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A Problem of Enumeration of Two-color Bracelets with Several Variations

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Abstract. We consider the problem of enumeration of incongruent two-color bracelets of n beads, k of which are black, and study several natural variations of this problem. We also give recursion formulas for enumeration of t-color bracelets, $t \geq 3$.

1. Introduction

Professor Richard H. Reis (South-East University of Massachusetts, USA) in 1978 put the problem: "Let a circumference is split by the same n parts. It is required to find the number $R(n,k)$ of the incongruent convex k-gons, which could be obtained by connection of some k from n dividing points. Two k-gons are considered congruent if they are coincided at the rotation of one relatively other along the circumference and (or) by reflection of one of the k-gons relatively some diameter".

In 1979 Hansraj Gupta [1] gave the solution of the Reis problem.

Theorem 1 (H. Gupta).

$$R(n,k) = \frac{1}{2}\left(\binom{\lfloor \frac{n-h_k}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} + \frac{1}{k}\sum_{d|(k,n)} \varphi(d)\binom{\frac{n}{d}-1}{\frac{k}{d}-1}\right), \tag{1.1}$$

where $h_k \equiv k(\mod 2)$*,* $h_k = 0$ *or* 1*,* (n,k) *is* $\gcd(n,k)$*,* $\varphi(n)$ *- the Euler function.*

Consider some convex polygon with the tops in the circumference splitting points, "1" or "0" is put in accordance to each splitting point depending on whether a top of the polygon is in the point. Thus, there is the mutual one-to-one correspondence between the set of convex polygons with the tops in the circumference splitting points and the set of all $(0,1)$-configurations with the elements in these points.

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Using this bijection and calculating the cyclic index of the Dihedral group appearing here, the author [4] gave a short proof of Theorem 1. Besides, this bijection shows that formula (1.1) solves the problem of enumeration of two-color bracelets of n beads, k of which are black and $n-k$ are white. In turn, it allows to obtain several simple formulas for series of sequences in [6] (see, e.g., the author's explicit formulas for sequences A032279-A032282, A005513-A005516).

Note also that, quite recently, the author found an application of enumeration of two-color bracelets to some questions of the permanent theory (see [5], Section 5).

Let n beads of a bracelets are located in n dividing points of a circumference which is split by the same n parts. Let $\mathscr{T}$ be cyclic group of turns with the generating element $\tau = e^{\frac{2\pi i}{n}}$.

Definition 1. We call two-color bracelet of n beads *symmetric respectively rotation* (*two-color SR-bracelet*) if its cyclic grope of turns is a proper subgroup of $\mathscr{T}$.

Remark 1. When we say about a two-color bracelet, we exclude cases when it contains only white (black) color.

Consider several variations of enumeration of two-color bracelets.

Variation 1. To find the number N_n of all incongruent two-color SR-bracelets.

Variation 2. To find the number $N^{(k)}(n)$ of those two-color SR-bracelets which have exactly k black beads.

Variation 3. To find the number S_n of those two-color SR-bracelets which have a diameter of symmetry.

Variation 4. To find the number $S^{(k)}(n)$ of those two-color SR-bracelets which have exactly k black beads and a diameter of symmetry.

Variation 5. Let m be a positive integer. To find the number $N_{n,m}$ of all incongruent two-color SR-bracelets with isolated black beads such that between every two black beads there exist at least m white ones.

Variation 6. For $m \geq 1$, to find the number $S_{n,m}$ of those two-color SR-bracelets which have a diameter of symmetry.

Variation 7. For $m \geq 1$, to find the number $N_{n,m}^{(k)}$ of those two-color SR-bracelets in Variation 2 which have exactly k black beads.

Variation 8. For $m \geq 1$, to find the number $S_{n,m}^{(k)}$ of those two-color SR-bracelets in Variation 3 which have exactly k black beads.

