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Research Article

A Uniform Convergent Method for Singularly Perturbed Nonlinear Differential-Difference Equation

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Abstract. In this paper, the singularly perturbed boundary-value problem for a nonlinear second order delay differential equation is considered. For the numerical solution of this problem, we use an exponentially fitted difference scheme on a uniform mesh which is succeeded by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder term in integral form. Also, the method is proved to be first-order convergent in the discrete maximum norm uniformly in the perturbation parameter. Furthermore, numerical illustration provide support of the theoretical results.

Keywords. Singular perturbation; Boundary value problem; Fitted difference method; Delay differential equation; Uniform convergence

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1. Introduction

A singularly perturbed *differential-difference equation* (DDE) is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and including at least one delay term. *Singularly perturbed delay differential initial or boundary value problems* (SPDPs) are a very important role in science and engineering field. For instance, they occur in the study of human pupil light reflex [15], first-exit problems in neurobiology [22], models of physiological processes and diseases [16], optimal control theory [11], optically bistable devices [5] and signal transmission [6], and other models [7].

On the other hand, it is quite difficult to find exact solutions to SPDPs. So, numerical methods play an important role in this work area. In [12, 13], the authors have considered some asymptotic analysis of *boundary value problems* (BVPs) for second order singularly perturbed DDEs and some numerical techniques for solving of this type of problems with large and small shifts were considered in [9, 14, 18, 19, 24]. Recently, there has been a growing interest in the numerical solution of SPDPs. For example, reproducing kernel method [9], initial value technique [23], some special finite element methods [17, 21, 25] have been used for solving SPDPs.

Motivated by the previous works, we consider following nonlinear second order singularly perturbed delay differential problem

$$Lu := \varepsilon u''(x) + a(x)u'(x) + f(x,u(x), \quad u(x-r)) = 0,\ x \in \Omega, \tag{1.1}$$

subject to the interval and boundary conditions,

$$u(x) = \varphi(x), \quad x \in \Omega_0; \quad u(l) = A, \tag{1.2}$$

where $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 = (0,r]$, $\Omega_2 = (r,l)$, $\bar{\Omega} = [0,l]$, $\Omega_0 = [-r,0]$ and $0 < \varepsilon \le 1$ is the perturbation parameter, $a(x) \ge \alpha > 0$, $f(x,u,v)$, and $\varphi(x)$ are given sufficiently smooth functions satisfying certain regularity conditions in $\bar{\Omega}$, $\bar{\Omega} \times \mathbb{R}^2$ and Ω_0 respectively, to be specified and r is a constant delay, which is independent of ε, and A is a given constant and furthermore

$$0 \le \frac{\partial f}{\partial u} \le b^* < \infty, \quad \left|\frac{\partial f}{\partial v}\right| \le c^* .$$

It is well known that, for small values of ε, standard numerical methods for solving such problems are unstable and do not give accurate results. For that reason, it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the parameter value ε, i.e., methods that are convergent ε-uniformly. These include fitted finite difference methods, finite element methods using special elements such as exponential elements, and methods which use a priori refined or special non-uniform grids which condense in the boundary layers in a special manner. The various approaches to the design and analysis of appropriate numerical methods for singularly perturbed differential equations can be found in [2, 8, 10, 20] and the references therein. The numerical method presented here comprises a fitted difference scheme on a uniform mesh. We have derived this approach on the basis of the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. This results in a local truncation error containing only first order derivatives of exact solution and hence facilitates examination of the convergence. The solution of a singularly perturbed problem of the form (1.1)-(1.2) normally has a boundary layer (at $x = 0$ for $a(x) \ge \alpha_1 > 0$ or at $x = l$ for $a(x) \le \alpha_2 < 0$) [8, 12].

The present paper is organized as follows. In Section 2, we state some important properties of the exact solution. The description the finite difference discretization have been given in Section 3. In Section 4, we present the error analysis for the approximate solution. In Section 5, we formulate the iterative algorithm for solving the discrete problem and present numerical results which validate the theoretical analysis computationally. The paper ends with a summary of the main conclusion.

