z}$, then $\{\widetilde{h}_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a frame of all the space.

It would be very interesting, to find conditions, if any, for that $\{\widetilde{h}_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ be a family of frames (not necessarily a frame of all the space). We note that this is not the goal of this work.

3.1.1 Formula of Reconstruction for A Particular Case of Family of Frames

If in a Hilbert space H we consider a family of frames $\{h_{jk}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ linked to subspaces $\{\mathscr{W}_j\}_{j\in\mathbb{Z}}$, and $\forall j$, the outer frame $\{h_{jk}\}_{k\in\mathbb{Z}}$ of $\mathscr{W}_j$ is such that $\{P_j h_{jk}\}_{k\in\mathbb{Z}}$ is a Parseval frame of $\mathscr{W}_j$, of (13):

$$f = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} \langle f_j, h_{j,k}\rangle P_j h_{j,k}. \tag{15}$$

The expression (15) is due to $\{P_j h_{jk}\}_{k\in\mathbb{Z}}$ is dual of itself.

3.2 Analysis and Synthesis in L^2 from Generators of MRA Subspaces

The classic definition of MRA in $L^2(\mathbb{R})$ introduced by Mallat (1989) ([6]), was later generalized in different ways:

(1) to $L^2(\mathbb{R}^d)$ with dyadic dilation $A = 2.I_d$;

(2) to $L^2(\mathbb{R}^d)$ but with arbitrary dilation (being A expansive but not dyadic),

(3) considering Riesz bases instead of orthogonal bases of subspaces of the MRA;

(4) taking frames for the generation of these subspaces of the MRA. The last ones, introduced by Benedetto and Treiber ([2]) are known as Analysis of Multiresolution Frame;

(5) admitting a finite number of scaling functions (the wavelets associated with these MRA, are called multiwavelets), in this case were exposed necessary and sufficient conditions for the existence of wavelet basis associated ([3]).

(6) Baggett et al. (1999) ([1]) extended the definition to Generalized Analysis of Multiresolution considering an abelian group of unitary operators Γ over H called *group of translations*, with the property that the subspace V_0 is invariant under the action of Γ.

We introduce here a new generalization of the concept of MRA.

Definition 3.7. Given $\{\mathscr{V}, A, \mathbb{Z}^d\}$, with $\mathscr{V} = \{\mathscr{V}_j\}_{j\in\mathbb{Z}}$ a family of closed subspaces of $L^2(\mathbb{R}^d)$, expansive $A \in GL_d(\mathbb{R})$, such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, we say that it is a *Affine Generalized Multiresolution Analysis* (AGMRA) of $L^2(\mathbb{R}^d)$), if:

(1) $\mathscr{V}_j \subset \mathscr{V}_{j+1}$ for each $j \in \mathbb{Z}$,

(2) $g \in \mathscr{V}_j \; sii \; g \circ A \in \mathscr{V}_{j+1}$ for each $j \in \mathbb{Z}$,

(3) $\bigcap\limits_{j\in\mathbb{Z}} \mathscr{V}_j = \{0\}$,

(4) $\overline{\bigcup\limits_{j\in\mathbb{Z}} \mathscr{V}_j} = L^2(\mathbb{R}^d)$, and

(5) there exist $\varphi \in \mathscr{V}_0$ such that $\{\varphi(x-k)\}_{k\in\mathbb{Z}^d}$ is an outer frame of the subspace $\mathscr{V}_0$.

The function φ is also called *scale function* of the AGMRA, in this work.

Remark 3.8. Due to (2) and (5) the set $\{|\det A|^{j/2}\varphi(A^j x-k)\}_{k\in\mathbb{Z}^d}$ results outer frame of the subspace $\mathscr{V}_j$. The Definition 3.7 allows, as in the classic case, the decomposition of the space $L^2(\mathbb{R}^d)$ in direct sum of mutually orthogonal subspaces. Defining $\mathscr{W}_j$ for each $j \in \mathbb{Z}$ as the orthogonal complement of $\mathscr{V}_j$ in $\mathscr{V}_{j+1}$, using (4) of the Definition 3.7 we have

$$L^2(\mathbb{R}^d) = \overset{\perp}{\bigoplus_{j\in\mathbb{Z}}} \mathscr{W}_j. \tag{16}$$

As in the classic case we define here the concept of *association* a family of frames to AGMRA.

