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Research Article

Some Fixed Point Theorems for Generalized α-η-ψ-Geraghty Contractive Type Mappings in Partial b-Metric Spaces

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Abstract. In this paper, we introduce the notion of generalized α-η-ψ-Geraghty contractive type mappings in the set up of partial b-metric spaces and α-orbital attractive mappings with respect to η. Furthermore, the fixed point theorems for such mappings in complete partial b-metric spaces are proven without assuming the subadditivity of ψ. Some examples are also provided for supporting of our main results.

Keywords. Generalized α-η-ψ-Geraghty contractive type mappings; α-orbital attractive mappings with respect to η; Complete partial b-metric spaces; Fixed points

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1. Introduction and Preliminaries

One of the most important results in fixed point theory is the Banach contraction principle [3] because of its application in many branches of mathematics and mathematical sciences. In 1993, Czerwik [4] introduced the concept of b-metric spaces afterward the concept of partial metric spaces is introduced by Matthews [13] in 1994. In 2013, Shukla [21] introduced the partial b-metric spaces by unification two notions of b-metric spaces and partial metric spaces. Mustafa [15] gave a modified version of partial b-metric spaces which it is dependent on b-metric spaces and proved some common fixed point results for (ψ,φ)-weakly contractive mappings in the set up of ordered partial b-metric spaces.

Generalization of the Banach contraction principle given by Geraghty [7] is one of the most interesting results. Later, Harandi and Emami [1] characterized the result of Geraghty [7] in the context of a partially ordered complete metric space. In 2013, Cho *et al*. [5] defined the concept of α-Geraghty contractive type mappings in the setting of metric spaces. On the other hand, Karapinar [10] investigated the existence and uniqueness of fixed point of generalization of α-ψ-Geraghty contractive type mappings under new conditions concerning with triangular α-admissible mappings. In 2014, Mukheimer [14] introduced the concept of α-ψ-φ-contractive mappings in complete ordered partial b-metric spaces and studied fixed points for such mappings. Recently, Popescu [16] generalized the results obtained in [5] and gave triangular α-orbital admissible conditions to prove fixed point theorems.

For the sake of completeness, we recall some basic definitions and fundamental results.

Let $\mathscr{F}$ be the class of all functions $\beta:[0,\infty)\to[0,1)$ satisfying the following condition:

$$\lim_{n\to\infty}\beta(t_n)=1 \quad \text{implies} \quad \lim_{n\to\infty}t_n=0.$$

Remark 1.1. We illustrate some interesting properties of functions in $\mathscr{F}$.

(1) The class $\mathscr{F}$ is nonempty. Indeed, for each $\alpha\in[0,1)$ we define $\beta:[0,\infty)\to[0,1)$ by $\beta_\alpha(t)=\alpha$ for all $t\in[0,\infty)$. We obtain that $\beta_\alpha\in\mathscr{F}$ and $\mathscr{F}$ is uncountable.

(2) There exists a differentiable function which does not belong to the class $\mathscr{F}$. For example, take $\beta(t)=\frac{t}{1+t}$ for all $t\in[0,\infty)$. If we put $t_n=n$ for all $n\in\mathbb{N}$, then we have $\lim\limits_{n\to\infty}\frac{t_n}{1+t_n}=1$ but $\lim\limits_{n\to\infty}t_n\neq 0$. Therefore $\beta\notin\mathscr{F}$.

(3) There exists a function in $\mathscr{F}$ which is not continuous. For instance,

$$\beta(t)=\begin{cases}\frac{1}{1+t}, & t>0;\\ 0, & t=0.\end{cases}$$

It is obviously that $\beta\in\mathscr{F}$ but it is not continuous from the right at $x=0$.

Theorem 1.2 (Geraghty [7]). *Let (X,d) be a complete metric space and $T:X\to X$ be a mapping. If T satisfies the following inequality:*

$$d(Tx,Ty)\le\beta(d(x,y))d(x,y),$$

for any $x,y\in X$, where $\beta\in\mathscr{F}$, then T has a unique fixed point.

Notice that T is a nonexpansive mapping and moreover, it is also a continuous function.

The results of Geraghty have attracted a numbers of authors [1,5,10,12,20,21].

Shukla [21] unified partial metrics and b-metric spaces by introducing the concept of partial b-metric space as follows.

Definition 1.3 ([21]). A partial b-metric on a nonempty set X is a mapping $p_b:X\times X\to[0,\infty)$ satisfying the following conditions for all $x,y,z\in X$:

(p_{b1}) $x=y$ if and only if $p_b(x,x)=p_b(x,y)=p_b(y,y)$;

(p_{b2}) $p_b(x,x) \le p_b(x,y)$;

(p_{b3}) $p_b(x,y) = p_b(y,x)$;

(p_{b4}) $p_b(x,y) \le s[p_b(x,z)+p_b(z,y)] - p_b(z,z)$.

A partial b-metric space is a pair (X,p_b) such that X is a nonempty set and p_b is a partial b-metric on X. The number $s \ge 1$ is called the coefficient of (X,p_b).

It is clear that every partial metric space is a partial b-metric space with the coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of these facts need not hold.

Example 1.4 ([15]). Let (X,d) be a metric space and $p_b(x,y) = d(x,y)^q + a$, where $q > 1$ and $a \ge 0$ are real numbers. Then p_b is a partial b-metric with the coefficient $s = 2^{q-1}$, but it is neither a b-metric nor a partial metric.

Note that in a partial b-metric space, the limit of a convergent sequence may not be unique (see [21, Example 2]). Some more examples of partial b-metrics can be constructed by using of the following propositions.

Proposition 1.5 ([21]). *Let X be a nonempty set, and let p be a partial metric and d be a b-metric with the coefficient $s \ge 1$ on X. Then the function $p_b : X \times X \to [0,\infty)$, defined by $p_b(x,y) = p(x,y)+d(x,y)$ for all $x, y \in X$, is a partial b-metric on X with the coefficient s.*

Proposition 1.6 ([21]). *Let (X,p) be a partial metric space and $q \ge 1$. Then (X,p_b) is a partial b-metric space with the coefficient $s = 2^{q-1}$, where p_b is defined by $p_b(x,y) = [p(x,y)]^q$.*

In the following definition, Mustafa [15] modified the Definition 1.3 in order to obtain that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 1.7 ([15]). Let X be a nonempty set and $s \ge 1$ be given a real number. A function $p_b : X \times X \to [0,\infty)$ is a partial b-metric if the following conditions are satisfied for all $x, y, z \in X$:

(p_{b1}) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$;

(p_{b2}) $p_b(x,x) \le p_b(x,y)$;

(p_{b3}) $p_b(x,y) = p_b(y,x)$;

(p_{b4}) $p_b(x,y) \le s(p_b(x,z)+p_b(z,y) - p_b(z,z)) + \left(\frac{1-s}{2}\right)(p_b(x,x)+p_b(y,y))$.

The pair (X,p_b) is called a partial b-metric space. The number $s \ge 1$ is called the coefficient of (X,p_b).

Proposition 1.8 ([15]). *Every partial b-metric space p_b defines a b-metric d_{p_b}, where*

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y) \quad \text{for all } x, y \in X.$$

Definition 1.9 ([15]). A sequence $\{x_n\}$ in a partial b-metric space (X,p_b) is said to be:

(i) p_b-convergent to a point $x \in X$ if $\lim_{n\to\infty} p_b(x,x_n) = p_b(x,x)$;

(ii) A p_b-Cauchy sequence if $x \in X$ if $\lim_{n,m\to\infty} p_b(x_n,x_m)$ exists (and is finite);

(iii) A partial b-metric space (X,p_b) is said to be p_b-complete if every p_b-Cauchy sequence $\{x_n\}$ in X p_b-converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} p_b(x_n,x_m) = \lim_{n\to\infty} p_b(x_n,x) = p_b(x,x).$$

The following lemma shows the relationship between the concepts of p_b-convergent sequence, p_b-Cauchy sequence and p_b-completeness in (X,p_b) and (X,d_{p_b}).

Lemma 1.10 ([15]). (1) *A sequence $\{x_n\}$ is a p_b-Cauchy sequence in a partial b-metric space (X,p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X,d_{p_b}).*

(2) *A partial b-metric space (X,p_b) is p_b-complete if and only if the b-metric space (X,d_{p_b}) is b-complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x,x_n) = 0$ if and only if*

$$\lim_{n\to\infty} p_b(x,x_n) = \lim_{n,m\to\infty} p_b(x_n,x_m) = p_b(x,x).$$

Definition 1.11 ([15]). Let (X,p_b) and (X',p'_b) be two partial b-metric spaces, and let $f:(X,p_b)\to(X',p'_b)$ be a mapping. Then f is said to be p_b-continuous at a point $a \in X$ if for a given ε, there exists $\delta > 0$ such that $x \in X$ and $p_b(a,x) < \delta + p_b(a,a)$ imply that $p'_b(f(a),f(x)) < \varepsilon + p'_b(f(a),f(a))$. The mapping f is p_b-continuous on X if it is p_b-continuous at a all $a \in X$.