Notation. Throughout the paper, C denotes a generic positive constant independent of ε and the mesh parameter. Some specific, fixed constants of this kind are indicated by subscripting C. For any continuous function $g(x)$ denote norms which

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\Omega}} = \max_{0\le x\le l} |g(x)|, \quad \|g\|_1 \equiv \|g\|_{1,\Omega} = \int_0^l |g(x)|\,dx,$$

$$\|g\|_{\infty,k} \equiv \|g\|_{\infty,\bar{\Omega}_k}, \quad \|g\|_{1,k} \equiv \|g\|_{1,\Omega_k}, \quad k = 0,1,2.$$

2. Properties of the Exact Continuous Solution

Here we give some properties of the solution of (1.1)-(1.2), which are needed in later sections for the analysis of appropriate numerical solution.

Lemma 2.1. *Let* $a(x) \in C(\bar{\Omega})$, $f(x,\cdot,\cdot) \in C^1(\bar{\Omega},\mathbb{R}^2)$, $\varphi(x) \in C(\Omega_0)$ *and*

$$\rho := \alpha^{-1}c^*(l-r) < 1. \tag{2.1}$$

Then for the solution $u(x)$ *of the problem* (1.1)-(1.2) *the following estimates hold:*

$$\|u\|_\infty \le C_0, \tag{2.2}$$

where

$$C_0 = (|\varphi(0)| + |A| + \alpha^{-1}\|F\|_1 + \alpha^{-1}c^*\|\varphi\|_{1,0})(1-\rho)^{-1},$$

$$F(x) = -f(x,0,0),$$

$$|u'(x)| \le C\left(1 + \frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\right), \quad 0 \le x \le l, \tag{2.3}$$

providing that $\frac{\partial f}{\partial x}(x,u,v)$ *is bounded for* $x \in \bar{\Omega}$ *and* $|u|,|v| \le C_0$.

Proof. We rewrite (1.1) in the form

$$\varepsilon u'' + a(x)u' - b(x)u + c(x)u(x-r) = F(x), \tag{2.4}$$

with

$$b(x) = \frac{\partial f}{\partial u}(x,\tilde{u},\tilde{v}), \quad c(x) = \frac{\partial f}{\partial v}(x,\tilde{u},\tilde{v}),$$

$$\tilde{u} = \gamma u, \ \tilde{v} = \gamma u(x-r), \quad (0<\gamma<1)\text{-intermediate values.}$$

After using the Maximum Principle for the differential operator $L_* u = \varepsilon u'' + a(x)u' - b(x)u$, with first type boundary conditions, we get $|u(x)| \le w(x)$, where $w(x)$ is the solution of the BVP:

$$-\varepsilon w'' - a(x)w' = |c(x)u(x-r)| - |F(x)|,$$

$$|w(0)| = |\varphi(0)|, \quad |w(l)| = |A|.$$

The further analysis is similar to that of [1]. □

3. Discretization

In what follows, we denote by ω a uniform mesh on Ω :

$$\omega=\{x_i=ih,\ i=1,2,\dots,N-1;\ h=l/N\}$$

and $\bar{\omega}=\omega\cup\{x=0,l\}$. For simplicity, we will suppose that $\frac{r}{l}N=N_0$ is integer, i.e., $x_{N_0}=r$. Before describing our numerical method, we introduce some notation for mesh functions. For any mesh function $g(x)$, we use

$$g_i=g(x_i),\quad g_{\bar{x},i}=(g_i-g_{i-1})/h,\quad g_{x,i}=(g_{i+1}-g_i)/h,$$

$$g_{\overset{0}{x},i}=(g_{i+1}-g_{i-1})/(2h),\quad g_{\bar{x}x,i}=(g_{i+1}-2g_i+g_{i-1})/h^2,$$

$$\|g\|_\infty\equiv\|g\|_{\infty,\bar{\omega}}=\max_{0\le i\le N}|g_i|,\quad \|g\|_1=\|g\|_{1,\omega}=\sum_{i=1}^{N-1}h|g_i|\,.$$

