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(*Invited paper*)

An Embedding Theorem given by the Modulus of Variation

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Abstract. In [5] and [6] we extended an interesting theorem of Medvedeva [7] pertaining to the embedding relation $H^\omega \subset \Lambda BV$, where ΛBV denotes the set of functions of Λ-bounded variation. Our theorem proved in [6] unifies the notion of φ-variation due to Young [8] and that of the generalized Wiener class $BV(p(n)\uparrow)$ due to Kita and Yoneda [4].

In this note we generalize the theorem proved in [6] such that it will use the concept of the modulus of variation due to Chanturia [2]. For further references pertaining to the new notion mentioned above we refer to an interesting paper by Goginava and Tskhadaia [3].

We also show that our new theorem includes our previous result as a special case.

1. **Introduction**

Let $\omega(\delta)$ be a nondecreasing continuous function on the interval $[0,1]$ having the following properties:

$$\omega(0)=0,\ \omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_1) \ \text{ for } \ 0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 1.$$

Such a function is called a modulus of continuity, and it will be denoted by $\omega(\delta) \in \Omega$.

The modulus of continuity of a continuous function will be denoted by $\omega(f,\delta)$, that is,

$$\omega(f;\delta) := \sup_{\substack{0\leqq h\leqq \delta \\ 0\leqq x\leqq 1-h}} |f(x+h)-f(x)|.$$

As usual, set

$$H^\omega := \{f \in C : \omega(f;\delta) = O(\omega(\delta))\}.$$

Chanturia [2] introduced the concept of the *modulus of variation* for everywhere bounded 1-periodic functions.

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The modulus of variation of a function f is the function $v(f,n)$ with domain the positive integers, defined by

$$v(f,n) := \sup_{\pi_n} \sum_{k=1}^{n} |f(b_k) - f(a_k)|, \tag{1.1}$$

where π_n is an arbitrary system of n disjoint subintervals (a_k, b_k) of $(0,1)$.

Let $v(n)$ be a nondecreasing and upwards convex function on $[0,\infty)$. Such a function is called a modulus of variation, and it will be denoted by $v(n) \in \mathscr{V}$. It is clear that any $v(f,n) \in \mathscr{V}$.

If $v(n) \in \mathscr{V}$ is given, then $V[v(n)]$ denotes the class of functions for which $v(f,n) = O(v(n))$.

Chanturia has extensively investigated the uniform and absolute convergence of Fourier series if $f \in V[(v,n)]$ (see e.g. the references given in [3]).

Next we define a generalization of the class $V[v(n)]$.

Let $\Phi := \{\varphi_k\}$ be a sequence of nondecreasing functions $\varphi_k \colon [0,\infty) \to \mathbb{R}$ and $\varphi_k(0) = 0$; and let $\Lambda := \{\lambda_k\}$ be a nondecreasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

If $v(n) \in \mathscr{V}$ is given, then $V[\Phi, \Lambda, v(n)]$ denotes the class of functions $f : [0,1] \to \mathbb{R}$ for which the condition

$$\sup_{\pi_n} \sum_{k=1}^{n} \varphi_k(|f(b_k) - f(a_k)|)\lambda_k^{-1} = O(v(n)), \tag{1.2}$$

holds, where π_n is defined at (1.1).

A sequence $t := \{t_k\}$ satisfying the conditions

$$t_k \geqq 0 \quad \text{and} \quad \sum_{k=1}^{\infty} t_k \leqq 1,$$

will be denoted by $t \in T$.

In the sequel, Φ, Λ and T always have these properties.

Our theorem to be generalized reads as follows, where the class $\Lambda\{\varphi_n\}BV$ means the class $\Lambda[\Phi, \Lambda, v(n) \equiv 1]$ introduced now.

Theorem 1.1 ([6]). *Assume that $\omega(t) \in \Omega$ and for every $k \in \mathbb{N}$, $\varphi_k(\omega(\delta)) \in \Omega$. Then the embedding relation $H^\omega \subset \Lambda\{\varphi_k\}BV$ holds if and only if for any $t \in T$*

$$\sum_{k=1}^{\infty} \varphi_k(\omega(t_k))\lambda_k^{-1} < \infty. \tag{1.3}$$

2. **Results**

Our new theorem generalizes Theorem 1.1 proved in [6], which unified all of the former results mentioned in the Abstract.