Notice that, $N_{n,m} - S_{n,m}$ is the number of those two-color SR-bracelets, none of which has a diameter of symmetry; $N_{n,m}^{(k)} - S_{n,m}^{(k)}$ is the same two-color SR-bracelets having exactly k 1's.

Some words about structure of the article. Section 2 is devoted to solutions of Variations 1-4. In Section 3 we introduce two different generalizations of the Fibonacci numbers. In Section 4 we solve Variations 5-8. In Section 5 we consider an interesting example and the connected with it some numerical results. In conclusion, in Section 6 we discuss two open questions (Variation 9-10) and give enumeration of t-color SR-bracelets (Variation 11-12) and Theorems 4-6.

2. **Variations 1-4**

Theorem 2. *The following formulas take place:*

$$N_n = -2 - \sum_{d|n, d\geq 2} \mu(d)\alpha_{\frac{n}{d}}\,, \tag{2.1}$$

$$S_n = -2 - \sum_{d|n, d\geq 2} \mu(d)\beta_{\frac{n}{d}}\,, \tag{2.2}$$

$$N_n^{(k)} = -\sum_{d|(k,n), d\geq 2} \mu(d) R\left(\frac{n}{d}, \frac{k}{d}\right), \tag{2.3}$$

$$S_n^{(k)} = -\sum_{d|(k,n), d\geq 2} \mu(d) R^1\left(\frac{n}{d}, \frac{k}{d}\right), \tag{2.4}$$

where $\mu(n)$ *is the Mobius function,*

$$\beta_n = (5 + (-1)^n) 2^{\lfloor \frac{n-3}{2} \rfloor}, \tag{2.5}$$

$$\alpha_n = \frac{1}{n} \sum_{d|n} \varphi(d) 2^{\frac{n}{d}-1} + \frac{\beta_n}{2}\,, \tag{2.6}$$

$$R^1(n,k) = \binom{\left\lfloor \frac{n-h_k}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} \tag{2.7}$$

and $R(n,k)$ *is defined by* (1.1).

Proof. (1) Summing (1.1) by k from 1 to $n-1$ we find the number λ_n of all incongruent of all incongruent two-color bracelets:

$$\lambda_n = \sum_{k=1}^{n-1} R(n,k). \tag{2.8}$$

Let d, $1 \leq d \leq n$, be a divisor of n. Denote via $\nu_d = \nu_d(n)$ the number of incongruent bracelets with the minimal angle of self-coincidence equals to $\frac{2\pi}{n}d$. Then

$$\lambda_n = \sum_{d|n} \nu_d \tag{2.9}$$

and, by definition of N_n, we have

$$N_n = \lambda_n - \nu_1 - \nu_n\,. \tag{2.10}$$

Using the Mobius inverse formula (cf., e.g., [3]), we find from (2.9)

$$\nu_n = \sum_{d|n} \mu(d)\lambda_{\frac{n}{d}} = \lambda_n + \sum_{d|n, d\geq 2} \mu(d)\lambda_{\frac{n}{d}}\,.$$

Now (2.8) implies

$$N_n = -\sum_{d|n, 2\leq d\leq \frac{n}{2}} \mu(d)\lambda_{\frac{n}{d}}\,. \tag{2.11}$$

To complete the proof of (2.1), we need two technical lemmas.

Lemma 1. *For* $h_k \equiv k \pmod 2$, $h_k = 0$ *or* 1, *we have*

$$\sum_{k=1}^{n-1} \binom{\lfloor \frac{n-h_k}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} = (5+(-1)^n)2^{\lfloor \frac{n-3}{2} \rfloor} - 2\,. \tag{2.12}$$