The approach of generating difference method is through the integral identity

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}}Lu(x)\psi_i(x)dx=0,\quad i=1,2,\dots,N-1$$

with basis functions

$$\psi_i(x)=\begin{cases}\psi_i^{(1)}(x), & x_{i-1}<x<x_i,\\ \psi_i^{(2)}(x), & x_i<x<x_{i+1},\\ 0, & x\notin(x_{i-1},x_{i+1}),\end{cases}$$

where $\psi_i^{(1)}(x)$ and $\psi_i^{(2)}(x)$ are the solutions of the following problems, respectively

$$\varepsilon\psi''(x)-a_i\psi'(x)=0,\quad x_{i-1}<x<x_i,$$

$$\psi(x_{i-1})=0,\ \psi(x_i)=1,$$

$$\varepsilon\psi''(x)-a_i\psi'(x)=0,\quad x_i<x<x_{i+1},$$

$$\psi(x_i)=1,\ \psi(x_{i+1})=0.$$

The functions $\psi_i^{(1)}(x)$ and $\psi_i^{(2)}(x)$ can be explicitly expressed as

$$\psi_i^{(1)}(x)=\frac{e^{a_i(x-x_{i-1})/\varepsilon}-1}{e^{a_ih/\varepsilon}-1},\quad \psi_i^{(2)}(x)=\frac{1-e^{-a_i(x_{i+1}-x)/\varepsilon}}{1-e^{-a_ih/\varepsilon}},$$

which, clearly, satisfy

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}}\psi_i(x)dx=1.$$

To be consistent with [1], we obtain

$$\varepsilon\theta_iu_{\bar{x}x,i}+a_iu_{\overset{0}{x},i}+f(x_i,u_i,u_{i-N_0})+R_i=0,\quad i=1,2,\dots,N-1,$$

where

$$\theta_i=\gamma_i\coth(\gamma_i),\quad \gamma_i=a_ih/(2\varepsilon),\tag{3.1}$$

and with remainder term

$$R_i=h^{-1}\int_{x_{i-1}}^{x_{i+1}}[a(x)-a(x_i)]u'(x)\psi_i(x)dx+h^{-1}\int_{x_{i-1}}^{x_{i+1}}dx\psi_i(x)\int_{x_{i-1}}^{x_{i+1}}\frac{d}{dx}f(\xi,u(\xi),u(\xi-r))K_{0,i}^*(x,\xi)d\xi,\tag{3.2}$$

$$K^{*}_{0,i}(x,\xi) = T_0(x-\xi) - T_0(x_i-\xi), \quad 1 \le i \le N-1,$$

$$T_0(\lambda) = 1,\ \lambda \ge 0; \quad T_0(\lambda) = 0,\ \lambda < 0.$$

Based on foregoing, we propose the following difference scheme for approximating (1.1)-(1.2)

$$\varepsilon\theta_i y_{\bar{x}x,i} + a_i y_{\overset{0}{x},i} + f(x_i, y_i, y_{i-N_0}) = 0, \quad 0 < i < N, \tag{3.3}$$

$$y_i = \varphi_i, \quad -N_0 \le i \le 0,\ y_N = A, \tag{3.4}$$

where θ is given by (3.1).

4. Error Analysis

Let $z_i = y_i - u_i$. Then the error in the numerical solution satisfies

$$\varepsilon\theta_i z_{\bar{x}x,i} + a_i z_{\overset{0}{x},i} + f(x_i, y_i, y_{i-N_0}) - f(x_i, u_i, u_{i-N_0}) = R_i, \quad 0 < i < N, \tag{4.1}$$

$$z_i = 0,\ -N_0 \le i \le 0;\ z_N = 0. \tag{4.2}$$

where the truncation error R_i is given by (3.2).