Proposition 1.12 ([15]). *Let (X,p_b) and (X',p'_b) be two partial b-metric spaces. Then the mapping $f: X \to X'$ is p_b-continuous at a point $x \in X$ if and only if it is p_b-sequentially continuous at x; that is, whenever $\{x_n\}$ is p_b-convergent to x, $\{f(x_n)\}$ is p'_b-convergent to $f(x)$.*

The following vital lemma is useful in proving our main results.

Lemma 1.13 ([15]). *Let (X,p_b) be a partial b-metric space with the coefficient $s > 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then we have*

$$\begin{aligned}\frac{1}{s^2}p_b(x,y) - \frac{1}{s}p_b(x,x) - p_b(y,y) &\le \liminf_{n\to\infty} p_b(x_n,y_n)\\ &\le \limsup_{n\to\infty} p_b(x_n,y_n)\\ &\le s p_b(x,x) + s^2 p_b(y,y) + s^2 p_b(x,y).\end{aligned}$$

In particular, if $p_b(x,y) = 0$, then we have $\lim_{n\to\infty} p_b(x_n,y_n) = 0$.
Moreover, for each $z \in X$, we have

$$\begin{aligned}\frac{1}{s}p_b(x,z) - p_b(x,x) \le \liminf_{n\to\infty} p_b(x_n,z) &\le \limsup_{n\to\infty} p_b(x_n,z)\\ &\le s p_b(x,z) + s p_b(x,x).\end{aligned}$$

In particular, if $p_b(x,x) = 0$, then we have

$$\frac{1}{s}p_b(x,z) \le \liminf_{n\to\infty} p_b(x_n,z) \le \limsup_{n\to\infty} p_b(x_n,z) \le s p_b(z,z).$$

On the other hand, in 2012, Samet *et al.* [3] introduced the concept of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces. Afterward Salimi *et al.* [18] modified the notion of α-ψ-contractive and α-admissible mappings and established fixed point theorems which are proper generalizations of the recent results in [19], [8].

Definition 1.14 ([18])**.** Let T be a self mapping on X and $\alpha,\eta : X\times X \to [0,\infty)$ be two functions. We say that T is α-admissible with respect to η if for all $x,y\in X$,

$$\alpha(x,y)\ge \eta(x,y) \quad \text{implies} \quad \alpha(Tx,Ty)\ge \eta(Tx,Ty).$$

We say that T is α-admissible if for all $x,y\in X$,

$$\alpha(x,y)\ge 1 \quad \text{implies} \quad \alpha(Tx,Ty)\ge 1.$$

Karapinar *et al.* [10] introduced the new concept of triangular α-admissible mappings to investigate fixed points for such mappings in metric spaces.

Definition 1.15 ([10])**.** Let $T : X \to X$ and $\alpha : X\times X \to [0,\infty)$. We say that T is a triangular α-admissible mapping if

(T1) T is α-admissible;

(T2) $\alpha(x,z)\ge 1$ and $\alpha(z,y)\ge 1$ imply $\alpha(x,y)\ge 1$.

Definition 1.16 ([11])**.** Let Ψ' be a family of function $\psi : [0,\infty)\to[0,\infty)$ satisfies the following properties:

(i) ψ is continuous and nondecreasing;

(ii) $\psi(t)=0$ if and only if $t=0$;

(iii) ψ is subadditive, $\psi(s+t)\le\psi(s)+\psi(t)$.

Definition 1.17 ([10])**.** Let (X,d) be a metric space and $\alpha : X\times X\to[0,\infty)$. A mapping $T : X\to X$ is said to be a generalized α-ψ-Geraghty contractive type mapping if there exists $\beta\in\mathscr{F}$ such that

$$\alpha(x,y)\psi(d(Tx,Ty))\le \beta(\psi(M(x,y)))\psi(M(x,y)) \quad \text{for any } x,y\in X,$$

where $\quad M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}\quad$ and $\quad \psi\in\Psi'$.

Theorem 1.18 ([10])**.** *Let (X,d) be a complete metric space, $\alpha : X\times X\to[0,\infty)$ be a function, and let $T : X\to X$ be a mapping. Suppose that the following conditions are satisfied:*

(i) *T is a generalized α-ψ-Geraghty contractive type mapping;*

(ii) *T is a triangular α-admissible mapping;*

(iii) *there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\ge 1$;*

(iv) *T is a continuous mapping.*

Then T has a fixed point $x^*\in X$ and $\{T^n x_1\}$ converges to x^*.

We are interesting in a class of Ψ by omitting the subadditivity of ψ.

Definition 1.19. Let Ψ be a family of function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and nondecreasing;

(ii) $\psi(t)=0$ if and only if $t=0$.

The family Ψ is convex. Moreover, condition (i) is independent of (ii) and conversely. For example, $\psi(t)=\ln(t+2)$ satisfies condition (i), but $\psi(t)\neq 0$ when $t=0$ and $\psi(t)=\frac{t}{t-1}$ fails at $t=1$ which implies ψ is not a continuous function but $\psi(t)=0$ if and only if $t=0$.

In 2014, Popescu [16] introduced three new concepts of α-orbital admissible, triangular α-orbital admissible and α-orbital attractive mappings.

Definition 1.20 ([16])**.** Let $T:X\to X$ be a mapping and $\alpha:X\times X\to[0,\infty)$ be a function. Then T is said to be triangular α-orbital admissible if

(O1) T is α-orbital admissible, that is, $\alpha(x,Tx)\geq 1$ implies $\alpha(Tx,T^2x)\geq 1$;

(O2) $\alpha(x,y)\geq 1$ and $\alpha(y,Ty)\geq 1$ imply $\alpha(x,Ty)\geq 1$.

Definition 1.21 ([16])**.** $T:X\to X$ be a mapping and $\alpha:X\times X\to[0,\infty)$ be a function. Then T is said to be α-orbital attractive if

$$\alpha(x,Tx)\geq 1 \quad \text{implies} \quad \alpha(x,y)\geq 1 \text{ or } \alpha(y,Tx)\geq 1$$

for every $y\in X$.

Theorem 1.22 ([16])**.** *Let (X,d) be a complete metric space, $\alpha:X\times X\to[0,\infty)$ be a function, and let $T:X\to X$ be a mapping. Suppose that the following conditions are satisfied:*

(1) *T is a generalized α-Geraghty contractive type mapping;*

(2) *T is an α-orbital admissible mapping;*

(3) *there exists $x_*\in X$ such that $\alpha(x_*,Tx_*)\geq 1$;*

(4) *T is an α-orbital attractive mapping.*

Then T has a fixed point $x_*\in X$ and $\{T^nx_*\}$ converges to x_*.

In 2016, Chuadchawna *et al*. [6] introduced the concept of triangular α-orbital admissible mappings with respect to η and proved the lemma which will be used efficiently for proving our main results.

Definition 1.23 ([6])**.** Let $T:X\to X$ be a mapping and $\alpha,\eta:X\times X\to[0,\infty)$ be functions. Then T is said to be α-orbital admissible with respect to η if

$$\alpha(x,Tx)\geq\eta(x,Tx) \quad \text{implies} \quad \alpha(Tx,T^2x)\geq\eta(Tx,T^2x).$$

Definition 1.24 ([6])**.** Let $T:X\to X$ be a mapping and $\alpha,\eta:X\times X\to[0,\infty)$ be functions. Then T is said to be triangular α-orbital admissible with respect to η if

(T1) T is α-orbital admissible with respect to η;

(T2) $\alpha(x,y)\geq\eta(x,y)$ and $\alpha(y,Ty)\geq\eta(y,Ty)$ imply $\alpha(x,Ty)\geq\eta(x,Ty)$.

Remark 1.25. If we suppose that $\eta(x,y) = 1$ for all $x,y \in X$, then Definition 1.24 reduces to Definition 1.20.

Lemma 1.26 ([6])**.** *Let $T : X \to X$ be a triangular α-orbital admissible mapping with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n,x_m) \geq \eta(x_n,x_m)$ for all $m,n \in \mathbb{N}$ with $n < m$.*

In this paper, we introduce the notion of generalized α-η-ψ-Geraghty contractive type mappings and α-orbital attractive mappings with respect to η in the set up of partial b-metric spaces. Furthermore, the fixed point theorems for such mappings which are triangular α-orbital admissible with respect to η in complete partial b-metric spaces are proven without assuming the subadditivity of ψ. Examples are also provided for supporting of our main results. Our results generalize and extend the results proved by [6], [10], [16].

