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Some Remarks on Positive Solutions of Nonlinear Problems at Resonance

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Abstract. The proof of a result of J.J. Nieto [3] appeared in "*Acta Math. Hung.*" (1992) concerning the positive solutions of nonlinear problems at resonance is corrected and improved.

1. Introduction

The Method of differential inequalities or the method of upper and lower solutions has been used by Nieto [3] to show the existence of positive periodic solutions for a second order nonlinear differential equation. Nieto [3] has obtained two existence results of positive and negative solutions for a class of nonlinear problems at resonance. However we would like to point out that the proof of the first main result (Theorem 6) in [3] is not correct. We also improve Theorem 7 of [3]. The correction of the proof of Theorem 6 in [3] is the motivation of this brief paper.

2. Positive Solutions and the Method of Upper and Lower Solutions

J.J. Nieto in the paper [3] studied the existence of positive periodic solutions of the equation

$$u'' + u + \mu u^2 = h(t), \quad u(0) = u(\tau), \quad u'(0) = u'(\tau), \tag{2.1}$$

where $h(t) = \epsilon \cos \omega t$ is $\tau = 2\pi\omega^{-1}$ periodic, $\mu \neq 0$, $\epsilon \neq 0$ and $\omega > 0$.

Nieto and Rao in [2] gave the following result:

Theorem 2.1. *Equation* (2.1) *has a periodic solution if* $4|\mu\epsilon| < 1$.

Making $s = \omega t$, (2.1) becomes

$$u'' + \omega^{-2}[u + \mu u^2 - \epsilon \cos s] = 0, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{2.2}$$

where $u = u(s)$ and $u'' = \frac{d^2u}{ds^2}$.

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Thus we are interested in the existence of 2π-periodic solutions of (2.2) and note that it is of the form

$$-u''(t) = f(t,u),\ t \in [0,2\pi],\ \ u(0) = u(2\pi),\ \ u'(0) = u'(2\pi). \tag{2.3}$$

As usual, we say that $\alpha \in C^2([0,2\pi],\mathbb{R})$ is a lower solution of (2.3) if

$$\begin{cases} -\alpha''(t) \le f(t,\alpha(t)), \text{ for } t \in [0,2\pi], \\ \alpha(0) = \alpha(2\pi), \text{ and } \alpha'(0) \ge \alpha'(2\pi). \end{cases} \tag{2.4}$$

Similarly, $\beta \in C^2([0,2\pi],\mathbb{R})$ is an upper solution of (2.3) if

$$\begin{cases} -\beta''(t) \ge f(t,\beta(t)), \text{ for } t \in [0,2\pi], \\ \beta(0) = \beta(2\pi), \text{ and } \beta'(0) \le \beta'(2\pi). \end{cases} \tag{2.5}$$

Theorem 2.2. [1]. *If* (2.3) *has an upper solution* β *and a lower solution* α *such that* $\alpha \le \beta$ *in* $[0,2\pi]$*, then there exists at least one solution* u *of* (2.3) *with* $\alpha \le u \le \beta$ *in* $[0,2\pi]$.

We are now in a position to prove the following result due to Nieto [3] and then we critically observe that it corrects the proof of Theorem 6 of [3] and improves it since we do not impose any condition on the sign of the real parameter ϵ.

Theorem 2.3. *If* $\mu < 0$ *and* $4|\mu\epsilon| < 1$*, then there exists a positive* $(2\pi\omega^{-1})$*-periodic solution of* (2.1).

Proof. Note that equation (2.2) can be written in the form

$$-u''(s) = f(s,u), \tag{2.6}$$

where $f(s,u) = \omega^{-2}[u + \mu u^2 - \epsilon\cos s]$.

For all arbitrary $\epsilon \neq 0$ and $\mu < 0$, let $0 < a_2 < a_1$ be the real roots of $\mu a^2 + a - |\epsilon| = 0$ and $b_2 < 0 < b_1$ the real roots of $\mu b^2 + b + |\epsilon| = 0$. Note that $b_2 < 0 < a_2 < a_1 < b_1$.

Choose $r \in [a_2, a_1]$ and $R \ge b_1$ and define $\alpha(s) = r$ and $\beta(s) = R$ $(r < R)$ for $s \in [0,2\pi]$. Since $-|\epsilon| \le -\epsilon\cos s \le |\epsilon|$; we obtain

$$\begin{aligned} f(s,\beta(s)) &= \omega^{-2}(R + \mu R^2 - \epsilon\cos s) \\ &\le \omega^{-2}(R + \mu R^2 + |\epsilon|) \\ &\le 0 = -\beta''(s), \end{aligned}$$

$$\begin{aligned} f(s,\alpha(s)) &= \omega^{-2}(r + \mu r^2 - \epsilon\cos s) \\ &\ge \omega^{-2}(r + \mu r^2 - |\epsilon|) \\ &\ge 0 = -\alpha''(s). \end{aligned}$$

Therefore, by Theorem 2.2, there exists a solution u of (2.2) such that $u \ge r > 0$. This complete the proof. □

Now, we shall improve Theorem 7 in [3] since we do not require $\epsilon < 0$.

Theorem 2.4. *If* $\mu > 0$ *and* $4|\mu\epsilon| < 1$*, then* (2.1) *has a negative* $(2\pi\omega^{-1})$*-periodic solution.*

Proof. The same argument as in Theorem 7 of [3] will be used. □

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