Lemma 4.1. *If $a(x) \in C(\bar{\Omega})$, $f(x,\cdot,\cdot) \in C^1(\bar{\Omega}, \mathbb{R}^2)$ and $\varphi(x) \in C^1(\Omega_0)$, then for the truncation error R_i we have*

$$\|R\|_1 \le Ch. \tag{4.3}$$

Proof. From

$$|R_i| \le h^{-1}\int_{x_{i-1}}^{x_{i+1}} |[a(x)-a(x_i)]u'(x)\psi_i(x)|dx$$
$$+ h^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx}|f(\xi,u(\xi),u(\xi-r))|\,d\xi,$$

taking also into account that $0 \le \psi_i(x) \le 1$, it is not hard to get

$$|R_i| \le \left\{Ch(h^{-1}\int_{x_{i-1}}^{x_{i+1}} |u'(x)|dx) + \int_{x_{i-1}}^{x_{i+1}} \left|\frac{\partial f(\xi,u(\xi),u(\xi-r))}{\partial \xi}\right| d\xi\right.$$
$$\left. + \int_{x_{i-1}}^{x_{i+1}} \left|\frac{\partial f}{\partial u}\frac{du(\xi)}{d\xi} + \frac{\partial f}{\partial v}\frac{du(\xi-r)}{d\xi}\right| d\xi\right\}$$

and

$$|R_i| \le C\left\{h + \int_{x_{i-1}}^{x_{i+1}} (|u'(\xi)| + |u'(\xi-r)|)d\xi\right\}.$$

Hence,

$$\|R\|_1 \le Ch\sum_{i=1}^{N-1}\left(h + \int_{x_{i-1}}^{x_{i+1}} (|u'(\xi)| + |u'(\xi-r)|)d\xi\right)$$
$$\le Ch\left(1 + \int_0^l |u'(x)|dx + \int_0^l \left|u'(\xi-r)\right| d\xi\right)$$

and, after replacing $s = \xi - r$ in second integral, this reduces to

$$\|R\|_1 \le Ch\left(1 + \int_0^l |u'(x)|dx + \int_{-r}^{l-r} |u'(s)|ds\right)$$

$$\leq Ch\left(1+\int_0^l |u'(x)|dx+\int_{-r}^0 |\varphi'(x)|dx+\int_0^{l-r} |u'(x)|dx\right)$$

and using Lemma 2.1, we obtain

$$\|R\|_1 \leq Ch\left(1+\frac{1}{\varepsilon}\int_0^l e^{-\frac{\alpha x}{\varepsilon}}dx+\int_{-r}^0 |\varphi'(x)|dx+\frac{1}{\varepsilon}\int_0^{l-r} e^{-\frac{\alpha x}{\varepsilon}}ds\right)$$
$$\leq Ch\left(1+\alpha^{-1}(1-e^{-\frac{\alpha l}{\varepsilon}})+\|\varphi'\|_{1,0}+\alpha^{-1}(1-e^{-\frac{\alpha(l-r)}{\varepsilon}})\right)=O(h).$$ □

Lemma 4.2. *Let z_i be the solution* (4.1)-(4.2) *and* (2.1) *holds true. Then*

$$\|z\|_{\infty,\omega} \leq \alpha^{-1}(1-\rho)^{-1}\|R\|_{1,\omega}. \tag{4.4}$$

Proof. (4.1) can be rewritten as

$$\varepsilon\theta_i z_{\bar{x}x,i}+a_i z_{\overset{0}{x},i}-\widetilde{b}_i z_i+\widetilde{c}_i z_{i-N_0}=R_i, \quad 0<i<N,$$

where

$$\widetilde{b}_i=\frac{\partial f}{\partial u}(x_i,\widetilde{y}_i,\widetilde{y}_{i-N_0}), \quad \widetilde{c}_i=\frac{\partial f}{\partial v}(x_i,\widetilde{y}_i,\widetilde{y}_{i-N_0}),$$

$\widetilde{y}_i$, $\widetilde{y}_{i-N_0}$ intermediate points called for by the mean value theorem.