2. Main Results

2.1 Generalized α-η-ψ-Geraghty Contractive Type Mappings with Fixed Point Theorems

We now introduce the concept of generalized α-η-ψ-Geraghty contractive type mappings on partial b-metric spaces.

Definition 2.1. Let (X,p_b) be a partial b-metric space with the coefficient $s \geq 1$. A mapping $T : X \to X$ is said to be a generalized α-η-ψ-Geraghty contractive type mapping if there exist $\psi \in \Psi$, $\alpha,\eta : X \times X \to [0,\infty)$ and $\beta \in \mathscr{F}$ such that

$$\alpha(x,y) \geq \eta(x,y) \text{ implies } \psi(sp_b(Tx,Ty)) \leq \beta(\psi(M_s^T(x,y)))\psi(M_s^T(x,y)) \tag{1}$$

for all $x,y \in X$, where

$$M_s^T(x,y) = \max\left\{p_b(x,y), p_b(x,Tx), p_b(y,Ty), \frac{p_b(x,Ty)+p_b(y,Tx)}{2s}\right\}.$$

If we suppose that $\eta(x,y) = 1$ for all $x,y \in X$ and let

$$M_s^T(x,y) = M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$

then Definition 2.1 reduces to Definition 1.17 in the setting of metric spaces.

Theorem 2.2. *Let (X,p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. Let $T : X \to X$ be a generalized α-η-ψ-Geraghty contractive type mapping. Suppose that the following conditions hold:*

(i) *T is a triangular α-orbital admissible mapping with respect to η;*

(ii) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$;*

(iii) *if $\{x_n\}$ is a p_b-convergent sequence to z in X and $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for each $n \in \mathbb{N}$, then $\alpha(z,z) \geq \eta(z,z)$;*

(iv) *T is continuous.*

Then T has a fixed point.

Proof. Let $x_1 \in X$ such that $\alpha(x_1,Tx_1) \ge \eta(x_1,Tx_1)$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. By Lemma 1.26, we get that

$$\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1}), \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Assume that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. We first prove that the sequence $\{p_b(x_n,x_{n+1})\}$ is nonincreasing and tends to 0 as $n \to \infty$. By using (2), for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\psi(sp_b(x_{n+1},x_{n+2})) &= \psi(sp_b(Tx_n,Tx_{n+1}))\\ &\le \beta(\psi(M_s^T(x_n,x_{n+1})))\psi(M_s^T(x_n,x_{n+1}))\\ &< \psi(M_s^T(x_n,x_{n+1})),\end{aligned} \tag{3}$$

where

$$\begin{aligned}M_s^T(x_n,x_{n+1}) &= \max\left\{p_b(x_n,x_{n+1}),p_b(x_n,Tx_n),p_b(x_{n+1},Tx_{n+1}),\frac{p_b(x_n,Tx_{n+1})+p_b(x_{n+1},Tx_n)}{2s}\right\}\\ &= \max\left\{p_b(x_n,x_{n+1}),p_b(x_n,x_{n+1}),p_b(x_{n+1},x_{n+2}),\frac{p_b(x_n,x_{n+2})+p_b(x_{n+1},x_{n+1})}{2s}\right\}\\ &\le \max\Big\{p_b(x_n,x_{n+1}),p_b(x_n,x_{n+1}),p_b(x_{n+1},x_{n+2}),\\ &\qquad \frac{sp_b(x_n,x_{n+1})+sp_b(x_{n+1},x_{n+2})+(1-s)p_b(x_{n+1},x_{n+1})}{2s}\Big\}\\ &= \max\{p_b(x_n,x_{n+1}),p_b(x_{n+1},x_{n+2})\}.\end{aligned} \tag{4}$$

If $\max\{p_b(x_n,x_{n+1}),p_b(x_{n+1},x_{n+2})\} = p_b(x_{n+1},x_{n+2})$, then $\psi(sp_b(x_{n+1},x_{n+2})) < \psi(p_b(x_{n+1},x_{x+2}))$ which contradicts to $\psi(sp_b(x_{n+1},x_{n+2})) \ge \psi(p_b(x_{n+1},x_{x+2}))$.
This implies that $\max\{p_b(x_n,x_{n+1}),p_b(x_{n+1},x_{n+2})\} = p_b(x_n,x_{n+1})$.
It follows that $0 < p_b(x_{n+1},x_{n+2}) \le p_b(x_n,x_{n+1})$. Hence the sequence $\{p_b(x_n,x_{n+1})\}$ is nonnegative nonincreasing and bounded below.
It follows that there exists $r \ge 0$ such that

$$\lim_{n\to\infty} p_b(x_n,x_{n+1}) = r.$$

Suppose that $r > 0$. By using (3), we have

$$\begin{aligned}\frac{\psi(p_b(x_{n+1},x_{n+2}))}{\psi(p_b(x_n,x_{n+1}))} &\le \frac{\psi(sp_b(x_{n+1},x_{n+2}))}{\psi(p_b(x_n,x_{n+1}))}\\ &\le \beta(\psi(M_s^T(x_n,x_{n+1}))) < 1,\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} \beta(\psi(M_s^T(x_n,x_{n+1}))) = 1.$$

Since $\beta \in \mathscr{F}$, we have $\lim_{n\to\infty} \psi(M_s^T(x_n,x_{n+1})) = 0$ and so

$$r = \lim_{n\to\infty} p_b(x_n,x_{n+1}) = 0. \tag{5}$$

We next prove that $\{x_n\}$ is a p_b-Cauchy sequence in (X,p_b) by proving that $\{x_n\}$ is a b-Cauchy sequence in (X,d_{p_b}). Suppose that $\{x_n\}$ is not a b-Cauchy sequence in (X,d_{p_b}). Then there exists $\varepsilon > 0$ such that for all $k > 0$, there exist $n(k) > m(k) > k$ for which we can find two subsequences

$\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$d_{p_b}(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \tag{6}$$

and

$$d_{p_b}(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{7}$$

Therefore

$$\begin{aligned}\varepsilon \leq d_{p_b}(x_{m(k)}, x_{n(k)}) &\leq s d_{p_b}(x_{m(k)}, x_{n(k)-1}) + s d_{p_b}(x_{n(k)-1}, x_{n(k)}) \\ &< s\varepsilon + s d_{p_b}(x_{n(k)-1}, x_{n(k)}).\end{aligned} \tag{8}$$

Taking the lower limit for (8) as $k \to \infty$, we have

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}) \leq \limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon. \tag{9}$$

From (8) and (9), we obtain that

$$\varepsilon \leq \limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)}) \leq s\varepsilon.$$

By using the triangular inequality, we have

$$\begin{aligned}d_{p_b}(x_{m(k)+1}, x_{n(k)}) &\leq s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s d_{p_b}(x_{m(k)}, x_{n(k)}) \\ &\leq s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s^2 d_{p_b}(x_{m(k)}, x_{n(k)-1}) + s^2 d_{p_b}(x_{n(k)-1}, x_{n(k)}) \\ &\leq s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s^2\varepsilon + s^2 d_{p_b}(x_{n(k)-1}, x_{n(k)}).\end{aligned}$$

By taking the upper limit as $k \to \infty$ in the above inequality, we obtain that

$$\limsup_{k\to\infty} d_{p_b}(x_{m(k)+1}, x_{n(k)}) \leq s^2\varepsilon.$$

Similarly, we also have

$$\begin{aligned}d_{p_b}(x_{m(k)+1}, x_{n(k)-1}) &\leq s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s d_{p_b}(x_{m(k)+1}, x_{n(k)-1}) \\ &\leq s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s\varepsilon.\end{aligned}$$

By taking the upper limit as $k \to \infty$ in the above inequality, this yields

$$\limsup_{k\to\infty} d_{p_b}(x_{m(k)+1}, x_{n(k)-1}) \leq s\varepsilon.$$

By using the definition of d_{p_b} and (9), we obtain that

$$2\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) = \limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}).$$

It follows that

$$\frac{\varepsilon}{2s} \leq \liminf_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) \leq \limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) \leq \frac{\varepsilon}{2}. \tag{10}$$

Similarly, we can prove that,

$$\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)}) \leq \frac{s\varepsilon}{2}, \tag{11}$$

$$\frac{\varepsilon}{2s} \leq \limsup_{k\to\infty} p_b(x_{m(k)+1}, x_{n(k)}), \tag{12}$$

$$\limsup_{k\to\infty} p_b(x_{m(k)+1}, x_{n(k)-1}) \leq \frac{s\varepsilon}{2}. \tag{13}$$