Let $\{\mathscr{V}, A, \mathbb{Z}^d\}$ be a AGMRA of $L^2(\mathbb{R}^d)$, let M be a finite or numerable set of indexes, and $\{\psi_i\}_{i\in M}$ be a subset of $L^2(\mathbb{R}^d)$. If the affine system:

$$\{|\det A|^{j/2}\psi_i(A^j x-k)\}_{i\in M, j\in\mathbb{Z}, k\in\mathbb{Z}^d} \tag{17}$$

is a family of frames of $L^2(\mathbb{R}^d)$, we say that it is a *family of frames of wavelets* associated to AGMRA $\{\mathscr{V}, A, X\}$ if, for each $j \in \mathbb{Z}$

$$\{|det\, A|^{j/2}\psi_i(A^j x-x_k)\}_{i\in M, k\in\mathbb{Z}}$$

is an outer frame of the subspace $\mathscr{W}_j$ orthogonal complement of $\mathscr{V}_j$ in $\mathscr{V}_{j+1}$.

We present a significant result using this definition. Under certain conditions for the bounds of the frames of subspaces of the AGMRA, it is possible to have a particular process of decomposition or analysis, and a process of reconstruction or synthesis of the vector space from the generating family of the AGMRA.

Theorem 3.9. *Let $\{\mathscr{V}, A, \mathbb{Z}^d\}$ be a AGMRA of $L^2(\mathbb{R}^d)$ with φ scale function, and let m_j and M_j be the bounds of the outer frame $\{|det\ A|^{j/2}\varphi(A^j x-k)\}_{k\in\mathbb{Z}^d}$ of $\mathscr{V}_j$. If:*

$$0 < m =: \inf_j m_j \le M =: \sup_j M_j < \infty$$

then:

(1) *$\{|\det\ A|^{j/2}\varphi(A^j x-k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a family of frames for $L^2(\mathbb{R}^d)$ associated to the AGMRA $\{\mathscr{V}, A, \mathbb{Z}^d\}$ with bounds m and M; and*

(2) *$\{P_{j-1}\circ\varphi_{jk}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ is frame for $L^2(\mathbb{R}^d)$ with bounds m and M (P_{j-1} is the orthogonal projection over $\mathscr{W}_{j-1}$, with $\mathscr{V}_j = \mathscr{W}_{j-1}\oplus^{\perp}\mathscr{V}_{j-1}$, and $\varphi_{jk}(x) := |\det\ A|^{j/2}\varphi(A^j x-k)$).*

Proof. Let $f \in L^2(\mathbb{R}^d)$, then $f = \sum_{j\in\mathbb{Z}} f_j$, where $f_j \in \mathscr{W}_j$ is the orthogonal projection of f over the closed subspace $\mathscr{W}_j$ orthogonal complement of $\mathscr{V}_j$ in $\mathscr{V}_{j+1}$, with

$$\{\varphi_{j+1,k} = |\det\ A|^{(j+1)/2}\varphi(A^{j+1}x-k)\}_{k\in\mathbb{Z}^d}$$

an outer frame of $\mathscr{V}_{j+1}$ with bounds m_{j+1} and M_{j+1}, then $\{\varphi_{j+1,k}\}_{k\in\mathbb{Z}^d}$ is outer frame of $\mathscr{W}_j$ with the same bounds:

$$m_{j+1}\|f_j\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f_j, \varphi_{j+1,k}\rangle|^2 \le .M_{j+1}\|f_j\|^2 \tag{18}$$

Summing over j in the above expression, considering that $\|f\|^2 = \sum_{j\in\mathbb{Z}}\|f_j\|^2$ and assumptions about m and M:

$$m\|f\|^2 \le \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} |\langle f_j, \varphi_{j+1,k}\rangle|^2 \le M\|f\|^2. \tag{19}$$

(19) implies that $\{|\det\ A|^{j/2}\varphi(A^j x-k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a family of frames of $L^2(\mathbb{R}^d)$ with bounds m and M associated to the AGMRA $\{\mathscr{V}, A, \mathbb{Z}^d\}$.

$\{\langle f_j, \varphi_{j+1,k}\rangle\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is the set *analysis* or coefficients of f in the family of frames.

Moreover (19) is equivalent to:

$$m\|f\|^2 \le \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} |\langle f, P_j\ \varphi_{j+1,k}\rangle|^2 \le M\|f\|^2 \tag{20}$$

and (20), states that the set $\{P_{j-1}\ \varphi_{jk}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is frame of $L^2(\mathbb{R}^d)$ with bounds m and M. □

Example. Consider the Haar MRA in $L^2(\mathbb{R})$ whose scaling function is $\varphi(x) = \chi_{[0,1)}(x)$. Being $\{2^{j/2}\chi_{[0,1)}(2^j\cdot -k)\}_{k\in\mathbb{Z}}$ an orthonormal basis of the subspace $\mathscr{V}_j$ of MRA, then it is an outer frame of the orthogonal complement $\mathscr{V}_{j-1}$ in $\mathscr{V}_j$. Applying Theorem 3.9:

$$\{2^{j/2}\chi_{[0,1)}(2^j\cdot -k)\}_{k\in\mathbb{Z},j\in\mathbb{Z}}$$

is a Parseval family of frames of $L^2(\mathbb{R})$, associated with the MRA Haar, and the set:

$$\{P_{j-1}\, 2^{j/2}\chi_{[0,1)}(2^j\cdot -k)\}_{k\in\mathbb{Z},j\in\mathbb{Z}}$$

is a Parseval frame of $L^2(\mathbb{R})$.