We here will use the discrete Green's function $G^h(x_i,\xi_j)$ for the operator

$$\ell_* z_i := -\varepsilon\theta_i z_{\bar{x}x,i}-a_i z_{\overset{0}{x},i}, \quad 1\leq i\leq N-1,$$
$$z_0=z_N=0.$$

As a function of x_i for fixed ξ_j this function is being defined as

$$\ell_* G^h(x_i,\xi_j)=\delta^h(x_i,\xi_j), \quad x_i\in\omega,\ \xi_j\in\omega,$$
$$G^h(0,\xi_j)=G^h(l,\xi_j), \quad \xi_j\in\omega,$$

where $\delta^h(x_i,\xi_j)=h^{-1}\delta_{ij}$ and δ_{ij} is the Kronecker delta. In the analogous manner as in [3] one can show that $0\leq G^h(x_i,\xi_j)\leq\alpha^{-1}$. Using Maximum Principle for the $\ell z_i := \varepsilon\theta_i z_{\bar{x}x,i}+a_i z_{\overset{0}{x},i}-\widetilde{b}_i z_i$ and also the Green's function $G^h(x_i,\xi_j)$, we obtain the following relation for solution of problem (4.1)-(4.2)

$$|z_i|\leq\sum_{j=1}^{N-1} hG^h(x_i,\xi_j)|\widetilde{c}_j z_{j-N_0}-R_j|,\ x_i\in\omega. \tag{4.5}$$

Then from (4.5) it follows that

$$\|z\|_{\infty,\omega}\leq\alpha^{-1}\left\{c^*\sum_{j=1}^{N-1} h|z_{j-N_0}|+\|R\|_{1,\omega}\right\}\leq\alpha^{-1}c^*\sum_{j=1}^{N-1} h|z_{j-N_0}|+\alpha^{-1}\|R\|_{1,\omega}$$

and after replacing $j-N_0=k$, we have

$$\|z\|_{\infty,\omega}\leq\alpha^{-1}c^*\sum_{k=1-N_0}^{N-N_0-1} h|z_k|+\alpha^{-1}\|R\|_{1,\omega}=\alpha^{-1}c^*\sum_{k=1}^{N-N_0-1} h|z_k|+\alpha^{-1}\|R\|_{1,\omega}$$
$$\leq\alpha^{-1}c^*(N-N_0-1)h\|z\|_{\infty,\omega}+\alpha^{-1}\|R\|_{1,\omega}$$
$$\leq\rho\|z\|_{\infty,\omega}+\alpha^{-1}\|R\|_{1,\omega},$$

which implies validity of (4.4). □

Now we give the main convergence result.

Theorem 4.3. *Let u be the solution of* (1.1)-(1.2) *and y the solution* (3.3)-(3.4)*. Then*

$$\|y-u\|_{\infty,\bar{\omega}} \le Ch.$$

Proof. This follows immediately by combining previous lemmas. □

5. Numerical Results

In this section, we present numerical experiments in order to illustrate the method described above. We solve the nonlinear problem (3.3)-(3.4) using the following quasi-linearization technique:

$$\varepsilon\theta_i y^{(n)}_{\bar{x}x,i} + a_i y^{(n)}_{\overset{0}{x},i} + f(x_i, y_i^{(n-1)}, y_{i-N_0}^{(n-1)}) + \frac{\partial f}{\partial y}(x_i, y_i^{(n-1)}, y_{i-N_0}^{(n-1)})[y_i^{(n)} - y_i^{(n-1)}] = 0, \quad 0 < i < N, \quad (5.1)$$

$$y_i^{(n)} = \varphi_i\,, -N_0 \le i \le 0, \quad y_N^{(n)} = A, \quad (5.2)$$

where θ is given by (3.1) and $n = 1,2,\ldots$; $y_i^{(0)}$ given $0 < i < N$. For the obtaining $y_i^{(n)}$, $n = 1,2,\ldots$ is being also used algorithm from [1].