Since T is a triangular α-orbital admissible mapping with respect to η and using (3), we obtain that $\alpha(x_{m(k)}, x_{n(k)-1}) \geq \eta(x_{m(k)}, x_{n(k)-1})$. By using (1), we have

$$\psi(sp_b(x_{m(k)+1}, x_{n(k)})) \leq \beta(\psi(M_s^T(x_{m(k)}, x_{n(k)-1})))\psi(M_s^T(x_{m(k)}, x_{n(k)-1})) \tag{14}$$

where

$$\begin{aligned}
M_s^T(x_{m(k)}, x_{n(k)-1}) &= \max\Big\{p_b(x_{m(k)}, x_{n(k)-1}), p_b(x_{m(k)}, Tx_{m(k)}), p_b(x_{n(k)-1}, Tx_{n(k)-1}), \\
&\qquad \frac{p_b(x_{m(k)}, Tx_{n(k)-1}) + p_b(x_{n(k)-1}, Tx_{m(k)})}{2s}\Big\} \\
&= \max\Big\{p_b(x_{m(k)}, x_{n(k)-1}), p_b(x_{m(k)}, x_{m(k)+1}), p_b(x_{n(k)-1}, x_{n(k)}), \\
&\qquad \frac{p_b(x_{m(k)}, x_{n(k)}) + p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\Big\}.
\end{aligned} \tag{15}$$

Taking the upper limit as $k \to \infty$ in the above inequality using (5), (10), (11) and (13), this yields

$$\begin{aligned}
\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1}) &= \max\Big\{\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}), \limsup_{k\to\infty} p_b(x_{m(k)}, x_{m(k)+1}), \\
&\qquad \limsup_{k\to\infty} p_b(x_{n(k)-1}, x_{n(k)}), \\
&\qquad \frac{\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)}) + \limsup_{k\to\infty} p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\Big\} \\
&= \max\Big\{\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}), 0, 0, \\
&\qquad \frac{\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)}) + \limsup_{k\to\infty} p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\Big\} \\
&\leq \max\Big\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big\} \\
&= \frac{\varepsilon}{2}.
\end{aligned} \tag{16}$$

By taking the upper limit in (14) as $k \to \infty$ and using (12) and (16), we have

$$\begin{aligned}
\psi(s\frac{\varepsilon}{2s}) &\leq \psi(\limsup_{k\to\infty} sp_b(x_{m(k)+1}, x_{n(k)})) \\
&\leq \beta(\psi(\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1})))\psi(\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1})) \\
&\leq \beta(\psi(\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1})))\psi(\frac{\varepsilon}{2}).
\end{aligned}$$

This implies that

$$\frac{\psi(\frac{\varepsilon}{2})}{\psi(\frac{\varepsilon}{2})} \leq \beta(\psi(\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1}))).$$

Since $\beta \in \mathscr{F}$, we have

$$\lim_{k\to\infty} \beta(\psi(\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1}))) = 1.$$

It follows that

$$\psi(\limsup_{k\to\infty} M_s^T(x_{m(k)},x_{n(k)-1})) = 0.$$

By using (14) we obtain,

$$\limsup_{k\to\infty} p_b(x_{m(k)},x_{n(k)-1}) = 0, \tag{17}$$

which contradicts to(10). Therefore the sequence $\{x_n\}$ is a b-Cauchy sequence in the b-metric space (X,d_{p_b}). Since (X,p_b) is p_b-complete, then (X,d_{p_b}) is b-complete. This implies that there exists $z \in X$ such that $\lim_{n\to\infty} d_{p_b}(x_n,z) = 0$. By applying Proposition 1.8, we have

$$2p_b(x_n,z) = d_{p_b}(x_n,z) + p_b(x_n,x_n) + p_b(z,z) \le d_{p_b}(x_n,z) + p_b(x_n,x_{n+1}) + p_b(x_n,z).$$

Therefore $p_b(x_n,z) \le d_{p_b}(x_n,z) + p_b(x_n,x_{n+1})$. By taking the limit as $n \to \infty$, we obtain that $\lim_{n\to\infty} p_b(x_n,z) = 0$. By Lemma 1.10, we have

$$0 = \lim_{n\to\infty} p_b(x_n,z) = \lim_{n\to\infty} p_b(x_n,x_m) = \lim_{n\to\infty} p_b(z,z).$$

We next prove that $z = Tz$. Suppose that $z \ne Tz$. By using the triangular inequality, we obtain that

$$p_b(z,Tz) \le s p_b(z,Tx_n) + s p_b(Tx_n,Tz).$$

By taking limit as $n \to \infty$ in the above inequality and using the continuity of T, we have

$$p_b(z,Tz) \le s \lim_{n\to\infty} p_b(z,x_{n+1}) + s \lim_{n\to\infty} p_b(Tx_n,Tz) = s p_b(Tz,Tz). \tag{18}$$

Since $\alpha(z,z) \ge \eta(z,z)$ and using (1), we have

$$\psi(s p_b(Tz,Tz)) \le \beta(\psi(M_s^T(z,z)))\psi(M_s^T(z,z)),$$

where

$$M_s^T(z,z) = \max\left\{p_b(z,z), p_b(z,Tz), p_b(z,Tz), \frac{p_b(z,Tz)+p_b(z,Tz)}{2s}\right\} = p_b(z,Tz). \tag{19}$$

Therefore

$$\psi(s p_b(Tz,Tz)) \le \beta(\psi(p_b(z,Tz)))\psi(p_b(z,Tz)) < \psi(p_b(z,Tz)). \tag{20}$$

Since ψ is nondecreasing, we have $s p_b(Tz,Tz) \le p_b(z,Tz)$. This implies that $s p_b(Tz,Tz) = p_b(z,Tz)$. From (20), we can deduce that

$$\frac{\psi(s p_b(Tz,Tz))}{\psi(p_b(z,Tz))} \le \beta(\psi(p_b(z,Tz))).$$

We obtain that

$$\lim_{n\to\infty} \beta(\psi(p_b(z,Tz))) = 1.$$

Therefore $p_b(z,Tz) = 0$. This implies that $p_b(z,z) = p_b(z,Tz) = p_b(Tz,Tz) = 0$. That is $Tz = z$ and thus z is a fixed point of T. □

We now investigate the fixed point result without continuity of a mapping T.

Definition 2.3. Let (X,p_b) be a p_b-complete partial b-metric space with the coefficient $s \ge 1$, $\alpha,\eta : X \times X \to [0,\infty)$ be functions, and let T be a self mapping on X. The sequence $\{x_n\}$ is α-regular with respect to η provided the following condition is satisfied: if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is p_b-convergent to x, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq \eta(x_{n(k)},x)$ for all $k \in \mathbb{N}$.

In the following theorem, we replace the continuity of the mapping T in Theorem 2.2 by α-regularity with respect to η.

Theorem 2.4. *Let (X,p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. Let $T: X \to X$ be a generalized α-η-ψ-Geraghty contractive type mapping. Suppose that the following conditions hold:*

(i) *T is a triangular α-orbital admissible mapping with respect to η;*

(ii) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$;*

(iii) *$\{x_n\}$ is α-regular with respect to η.*

Then T has a fixed point.

Proof. By the same proof as in Theorem 2.2, we can construct the sequence $\{x_n\}$ in X defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is p_b-convergent to z for some $z \in X$. By (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},z) \geq \eta(x_{n(k)},z)$ for all $n \in \mathbb{N}$. Since T is a generalized α-η-ψ-Geraghty contractive type mapping, we have

$$\psi(sp_b(Tx_{n(k)},Tz)) \leq \beta(\psi(M_s^T(x_{n(k)},z)))\psi(M_s^T(x_{n(k)},z)), \tag{21}$$

where,

$$\begin{aligned}
M_s^T(x_{n(k)},z) &= \max\left\{p_b(x_{n(k)},z), p_b(x_{n(k)},Tx_{n(k)}), p_b(z,Tz), \frac{p_b(x_{n(k)},Tz)+p_b(Tx_{n(k)},z)}{2s}\right\} \\
&= \max\left\{p_b(x_{n(k)},z), p_b(x_{n(k)},x_{n(k)+1}), p_b(z,Tz), \frac{p_b(x_{n(k)},Tz)+p_b(x_{n(k)+1},z)}{2s}\right\} \\
&\leq \max\Big\{p_b(x_{n(k)},z), p_b(x_{n(k)},x_{n(k)+1}), p_b(z,Tz), \\
&\qquad \frac{sp_b(x_{n(k)},z)+sp_b(z,Tz)+p_b(x_{n(k)+1},z)}{2s}\Big\}.
\end{aligned} \tag{22}$$