Proof. Indeed, for even n, we have

$$\sum_{k=1}^{n-1} \binom{\lfloor \frac{n-h_k}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} = \sum_{k=1,3,\dots,n-1} \binom{\frac{n-2}{2}}{\frac{k-1}{2}} + \sum_{k=2,4,\dots,n-2} \binom{\frac{n}{2}}{\frac{k}{2}} \tag{2.13}$$

$$= \sum_{t=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{t} + \sum_{t=1}^{\frac{n-2}{2}} \binom{\frac{n}{2}}{t} = 2^{\frac{n-2}{2}} + 2^{\frac{n}{2}} - 2 = 3\cdot 2^{\frac{n-2}{2}} - 2\,. \tag{2.14}$$

For odd n, we have

$$\sum_{k=1}^{n-1} \binom{\lfloor \frac{n-h_k}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} = \sum_{k=1,3,\dots,n-2} \binom{\frac{n-1}{2}}{\frac{k-1}{2}} + \sum_{k=2,4,\dots,n-1} \binom{\frac{n-1}{2}}{\frac{k}{2}} \tag{2.15}$$

$$= \sum_{t=0}^{\frac{n-3}{2}} \binom{\frac{n-1}{2}}{t} + \sum_{t=1}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{t} = 2\cdot 2^{\frac{n-1}{2}} - 2, \tag{2.16}$$

and the lemma follows from (2.14), (2.16). □

Lemma 2.

$$\sum_{k=1}^{n-1} \frac{1}{k} \sum_{d|(n,k)} \varphi(d) \binom{\frac{n}{d}-1}{\frac{k}{d}-1} = \frac{1}{n}\sum_{d|n} \varphi(d) 2^{\frac{n}{d}} - 2\,. \tag{2.17}$$

Proof. First of all, notice that

$$\frac{1}{k}\binom{\frac{n}{d}-1}{\frac{k}{d}-1} = \frac{1}{n}\cdot\frac{\frac{n}{d}}{\frac{k}{d}}\binom{\frac{n}{d}-1}{\frac{k}{d}-1} = \frac{1}{n}\binom{\frac{n}{d}}{\frac{k}{d}}\,.$$

Therefore, taking into account that, for $n \geq 1$, $\sum_{d|n} \varphi(d) = n$, in order to prove (2.17), it is sufficient to show that

$$\sum_{k=1}^{n-1} \sum_{d|(n,k)} \varphi(d) \binom{\frac{n}{d}}{\frac{k}{d}} = \sum_{d|n} \varphi(d)(2^{\frac{n}{d}} - 2). \tag{2.18}$$

Putting $k = dd_1$, we have

$$\begin{aligned}
\sum_{k=1}^{n-1}\sum_{d|(n,k)}\varphi(d)\binom{\frac{n}{d}}{\frac{k}{d}} &= \sum_{\substack{dd_1\le n-1\\ d|n}}\varphi(d)\binom{\frac{n}{d}}{d_1}\\
&= \sum_{\substack{1\le d\le \frac{n}{2}\\ d|n}}\varphi(d)\sum_{d_1=1}^{\frac{n}{d}-1}\binom{\frac{n}{d}}{d_1}\\
&= \sum_{\substack{1\le d\le \frac{n}{2}\\ d|n}}\varphi(d)(2^{\frac{n}{d}}-2)\\
&= \sum_{d|n}\varphi(d)(2^{\frac{n}{d}}-2).
\end{aligned}$$

This proves (2.17).

Now from (2.8), (1.1) and Lemmas 1, 2 we have

$$\begin{aligned}
\lambda_n &= (5+(-1)^n)2^{\lfloor\frac{n-5}{2}\rfloor}-1+\frac{1}{n}\sum_{d|n}\varphi(d)(2^{\frac{n}{d}-1}-1)\\
&= \alpha_n - 2\,.
\end{aligned}$$

Finally, from (2.11), taking into account, that for $n > 1$, $\sum_{d|n,d\ge 2}\mu(d) = -1$, we find

$$\begin{aligned}
N_n &= -\sum_{d|n,d\ge 2}\mu(d)\lambda_{\frac{n}{d}}\\
&= -\sum_{d|n,d\ge 2}\mu(d)(\alpha_{\frac{n}{d}}-2)\\
&= -2-\sum_{d|n,d\ge 2}\mu(d)\alpha_{\frac{n}{d}}\,.
\end{aligned}$$

Note that, as it was expected, if n is prime, then $N_n = 0$.