The reconstruction procedure given in Section 3.1.1 provides the following formula of synthesis for the vectors of $L^2(\mathbb{R})$:

$$f = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} \langle f_j, \varphi_{j+1,k}\rangle P_j .\varphi_{j+1,k}\ . \tag{21}$$

This is equivalent to:

$$f = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} \langle f, P_j\varphi_{j+1,k}\rangle P_j\varphi_{j+1,k} \tag{22}$$

be $\varphi_{j,k} = 2^{j/2}\chi_{[0,1)}(2^j\cdot -k)$, and

$$P_j\varphi_{j+1,k} = \begin{cases} 2^{-1/2}\psi_{j,\frac{k}{2}} & \text{if } k \text{ is even} \\ -2^{-1/2}\psi_{j,\frac{k-1}{2}} & \text{if } k \text{ is odd} \end{cases}$$

being $\psi_{j,k} = 2^{j/2}(\chi_{[\frac{2k}{2^{j+1}},\frac{2k+1}{2^{j+1}})} - \chi_{[\frac{2k+1}{2^{j+1}},\frac{2k+2}{2^{j+1}})})$.

According to Theorem 3.9, the set $\{P_{j-1}\, 2^{j/2}\chi_{[0,1)}(2^j\cdot -k)\}_{k\in\mathbb{Z},j\in\mathbb{Z}}$ is a Parseval frame of $L^2(\mathbb{R})$. This also follows from (22).

Interestingly, the set of previous example $\{\varphi_{jk} = 2^{j/2}\chi_{[0,1)}(2^j\cdot -k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is not frame of $L^2(\mathbb{R})$:

Let $f = \chi_{[0,1)}$. Let's see that $\sum_{j\in\mathbb{Z},k\in\mathbb{Z}} |\langle f,\varphi_{jk}\rangle|^2$ is not convergent:

$$\begin{aligned}\sum_{j\in\mathbb{Z},k\in\mathbb{Z}} |\langle f,\varphi_{jk}\rangle|^2 &= \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} |\langle \chi_{[0,1)}, 2^{j/2}\chi_{[\frac{k}{2^j},\frac{k+1}{2^j})}\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}}\sum_{k=0}^{2^j-1} 2^j\left|\int_{[0,1)\cap[\frac{k}{2^j},\frac{k+1}{2^j})} dx\right|^2 \\ &= \sum_{j\in\mathbb{Z}}\sum_{k=0}^{2^j-1} 2^j\frac{1}{2^{2j}} = \sum_{j\in\mathbb{Z}} 1\,.\end{aligned}$$

4. Conclusions

These new systems of representation of Hilbert spaces, families of frames, are even less restrictive than frames and therefore much less restrictive than the orthonormal and Riesz bases of such spaces. Moreover, the well-known wavelets are simple mathematical tools with a great variety of applications in diverse areas such as multidimensional signal processing and compression, numerical analysis, computer graphics and artificial vision, among others.

From this, the interest and extension of this research for the development of *families of wavelet frames* of the form $\{h_{jk} = |\det\, A|^{j/2}\psi(A^jx-k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$, such that $\{h_{jk}\}$ is a family of frames with properties adapted to each application, is extended.

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Competing Interests

Author declares that he has no competing interests.

Authors' Contributions

Author wrote, read and approved the final manuscript.

References

[1] L. Baggett, H. Medina and K. Merrill, Generalized multi-resolution analyses and a construction procedure for all wavelet sets in $\mathbb{R}^n$, *Journal Journal of Fourier Analysis and Applications* **5** (1999), 1531 – 5851.

[2] J.J. Benedetto and O. Treiber, Wavelets frames: multiresolution and extension principles, in *Wavets Trasnforms and Time Frequency Signal Analysis*, L. Debanath (editor), Birkausser, Boston (2001), 1 – 36.

[3] C. Cabrelli and M.L. Gordillo, Existence of multiwavelets in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* **130** (5) (2002), 1413 – 1424.

[4] R.J. Duffin and A.C. Schaeffer, A class of nonharmonic Fourier Series, *Trans. Amer. Math. Soc.* **72** (1952), 341 – 366.

[5] S. Li and H. Ogawa, A theory of pseudoframes for subspaces with applications, *Tech. Report* (1998).

[6] S. Mallat, Multiresolution approximation and wavelets, *Trans. Amer. Math. Soc.* **315** (1989).