By taking the upper limit as $k \to \infty$ in above inequality, we have

$$\limsup_{k\to\infty} M_s^T(x_{n(k)},z) \leq p_b(z,Tz). \tag{23}$$

From (21) and using Lemma 1.13, then taking the upper limit as $k \to \infty$, we obtain that

$$\begin{aligned}
\psi(p_b(z,Tz)) &= \psi(s\frac{1}{s}p_b(z,Tz) \\
&\leq \psi(s\liminf_{k\to\infty} p_b(x_{n(k)+1},Tz)) \\
&\leq \psi(s\limsup_{k\to\infty} p_b(x_{n(k)+1},Tz)) \\
&\leq \beta(\psi(\limsup_{k\to\infty} M_s^T(x_{n(k)},z)))\psi(\limsup_{k\to\infty} M_s^T(x_{n(k)},z)) \\
&\leq \beta(\psi(\limsup_{k\to\infty} M_s^T(x_{n(k)},z)))\psi(p_b(z,Tz)).
\end{aligned}$$

This implies that

$$\lim_{k\to\infty} \beta(\psi(\limsup_{k\to\infty} M_s^T(x_{n(k)},z))) = 1.$$

Therefore

$$\psi(\limsup_{k\to\infty} M_s^T(x_{n(k)},z)) = 0,$$

and then we have

$$\limsup_{k\to\infty} M_s^T(x_{n(k)},z) = 0. \tag{24}$$

Using Lemma 1.13 and (24), this yields

$$\begin{aligned}\frac{\frac{p_b(z,Tz)}{2s}}{s} &\le \liminf_{k\to\infty} \frac{p_b(x_{n(k)},Tz)}{2s}\\ &\le \liminf_{k\to\infty} \frac{p_b(x_{n(k)},Tz)+p_b(x_{n(k)+1},z)}{2s}\\ &\le \liminf_{k\to\infty} M_s^T(x_{n(k)},z)\\ &\le \limsup_{k\to\infty} M_s^T(x_{n(k)},z)\\ &\le p_b(z,Tz).\end{aligned}$$

Thus $p_b(z,Tz)=0$. Since $p_b(Tz,Tz)\le sp_b(Tz,z)+sp_b(z,Tz)$, we have $p_b(z,z)=p_b(z,Tz)=p_b(Tz,Tz)$ which implies that $z=Tz$. Hence z is a fixed point of T. □

We now give an example to support Theorem 2.4.

Example 2.5. Let $X=[0,\infty)$ and with the partial b-metric $p_b: X\times X\to[0,\infty)$ defined by $p_b(x,y)=[\max\{x,y\}]^2$ for all $x,y\in X$. Obviously, (X,p_b) is a partial b-metric space with $s=2$. Define the mapping $T:X\to X$ given by

$$Tx=\begin{cases}\frac{x}{9} & \text{if } x\in[0,1];\\ \ln x+3 & \text{if } x\in(1,\infty).\end{cases}.$$

Define $\psi:[0,\infty)\to[0,\infty)$ and $\beta:[0,\infty)\to[0,1)$ by $\psi(t)=t$ and

$$\beta(t)=\begin{cases}\frac{e^{-t}}{1+t} & \text{if } t\in(0,\infty);\\ \frac{1}{2} & \text{if } t=0.\end{cases}$$

Let $\alpha,\eta: X\times X\to[0,\infty)$ defined by

$$\alpha(x,y)=\begin{cases}6 & \text{if } x\in[0,1];\\ 0 & \text{if } x\in(1,\infty),\end{cases}$$

and

$$\eta(x,y)=\begin{cases}2 & \text{if } x\in[0,1];\\ 1 & \text{if } x\in(1,\infty).\end{cases}$$

Let $\alpha(x,Tx)\ge\eta(x,Tx)$. Thus $x,Tx\in[0,1]$ and so $T^2x=T(Tx)\in[0,1]$ which implies that $\alpha(Tx,T^2x)\ge\eta(Tx,T^2x)$ that is T is α-orbital admissible with respect to η. Now, let

$\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$, we get that $x,y,Ty \in [0,1]$ and so $\alpha(x,Ty) \geq \eta(x,Ty)$. Therefore T is triangular α-orbital admissible with respect to η. Let $\{x_n\}$ be a sequence such that $\{x_n\}$ is p_b-convergent to z and $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Then $\{x_n\} \subseteq [0,1]$ for any $n \in \mathbb{N}$ and so $z \in [0,1]$ which we have, $\alpha(x_n,z) \geq \eta(x_n,z)$. That is $\{x_n\}$ is α-regular with respect to η. The condition (ii) of Theorem 2.4 satisfied with $x_1 = 1 \in X$ since $\alpha(1,T1) = 6 \geq 2 = \eta(1,T1)$. We next prove that T is a generalized α-η-ψ-Geraghty contraction type mapping. Let $x,y \in X$ with $\alpha(x,y) \geq \eta(x,y)$. Thus $x,y \in [0,1]$. Without loss of generality, we may assume that $0 \leq y \leq x \leq 1$. Therefore

$$p_b(Tx,Ty) = \left[\max\left\{\frac{x}{9},\frac{y}{9}\right\}\right]^2 = \frac{x^2}{81}$$

and

$$M_s^T(x,y) = \max\left\{x^2, x^2, y^2, \frac{x^2 + [\max\{y, \frac{x}{9}\}]^2}{4}\right\} = x^2.$$

Since $\frac{2}{81} \leq \frac{1}{2e} \leq \frac{e^{-x^2}}{1+x^2}$, we obtain that

$$\begin{aligned}\psi(sp_b(Tx,Ty)) = \psi(2\frac{x^2}{81}) = \frac{2x^2}{81} &\leq \frac{e^{-x^2}}{1+x^2}\cdot x^2\\ &\leq \beta(\psi(x^2))\psi(x^2)\\ &\leq \beta(\psi(M_s^T(x,y)))\psi(M_s^T(x,y)).\end{aligned}$$

Thus T is a generalized α-η-ψ-Geraghty contraction type mapping. Hence all assumptions in Theorem 2.4 are satisfied and thus T has a fixed point which is $x = 0$.

2.2 α-orbital Attractive Mappings with Fixed Point Theorems

We now introduce the new concept of α-orbital attractive mappings with respect to η and investigate some fixed point theorems.

Definition 2.6. Let $T: X \to X$ be a mapping and $\alpha,\eta: X \times X \to [0,\infty)$ be functions. Then T is said to be an α-orbital attractive mapping with respect to η if

$$\alpha(x,Tx) \geq \eta(x,Tx) \text{ imply } \alpha(x,y) \geq \eta(x,y) \text{ or } \alpha(y,Tx) \geq \eta(y,Tx)$$

for every $y \in X$.

If we set $\eta(x,y) = 1$ for all $x,y \in X$, then it satisfies the Definition 1.21.

Theorem 2.7. *Let (X,p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. Let $T: X \to X$ be a generalized α-η-ψ-Geraghty contractive type mapping. Suppose that the following conditions hold:*

(i) *T is an α-orbital admissible mapping with respect to η;*

(ii) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$;*

(iii) *T is an α-orbital attractive mapping with respect to η.*

Then T has a fixed point.

Proof. Let $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Since T is an α-orbital admissible mapping with respect to η, we obtain that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{25}$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By applying (25) and since T is a generalized α-η-ψ-Geraghty contractive type mapping, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\psi(sp_b(x_{n+1}, x_{n+2})) &= \psi(sp_b(Tx_n, Tx_{n+1}))\\ &\leq \beta(\psi(M_s^T(x_n, x_{n+1})))\psi(M_s^T(x_n, x_{n+1}))\\ &< \psi(M_s^T(x_n, x_{n+1})),\end{aligned} \tag{26}$$

where

$$\begin{aligned}M_s^T(x_n, x_{n+1}) &= \max\left\{p_b(x_n, x_{n+1}), p_b(x_n, Tx_n), p_b(x_{n+1}, Tx_{n+1}), \frac{p_b(x_n, Tx_{n+1}) + p_b(x_{n+1}, Tx_n)}{2s}\right\}\\ &= \max\left\{p_b(x_n, x_{n+1}), p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{n+2}), \frac{p_b(x_n, x_{n+2}) + p_b(x_{n+1}, x_{n+1})}{2s}\right\}\\ &\leq \max\Big\{p_b(x_n, x_{n+1}), p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{n+2}),\\ &\qquad \frac{sp_b(x_n, x_{n+1}) + sp_b(x_{n+1}, x_{n+2}) + (1-s)p_b(x_{n+1}, x_{n+1})}{2s}\Big\}\\ &= \max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{n+2})\}.\end{aligned}$$