Theorem 2.1. *Assume that $\omega(t) \in \Omega$, $\nu(n) \in \mathcal{V}$ and for every $k \in \mathbb{N}$, $\varphi_k(\omega(\delta)) \in \Omega$. Then the embedding relation $H^\omega \subset V[\Phi, \Lambda, \nu(n)]$ holds if and only if for every $n \in \mathbb{N}$ and for any $t \in T$*

$$\sum_{k=1}^{n} \varphi_k(\omega(t_k))\lambda_k^{-1} \leqq M\nu(n), \tag{2.1}$$

where M is a positive constant.

Remark 2.2. We shall show that if $\nu(n) = O(1)$, then condition (2.1) is equivalent to (1.3).

Hereby, in this special case, our theorem reduces to Theorem 1.1.

3. Lemmas

We shall use the following three lemmas.

Lemma 3.1 ([1, p. 78]). *If $\omega(\delta) \in \Omega$ then there exists a concave function $\omega^*(\delta)$ such that*

$$\omega(\delta) \leqq \omega^*(\delta) \leqq 2\omega(\delta).$$

Lemma 3.2 ([7]). *If $\omega(\delta) \in \Omega$ and $t := \{t_k\} \in T$, then there exists a function $f \in H^\omega$ such that if*

$$x_0 = 0, \; x_1 = \frac{t_1}{2},$$

$$x_{2n} = \sum_{i=1}^{n} t_i \quad \text{and} \quad x_{2n+1} = x_{2n} + \frac{t_{n+1}}{2}, \quad n \geqq 1,$$

then

$$f(x_{2n}) = 0 \quad \text{and} \quad f(x_{2n+1}) = \omega(t_{n+1}) \quad for\ all \quad n \geqq 0.$$

In [7] a concrete function with these properties is also given. This lemma plays again a cardinal role in the proof.

Lemma 3.3 ([6]). *If $\omega(t) \in \Omega$ and for all $k \in \mathbb{N}$, $\varphi_k(\omega(t)) \in \Omega$ also holds, furthermore for any $t \in T$ the condition* (1.3) *stays, then there exists a positive number M such that for any $t \in T$*

$$\sum_{k=1}^{\infty} \varphi_k(\omega(t_k))\lambda_k^{-1} \leqq M \tag{3.1}$$

holds.

4. Proof of Theorem 2.1

Necessity. Suppose that $H^\omega \subset V[\Phi, \Lambda, \nu(n)]$. Without loss of generality, due to Lemma 3.1, we can assume that, for every k, the functions $\varphi_k(\omega(\delta))$ are concave moduli of continuity.

Indirectly, let us assume that there is no number M with property (2.1). Then for any $i \in \mathbb{N}$ there exists a sequence $t^{(i)} := \{t_{k,i}\} \in T$ such that

$$2^i \nu(n) < \sum_{k=1}^{n} \varphi_k(\omega(t_{k,i}))\lambda_k^{-1} . \tag{4.1}$$

Now define

$$t_k := \sum_{i=1}^{\infty} \frac{t_{k,i}}{2^i} . \tag{4.2}$$

It is easy to see that $t := \{t_k\} \in T$.

Since every $\varphi_k(\omega(\delta))$ is concave, thus by Jensen's inequality, we obtain that

$$\varphi_k(\omega(t_k)) = \varphi_k\left(\omega\left(\sum_{i=1}^{\infty} \frac{t_{k,i}}{2^i}\right)\right) \geqq \sum_{i=1}^{\infty} \frac{\varphi_k(\omega(t_{k,i}))}{2^i}. \tag{4.3}$$