We consider the test problem:

$$\varepsilon u''(x) + 8(x^2+9)u'(x) + \cosh(u(x-1)) = 0, \quad 0 < x < \frac{3}{2}$$

subject to the interval and boundary conditions

$$u(x) = x^2, \quad -1 \le x \le 0 \quad ; \quad u(3/2) = 2.$$

The initial guess in iteration process is taken as $y_i^{(0)} = x_i^2$ according to (2.2) and stopping criterion is

$$\max_i |y^{(n)} - y^{(n-1)}| \le 10^{-5}.$$

The exact solution of our test problem is unknown. Therefore, we use the double mesh principle to estimate the errors and compute the experimental rates of convergence in our computed solutions. That is, we compare the computed solutions with the solutions on a mesh that is twice as fine (see [4, 8]). The error estimates obtained in this way are denoted by:

$$e_\varepsilon^N = \max_i |y_i^{\varepsilon,N} - y_{2i}^{\varepsilon,2N}|.$$

The convergence rates are

$$p_\varepsilon^N = \log_2\left(e_\varepsilon^N / e_\varepsilon^{2N}\right).$$

Approximations to the ε-uniform rates of convergence are estimated by

$$e^N = \max_\varepsilon e_\varepsilon^N.$$

The corresponding ε-uniform convergence rates are computed using the formula

$$p^N = \log_2\left(e^N / e^{2N}\right).$$

The resulting errors and the corresponding numbers for $\varepsilon = 2^{-i}$, $i = 2,4,\ldots,16$ are listed in Table 1.

Table 1. Approximate errors, computed ε-uniform errors and convergence rates on ω_N

ε	$N_0 = 32$	$N_0 = 64$	$N_0 = 128$	$N_0 = 256$	$N_0 = 512$	$N_0 = 1024$
2^{-2}	6.25874E-5	2.75894E-5	8.62488E-6	2.32606E-6	5.92343E-7	1.48898E-7
	1.18	1.68	1.89	1.97	1.99	
2^{-4}	6.38614E-5	3.33762E-5	1.67126E-5	6.99265E-6	2.20640E-6	5.94527E-7
	0.94	1.00	1.26	1.66	1.89	
2^{-6}	6.38614E-5	3.33837E-5	1.70649E-5	8.62507E-6	4.24706E-6	1.76472E-6
	0.94	0.97	0.98	1.02	1.27	
2^{-8}	6.38614E-5	3.33837E-5	1.70649E-5	8.62698E-6	4.33728E-6	2.17412E-6
	0.94	0.97	0.98	0.99	1.00	
2^{-10}	6.38614E-5	3.33837E-5	1.70649E-5	8.62698E-6	4.33728E-6	2.17461E-6
	0.94	0.97	0.98	0.99	1.00	
2^{-12}	6.38614E-5	3.33837E-5	1.70649E-5	8.62698E-6	4.33728E-6	2.17461E-6
	0.94	0.97	0.98	0.99	1.00	
2^{-14}	6.38614E-5	3.33837E-5	1.70649E-5	8.62698E-6	4.33728E-6	2.17461E-6
	0.94	0.97	0.98	0.99	1.00	
2^{-16}	6.38614E-5	3.33837E-5	1.70649E-5	8.62698E-6	4.33728E-6	2.17461E-6
	0.94	0.97	0.98	0.99	1.00	
e^N	6.38614E-5	3.33837E-5	1.70649E-5	8.62698E-6	4.33728E-6	2.17461E-6
p^N	0.94	0.97	0.98	0.99	1.00	

6. Conclusion

In this paper, we have developed a finite difference method for solving the singularly perturbed boundary-value problem for a nonlinear second order delay differential equation. This method was based on an exponentially fitted difference scheme on a uniform mesh. From the method, first order convergence in the discrete maximum norm, independently of the perturbation parameter resulted. The approximate errors and the rates of convergence are computed for different values of ε and N in Table 1. Numerical results were carried out to show the efficiency and accuracy of the method. Theoretical results represented undergoing more complicated delay problems.

Competing Interests

Author declares that he has no competing interests.

Authors' Contributions

Author wrote, read and approved the final manuscript.

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