If $\max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{n+2})\} = p_b(x_{n+1}, x_{n+2})$. By (26), we obtain that $\psi(sp_b(x_{n+1}, x_{n+2})) < \psi(p_b(x_{n+1}, x_{x+2}))$ which contradicts to $\psi(sp_b(x_{n+1}, x_{n+2})) \geq \psi(p_b(x_{n+1}, x_{x+2}))$. This implies that $\max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{n+2})\} = p_b(x_n, x_{n+1})$. It follows that $0 < p_b(x_{n+1}, x_{n+2}) \leq p_b(x_n, x_{n+1})$. Hence the sequence $\{p_b(x_n, x_{n+1})\}$ is nonnegative nonincreasing and bounded below. Thus there exists some $r \geq 0$ such that

$$\lim_{n\to\infty} p_b(x_n, x_{n+1}) = r.$$

Suppose that $r > 0$. By (26), we have

$$\frac{\psi(p_b(x_{n+1}, x_{n+2}))}{\psi(p_b(x_n, x_{n+1}))} \leq \frac{\psi(sp_b(x_{n+1}, x_{n+2}))}{\psi(p_b(x_n, x_{n+1}))} \leq \beta(\psi(M_s^T(x_n, x_{n+1}))) < 1,$$

for all $n \in \mathbb{N}$. This yields that

$$\lim_{n\to\infty} \beta(\psi(M_s^T(x_n, x_{n+1}))) = 1.$$

Since $\beta \in \mathscr{F}$, we have $\lim_{n\to\infty} \psi(M_s^T(x_n, x_{n+1})) = 0$ and so

$$r = \lim_{n\to\infty} p_b(x_n, x_{n+1}) = 0. \tag{27}$$

We next prove that $\{x_n\}$ is a p_b-Cauchy sequence in (X, p_b) by proving that $\{x_n\}$ is a b-Cauchy sequence in (X, d_{p_b}). Suppose that $\{x_n\}$ is not a b-Cauchy sequence in (X, d_{p_b}). Then there exists $\varepsilon > 0$ such that for $k \in \mathbb{N}$, there exist $n(k) > m(k) > k$ for which we can find two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which,

$$d_{p_b}(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \tag{28}$$

and

$$d_{p_b}(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{29}$$

Then, we have

$$\begin{aligned}\varepsilon \le d_{p_b}(x_{m(k)}, x_{n(k)}) &\le s d_{p_b}(x_{m(k)}, x_{n(k)-1}) + s d_{p_b}(x_{n(k)-1}, x_{n(k)}) \\ &< s\varepsilon + s d_{p_b}(x_{n(k)-1}, x_{n(k)}).\end{aligned} \tag{30}$$

Taking the lower limit for (30) as $k \to \infty$, we have

$$\frac{\varepsilon}{s} \le \liminf_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}) \le \limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}) \le \varepsilon. \tag{31}$$

From (30) and (31), we obtain that

$$\varepsilon \le \limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)}) \le s\varepsilon.$$

By using the triangular inequality, we can deduce that

$$\begin{aligned}d_{p_b}(x_{m(k)+1}, x_{n(k)}) &\le s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s d_{p_b}(x_{m(k)}, x_{n(k)}) \\ &\le s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s^2 d_{p_b}(x_{m(k)}, x_{n(k)-1}) + s^2 d_{p_b}(x_{n(k)-1}, x_{n(k)}) \\ &\le s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s^2\varepsilon + s^2 d_{p_b}(x_{n(k)-1}, x_{n(k)}).\end{aligned}$$

By taking the upper limit as $k \to \infty$ in the above inequality, we have

$$\limsup_{k\to\infty} d_{p_b}(x_{m(k)+1}, x_{n(k)}) \le s^2\varepsilon.$$

We can also prove that

$$\begin{aligned}d_{p_b}(x_{m(k)+1}, x_{n(k)-1}) &\le s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s d_{p_b}(x_{m(k)+1}, x_{n(k)-1}) \\ &\le s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s\varepsilon.\end{aligned}$$

By taking the upper limit as $k \to \infty$ in the above inequality, we get that

$$\limsup_{k\to\infty} d_{p_b}(x_{m(k)+1}, x_{n(k)-1}) \le s\varepsilon.$$

By using the definition of d_{p_b}, we obtain that

$$2\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) = \limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}).$$

It follows that

$$\frac{\varepsilon}{2s} \le \liminf_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) \le \limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) \le \frac{\varepsilon}{2}. \tag{32}$$

Similarly, we can prove that.

$$\frac{\varepsilon}{2} \le \limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)}) \le \frac{s\varepsilon}{2}, \tag{33}$$

$$\frac{\varepsilon}{2s} \le \limsup_{k\to\infty} p_b(x_{m(k)+1}, x_{n(k)}) \le \frac{s^2\varepsilon}{2} \tag{34}$$

and

$$\limsup_{k\to\infty} p_b(x_{m(k)+1}, x_{n(k)-1}) \le \frac{s\varepsilon}{2}. \tag{35}$$

Since $\alpha(x_{n(k)-1}, x_{n(k)}) \ge \eta(x_{n(k)-1}, x_{n(k)})$ and T is an α-orbital attractive mapping with respect to η

and using (26), we obtain that $\alpha(x_{n(k)-1}, x_{m(k)}) \geq \eta(x_{n(k)-1}, x_{m(k)})$ or $\alpha(x_{m(k)}, x_{n(k)}) \geq \eta(x_{m(k)}, x_{n(k)})$.

We divide the proof in two cases as follows:

(1) There exists an infinite subset I of $\mathbb{N}$ such that $\alpha(x_{n(k)-1}, x_{m(k)}) \geq \eta(x_{n(k)-1}, x_{m(k)})$ for every $k \in I$.

(2) There exists an infinite subset J of $\mathbb{N}$ such that $\alpha(x_{m(k)}, x_{n(k)}) \geq \eta(x_{m(k)}, x_{n(k)})$ for every $k \in J$.

In the first case, since T is a generalized α-η-ψ-Geraghty contractive type mapping, we obtain that

$$\psi(sp_b(x_{n(k)}, x_{m(k)+1})) \leq \beta(\psi(M_s^T(x_{n(k)-1}, x_{m(k)})))\psi(M_s^T(x_{n(k)-1}, x_{m(k)})) \tag{36}$$

where

$$\begin{aligned} M_s^T(x_{n(k)-1}, x_{m(k)}) &= \max\Big\{p_b(x_{n(k)-1}, x_{m(k)}), p_b(x_{n(k)-1}, Tx_{n(k)-1}), p_b(x_{m(k)}, Tx_{m(k)}), \\ &\qquad \frac{p_b(x_{n(k)-1}, Tx_{m(k)}) + p_b(x_{m(k)}, Tx_{n(k)-1})}{2s}\Big\} \\ &= \max\Big\{p_b(x_{n(k)-1}, x_{m(k)}), p_b(x_{n(k)-1}, x_{n(k)}), p_b(x_{m(k)}, x_{m(k)+1}), \\ &\qquad \frac{p_b(x_{m(k)}, x_{n(k)}) + p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\Big\}. \end{aligned}$$

Taking the upper limit as $k \to \infty$ in the above inequality using (27), (32), (33) and (35), we get that

$$\begin{aligned} &\limsup_{k\to\infty, k\in I} M_s^T(x_{n(k)-1}, x_{m(k)}) \\ &= \max\Big\{\limsup_{k\to\infty, k\in I} p_b(x_{n(k)-1}, x_{m(k)}), \limsup_{k\to\infty, k\in I} p_b(x_{n(k)-1}, x_{n(k)}), \limsup_{k\to\infty, k\in I} p_b(x_{m(k)}, x_{m(k)+1}), \\ &\qquad \limsup_{k\to\infty, k\in I} \frac{p_b(x_{m(k)}, x_{n(k)}) + p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\Big\}. \\ &= \max\Big\{\limsup_{k\to\infty, k\in I} p_b(x_{n(k)-1}, x_{m(k)}), 0, 0, \frac{\limsup_{k\to\infty, k\in I} p_b(x_{m(k)}, x_{n(k)}) + \limsup_{k\to\infty, k\in I} p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\Big\} \\ &\leq \max\Big\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big\} \\ &= \frac{\varepsilon}{2}. \end{aligned} \tag{37}$$

By taking the upper limit in (36) as $k \to \infty$ and using (34) and (37), we have

$$\begin{aligned} \psi(s\frac{\varepsilon}{2s}) &\leq \psi(\limsup_{k\to\infty, k\in I} p_b(x_{n(k)}, x_{m(k)+1})) \\ &\leq \beta(\psi(\limsup_{k\to\infty, k\in I} M_s^T(x_{n(k)-1}, x_{m(k)})))\psi(\limsup_{k\to\infty, k\in I} M_s^T(x_{n(k)-1}, x_{m(k)})) \\ &\leq \beta(\psi(\limsup_{k\to\infty, k\in I} M_s^T(x_{n(k)-1}, x_{m(k)})))\psi(\frac{\varepsilon}{2}). \end{aligned}$$

Therefore

$$\frac{\psi(\frac{\varepsilon}{2})}{\psi(\frac{\varepsilon}{2})} \le \beta(\psi(\limsup_{k\to\infty,k\in I} M_s^T(x_{n(k)-1},x_{m(k)}))),$$

Since $\beta \in \mathscr{F}$, we obtain that

$$\lim_{k\to\infty,k\in I} \beta(\psi(\limsup_{k\to\infty,k\in I} M_s^T(x_{n(k)-1},x_{m(k)}))) = 1.$$

Therefore

$$\psi(\limsup_{k\to\infty,k\in I} M_s^T(x_{n(k)-1},x_{m(k)})) = 0.$$

By using (36), we obtain that

$$\limsup_{k\to\infty,k\in I} p_b(x_{n(k)-1},x_{m(k)}) = 0,$$

which contradicts to (32).