Utilizing (4.1) and (4.3) we get that

$$\begin{aligned} \sum_{k=1}^{n} \varphi_k(\omega(t_k))\lambda_k^{-1} &\geqq \sum_{k=1}^{n} \lambda_k^{-1} \sum_{i=1}^{\infty} \varphi_k(\omega(t_{k,i}))2^{-i} \\ &\geqq \sum_{i=1}^{n} 2^{-i} \sum_{k=1}^{n} \varphi_k(\omega(t_{k,i}))\lambda_k^{-1} \\ &\geqq n\nu(n) . \end{aligned} \tag{4.4}$$

Then, applying Lemma 3.2 with this t and $\omega(\delta)$, we obtain that there exists a function $f \in H^{\omega}$ such that

$$|f(x_{2k-1}) - f(x_{2k-2})| = \omega(t_k) \quad \text{for all } k .$$

Hence, by (4.4), we get that

$$\begin{aligned} \nu(n)^{-1} \sum_{k=1}^{n} \varphi_k(|f(x_{2k-1}) - f(x_{2k-1})|)\lambda_k^{-1} &= \nu(n)^{-1} \sum_{k=1}^{n} \varphi_k(\omega(t_k))\lambda_k^{-1} \\ &\geqq n, \end{aligned}$$

that is, (1.2) does not hold if $b_k = x_{2k-1}$ and $a_k = x_{2k-2}$, thus f does not belong to the class $V[\Phi, \Lambda, \nu(n)]$.

This and the assumption $H^{\omega} \subset V[\Phi, \Lambda, \nu(n)]$ contradict, whence the necessity of (2.1) follows.

Sufficiency. If we consider an arbitrary system of n disjoint subintervals (a_k, b_k) of $(0, 1)$ and take $t_k := (b_k - a_k)$, then $t := \{t_k\} \in T$; consequently for this t (2.1) also holds. Thus, if $f \in H^{\omega}$, by (2.1) we have that

$$\begin{aligned} \nu(n)^{-1} \sum_{k=1}^{n} \varphi_k(|f(b_k) - f(a_k)|)\lambda_k^{-1} &\leqq \nu(n)^{-1} \sum_{k=1}^{n} \varphi_k(\omega(b_k - a_k))\lambda_k^{-1} \\ &\leqq M , \end{aligned}$$

and, by (1.2), this shows that $f \in V[\Phi, \Lambda, \nu(n)]$.

Hereby we verified that the condition (2.1) is also sufficient to the embedding relation $H^{\omega} \subset V[\Phi, \Lambda, \nu(n)]$.

The proof is complete.

5. **Proof of Remark 2.2**

If $v(n) = O(1)$ and (2.1) holds for all n, then (2.1) is equivalent to (3.1). Clearly (3.1) implies (1.3), furthermore, by Lemma 3.3, (1.3) implies (3.1). Since, then (3.1) equivalent to (2.1), therefore the equivalence of (2.1) and (1.3) are proved, herewith the proof is complete.

This also shows that Theorem 2.1 in the special case $v(n) = O(1)$ includes Theorem 1.1.

References

[1] A.V. Efimov, Linear methods of approximation of continuous periodic functions, *Mat. Sb.* **54**(1961), 51–90 (in Russian).

[2] Z.A. Chanturia, The modulus of variation and its application in the theory of Fourier series, *Dokl. Akad. Nauk SSSR* **214**(1974), 63–66 (in Russian).

[3] U. Goginava and V. Tskhadaia, On the embedding $V[v(n)] \subset H_p^\omega$, *Proc. of A. Razmadze Math. Inst.* **136**(2004), 47–54.

[4] H. Kita and K. Yoneda, A generalization of bounded variation, *Acta Math. Hungar.* **56**(1990), 229–238.

[5] L. Leindler, A note on embedding of classes H^ω, *Analysis Math.* **27**(2001), 71–76.

[6] L. Leindler, On embedding of the class H^ω, *J. Inequal. Pure and Appl. Math.* **5**(2004) (4) (Article 105), 1–11.

[7] M.V. Medvedeva, On embedding H^ω, *Mat. Zametki* **64**(5)(1998), 713–719 (in Russian).

[8] L.C. Young, Sur une g n ralization de la notion de variation de Wiener et sur la convergence des s ries de Fourier, *C.R. Acad. Sci. Paris* **204**(1937), 470–472.

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