In the second case, since T is a generalized α-η-ψ-Geraghty contractive type mapping, we obtain that

$$\psi(sp_b(x_{m(k)+1},x_{n(k)+1})) \le \beta(\psi(M_s^T(x_{m(k)},x_{n(k)})))\psi(M_s^T(x_{m(k)},x_{n(k)})) \tag{38}$$

where

$$\begin{aligned} M_s^T(x_{m(k)},x_{n(k)}) &= \max\Big\{p_b(x_{m(k)},x_{n(k)}), p_b(x_{m(k)},Tx_{m(k)}), p_b(x_{n(k)},Tx_{n(k)}),\\ &\qquad \frac{p_b(x_{m(k)},Tx_{n(k)})+p_b(x_{n(k)},Tx_{m(k)})}{2s}\Big\}\\ &= \max\Big\{p_b(x_{m(k)},x_{n(k)}), p_b(x_{m(k)},x_{m(k)+1}), p_b(x_{n(k)},x_{n(k)+1}),\\ &\qquad \frac{p_b(x_{m(k)},x_{n(k)+1})+p_b(x_{n(k)},x_{m(k)+1})}{2s}\Big\}.\\ &\le \max\Big\{p_b(x_{m(k)},x_{n(k)}), p_b(x_{m(k)},x_{m(k)+1}), p_b(x_{n(k)},x_{n(k)+1}),\\ &\qquad \frac{sp_b(x_{m(k)},x_{n(k)})+sp_b(x_{n(k)},x_{n(k)+1})+p_b(x_{n(k)},x_{m(k)+1})}{2s}\Big\}. \end{aligned} \tag{39}$$

Taking the upper limit as $k\to\infty$ in the above inequality using (27), (32), (33) and (34), we get

$$\begin{aligned} \limsup_{k\to\infty,k\in J} M_s^T(x_{m(k)},x_{n(k)}) &= \max\Big\{\limsup_{k\to\infty,k\in J} p_b(x_{m(k)},x_{n(k)}), \limsup_{k\to\infty,k\in J} p_b(x_{m(k)},x_{m(k)+1}),\\ &\qquad \limsup_{k\to\infty,k\in J} p_b(x_{n(k)},x_{n(k)+1}),\\ &\qquad \limsup_{k\to\infty,k\in J} \frac{sp_b(x_{m(k)},x_{n(k)})+sp_b(x_{n(k)},x_{n(k)+1})+p_b(x_{n(k)},x_{m(k)+1})}{2s}\Big\}.\\ &= \max\Big\{\limsup_{k\to\infty,k\in J} p_b(x_{m(k)},x_{n(k)}), 0, 0,\\ &\qquad \frac{\limsup_{k\to\infty,k\in J} sp_b(x_{m(k)},x_{n(k)}) + \limsup_{k\to\infty,k\in J} p_b(x_{n(k)},x_{m(k)+1})}{2s}\Big\} \end{aligned}$$

$$\leq \max\left\{\frac{s\varepsilon}{2}, \frac{s\varepsilon}{2}\right\}$$
$$= \frac{s\varepsilon}{2}. \tag{40}$$

By taking the upper limit in (38) as $k \to \infty$ and using (33) and (40), we have

$$\psi(s\frac{\varepsilon}{2}) \leq \psi(\limsup_{k\to\infty, k\in J} p_b(x_{m(k)+1}, x_{n(k)+1}))$$
$$\leq \beta(\psi(\limsup_{k\to\infty, k\in J} M_s^T(x_{m(k)}, x_{n(k)})))\psi(\limsup_{k\to\infty, k\in J} M_s^T(x_{m(k)}, x_{n(k)}))$$
$$\leq \beta(\psi(\limsup_{k\to\infty, k\in J} M_s^T(x_{m(k)}, x_{n(k)})))\psi(\frac{s\varepsilon}{2}).$$

Therefore

$$\frac{\psi(\frac{s\varepsilon}{2})}{\psi(\frac{s\varepsilon}{2})} \leq \beta(\psi(\limsup_{k\to\infty, k\in J} M_s^T(x_{m(k)}, x_{n(k)}))).$$

Since $\beta \in \mathscr{F}$, we have

$$\lim_{k\to\infty, k\in J} \beta(\psi(\limsup_{k\to\infty, k\in J} M_s^T(x_{m(k)}, x_{n(k)}))) = 1.$$

Therefore

$$\psi(\limsup_{k\to\infty, k\in J} M_s^T(x_{m(k)}, x_{n(k)})) = 0.$$

By using (36), we obtain that

$$\limsup_{k\to\infty, k\in J} p_b(x_{n(k)}, x_{m(k)}) = 0.$$

which a contradiction to (33). This implies that the sequence $\{x_n\}$ is a b-Cauchy in the b-metric space (X, d_{p_b}). Since (X, p_b) is p_b-complete, then (X, d_{p_b}) is b-complete. It follows that there exists $z \in X$ such that $\lim_{n\to\infty} d_{p_b}(x_n, z) = 0$. We claim that $z = Tz$. Suppose on the contrary, that $z \neq Tz$. Since T is an α-orbital attractive mapping with respect to η, we have for each $n \in \mathbb{N}$ that $\alpha(x_n, z) \geq \eta(x_n, z)$ or $\alpha(z, x_{n+1}) \geq \eta(z, x_{n+1})$.

We divide the proof in two cases as follows.

(1) There exists an infinite subset I of $\mathbb{N}$ such that $\alpha(x_n, z) \geq \eta(x_n, z)$ for every $n \in I$.

(2) There exists an infinite subset J of $\mathbb{N}$ such that $\alpha(z, x_{n+1}) \geq \eta(z, x_{n+1})$ for every $n \in J$.

In the first case, since T is a generalized α-η-ψ-Geraghty contractive type mapping, we obtain that

$$\psi(sp_b(Tx_n, Tz)) \leq \beta(\psi(M_s^T(x_n, z)))\psi(M_s^T(x_n, z)), \tag{41}$$

where

$$M_s^T(x_n, z) = \max\left\{p_b(x_n, z), p_b(x_n, Tx_n), p_b(z, Tz), \frac{p_b(x_n, Tz) + p_b(Tx_n, z)}{2s}\right\}$$
$$= \max\left\{p_b(x_n, z), p_b(x_n, x_{n+1}), p_b(z, Tz), \frac{p_b(x_n, Tz) + p_b(x_{n+1}, z)}{2s}\right\}$$
$$\leq \max\left\{p_b(x_n, z), p_b(x_n, x_{n+1}), p_b(z, Tz), \frac{sp_b(x_n, z) + sp_b(z, Tz) + p_b(x_{n+1}, z)}{2s}\right\}. \tag{42}$$

By taking the upper limit in the above inequality, we obtain that

$$\limsup_{n\to\infty,n\in I} M_s^T(x_n,z) \le p_b(z,Tz).$$

From (41), using Lemma 1.13 and by taking the upper limit as $n\to\infty$, we obtain that

$$\begin{aligned}\psi(p_b(z,Tz)) &= \psi(s\frac{1}{s}p_b(z,Tz))\\ &\le \psi(s\liminf_{n\to\infty,n\in I} p_b(x_{n+1},Tz))\\ &\le \psi(s\limsup_{n\to\infty,n\in I} p_b(x_{n+1},Tz))\\ &\le \beta(\psi(\limsup_{n\to\infty,n\in I} M_s^T(x_n,z)))\psi(\limsup_{n\to\infty,n\in I} M_s^T(x_n,z))\\ &\le \beta(\psi(\limsup_{n\to\infty,n\in I} M_s^T(x_n,z)))\psi(p_b(z,Tz)).\end{aligned}$$

This implies that

$$\limsup_{n\to\infty,n\in I} \beta(\psi(\limsup_{n\to\infty,n\in I} M_s^T(x_n,z))) = 1.$$

Therefore

$$\psi(\limsup_{n\to\infty,n\in I} M_s^T(x_n,z)) = 0. \tag{43}$$

Using Lemma 1.13 and (43), we obtain that

$$\begin{aligned}\frac{\frac{p_b(z,Tz)}{2s}}{s} &\le \liminf_{n\to\infty}\frac{p_b(x_n,Tz)}{2s} \le \liminf_{n\to\infty}\frac{p_b(x_n,Tz)+p_b(x_{n+1},z)}{2s}\\ &\le \liminf_{n\to\infty} M_s^T(x_n,z)\\ &\le \limsup_{n\to\infty} M_s^T(x_n,z)\\ &\le p_b(z,Tz).\end{aligned}$$

This yields $p_b(z,Tz) = 0$. Since $p_b(Tz,Tz) \le sp_b(Tz,z) + sp_b(z,Tz)$, we have $p_b(z,z) = p_b(z,Tz) = p_b(Tz,Tz)$ which implies that $z = Tz$. Hence z is a fixed point of T.

In the second case, since T is a generalized α-η-ψ-Geraghty contractive type mapping, we obtain that

$$\psi(sp_b(Tz,Tx_{n+1})) \le \beta(\psi(M_s^T(z,x_{n+1})))\psi(M_s^T(z,x_{n+1})), \tag{44}$$

where

$$\begin{aligned}M_s^T(z,x_{n+1}) &= \max\left\{p_b(z,x_{n+1}), p_b(z,Tz), p_b(x_{n+1},Tx_{n+1}), \frac{p_b(z,Tx_{n+1})+p_b(x_{n+1},Tz)}{2s}\right\}\\ &= \max\left\{p_b(z,x_{n+1}), p_b(z,Tz), p_b(x_{n+1},x_{n+2}), \frac{p_b(z,Tx_{n+1})+p_b(x_{n+1},Tz)}{2s}\right\}\\ &\le \max\left\{p_b(z,x_{n+1}), p_b(z,Tz), p_b(x_{n+1},x_{n+2}), \frac{p_b(z,x_{n+2})+sp_b(x_{n+1},z)+sp_b(z,Tz)}{2s}\right\}.\end{aligned} \tag{45}$$

By taking the upper limit as above, we obtain

$$\limsup_{n\to\infty, n\in J} M_s^T(z,x_{n+1}) \le p_b(z,Tz).$$

From (44) and using Lemma 1.13, then taking the upper limit as $n\to\infty$, we obtain that

$$\begin{aligned}\psi(p_b(z,Tz)) &= \psi(s\frac{1}{s}p_b(z,Tz))\\ &\le \psi(s\liminf_{n\to\infty,n\in J} p_b(x_{n+2},Tz))\\ &\le \psi(s\limsup_{n\to\infty,n\in J} p_b(x_{n+2},Tz))\\ &\le \beta(\psi(\limsup_{n\to\infty,n\in J} M_s^T(z,x_{n+1})))\psi(\limsup_{n\to\infty,n\in J} M_s^T(z,x_{n+1}))\\ &\le \beta(\psi(\limsup_{n\to\infty,n\in J} M_s^T(z,x_{n+1})))\psi(p_b(z,Tz)).\end{aligned}$$

This implies that

$$\limsup_{n\to\infty,n\in J} \beta(\psi(\limsup_{n\to\infty,n\in J} M_s^T(z,x_{n+1}))) = 1.$$

Therefore

$$\psi(\limsup_{n\to\infty,n\in J} M_s^T(z,x_{n+1})) = 0. \tag{46}$$

Using Lemma 1.13 and (46), we get that

$$\begin{aligned}\frac{\frac{p_b(z,Tz)}{2s}}{s} &\le \liminf_{n\to\infty}\frac{p_b(x_{n+1},Tz)}{2s}\\ &\le \liminf_{n\to\infty}\frac{p_b(z,x_{n+2})+p_b(x_{n+1},Tz)}{2s}\\ &\le \liminf_{n\to\infty} M_s^T(z,x_{n+1})\\ &\le \limsup_{n\to\infty} M_s^T(z,x_{n+1})\\ &\le p_b(z,Tz).\end{aligned}$$

It follows that $p_b(z,Tz)=0$. Since $p_b(Tz,Tz)\le sp_b(Tz,z)+sp_b(z,Tz)$, we have $p_b(z,z)=p_b(z,Tz)=p_b(Tz,Tz)$ which implies that $z=Tz$. Hence z is a fixed point of T. □

The following example are given to support Theorem 2.7.

Example 2.8. Let $X=\{0,1,2,3\}$ with the partial b-metric $p_b: X\times X\to[0,\infty)$ define as $p_b(x,y)=|x-y|^2$. Obviously, (X,p_b) is a p_b-complete partial b-metric space with coefficient $s=2$ ([15, Example 3]). Define a mapping $T: X\to X$ by

$$T0=T1=2 \quad\text{and}\quad T2=T3=3.$$

Define $\psi:[0,\infty)\to[0,\infty)$ and $\beta:[0,\infty)\to[0,1)$ by $\psi(t)=\frac{t}{2}$ and $\beta(t)=\frac{1}{2}$, for each $t\in(0,\infty)$. Let $\alpha,\eta: X\times X\to[0,\infty)$ be defined by

$$\alpha(x,y)=\begin{cases}1 & \text{if } (x,y)\in\{(1,2),(2,1)\};\\ 6 & \text{otherwise,}\end{cases}$$

and

$$\eta(x,y)=\begin{cases}2 & \text{if } (x,y)\in\{(1,2),(2,1)\};\\ 0 & \text{otherwise.}\end{cases}$$

It is clear that T is α-orbital admissible with respect to η and also α-orbital attractive admissible with respect to η. Moreover, there exists $x_1=2$ and $\alpha(2,T2)=6\geq 0=\eta(2,T2)$. Let $\alpha(x,y)\geq\eta(x,y)$ and consider the following cases:

(1) If $x,y\in\{0,1\}$, then $Tx=Ty=2$. This implies that $\psi(sp_b(Tx,Ty))=0$;

(2) If $x,y\in\{2,3\}$, then $Tx=Ty=3$. This implies that $\psi(sp_b(Tx,Ty))=0$;

(3) If $x\in\{0,1\}, y\in\{2,3\}$ or $x\in\{2,3\}, y\in\{0,1\}$, then we divide the proof into the following cases:

(3.1) If $(x,y)\in\{(0,3),(3,0)\}$, then

$$\begin{aligned}M_s^T(0,3)&=\max\left\{p_b(0,3),p_b(0,2),p_b(3,3),\frac{p_b(0,3)+p_b(3,2)}{4}\right\}\\&=\max\left\{9,4,0,\frac{9+1}{4}\right\}\\&=9.\end{aligned}$$

We get that,

$$\begin{aligned}\psi(2p_b(T0,T3))&=1\\&\leq\frac{1}{2}\cdot\frac{9}{2}\\&\leq\beta(\psi(M_s^T(0,3)))\psi(M_s^T(0,3)).\end{aligned}$$

Since $p_b(x,y)=p_b(y,x)$ for all $x,y\in X$, we also obtain that

$$\psi(2p_b(T3,T0))\leq\beta(\psi(M_s^T(3,0)))\psi(M_s^T(3,0)).$$

(3.2) If $(x,y)\in\{(1,3),(3,1)\}$, then

$$M_s^T(1,3)=\max\left\{p_b(1,3),p_b(1,2),p_b(3,3),\frac{p_b(1,3)+p_b(3,2)}{4}\right\}=4.$$

We get that,

$$\begin{aligned}\psi(2p_b(T1,T3))&=1\\&\leq\frac{1}{2}\cdot\frac{4}{2}\\&\leq\beta(\psi(M_s^T(1,3)))\psi(M_s^T(1,3)).\end{aligned}$$

Since $p_b(x,y)=p_b(y,x)$ for all $x,y\in X$, we also obtain that

$$\psi(2p_b(T3,T1))\leq\beta(\psi(M_s^T(3,1)))\psi(M_s^T(3,1)).$$

(3.3) If $(x,y)\in\{(0,2),(2,0)\}$, then

$$M_s^T(0,2)=\max\left\{p_b(0,2),p_b(0,2),p_b(2,3),\frac{p_b(0,3)+p_b(2,2)}{4}\right\}=4.$$

We get that,

$$\psi(2p_b(T0,T2))=1$$

$$\leq \frac{1}{2}\cdot\frac{4}{2}$$
$$\leq \beta(\psi(M_s^T(0,2)))\psi(M_s^T(0,2)).$$

Since $p_b(x,y) = p_b(y,x)$ for all $x, y \in X$, we also obtain that

$$\psi(2p_b(T2,T0)) \leq \beta(\psi(M_s^T(2,0)))\psi(M_s^T(2,0)).$$

Hence all assumptions in Theorem 2.7 are satisfied and thus T has a fixed point which is $x = 3$.

In this work, we can relax the subadditivity of ψ in [10] and assure the existence of fixed point theorems for generalized α-η-ψ-Geraghty contractive type mappings in the setting of partial b-metric spaces. Our results generalize and extend the results proved by [6], [10], [16] as the aspect of generalized mappings and generalized metric spaces.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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