(2) As it showed in [1], $R^1(n,k)$ (2.7) gives the number of those k-gons or, by the bijection, those two-color bracelets having exactly k black beads, that are symmetric respectively a diameter. Therefore, by the same arguments for $R^1(n,k)$, we obtain (2.2).

(3) Let $(k,n) > 1$. For fixed $n,\ k$, let us consider the function $R(x,y)$ (1) on the set

$$\left\{\left(\frac{n}{(n,k)}m,\frac{k}{(n,k)}m\right)/m\,\Big|\,(n,k)\right\}.$$

The constriction of the $R(x,y)$ on this set is a function of m. Denote it by $R_{n,k}(m)$. Put

$$R_{n,k}(m) = \sum_{d|m}\nu_d\,, \tag{2.19}$$

where $\nu_d = \nu_d(n,k)$. In particular, $R_{n,k}((n,k)) = R(n,k)$. Using the Mobius inverse formula, according to (2.18) we have

$$\nu_m = \sum_{d|m} \mu(d) R_{n,k}\left(\frac{m}{d}\right) = \sum_{d|m} \mu(d) R\left(\frac{n}{(n,k)}\frac{m}{d}, \frac{k}{(n,k)}\frac{m}{d}\right).$$

In particular, for $m = (n,k)$

$$\nu_{(n,k)} = \sum_{d|(n,k)} \mu(d) R\left(\frac{n}{d}, \frac{k}{d}\right), \tag{2.20}$$

and (2.19), for $m = (n,k)$, has the form

$$R(n,k) = \sum_{d|(n,k)} \nu_d\,.$$

By the definition, we have now

$$N_n^{(k)} = R(n,k) - \nu_{(n,k)} \tag{2.21}$$

Finally, from (2.20)-(2.21) we deduce (2.3).

(4) From the same arguments it follows that, by replacing in (2.3) R by R^1, we obtain $S^{(k)}(n)$. □

For considerations of the further variations, we need two different generalizations of Fibonacci numbers.

3. **Two generalizations of Fibonacci numbers**

Definition 2. Let m be a positive integer. We call m-Fibonacci numbers of type 1 the sequence which is defined by the recursion

$$F_n^{(m)} = F_{n-1}^{(m)} + F_{n-m-1}^{(m)}, \tag{3.1}$$

with the initial conditions

$$F_0^{(m)} = F_1^{(m)} = \ldots = F_m^{(m)} = 1\,. \tag{3.2}$$

For $m = 1$ we obtain the very Fibonacci numbers:

$$F_n^{(1)} = F_n, \quad n \geq 0\,. \tag{3.3}$$

Definition 3. For a positive integer m, the sequence is defined by the same recursion

$$f_n^{(m)} = f_{n-1}^{(m)} + f_{n-m-1}^{(m)}, \tag{3.4}$$

but with other initial conditions

$$f_0^{(m)} = f_1^{(m)} = \ldots = f_{\lfloor \frac{m-1}{2} \rfloor}^{(m)} = 1,$$
$$f_{\lfloor \frac{m+1}{2} \rfloor}^{(m)} = f_{\lfloor \frac{m+1}{2} \rfloor + 1}^{(m)} = \ldots = f_m^{(m)} = 2, \tag{3.5}$$

we call m-Fibonacci numbers of type 2.

For $m=1$ we also obtain the very Fibonacci numbers, but, comparing with (3.3)), we have

$$f_n^{(1)} = F_{n+1}, \quad n \geq 0. \tag{3.6}$$

Now we prove several lemmas.

Lemma 3. *The following formulas hold*

$$F_n^{(m)} = \sum_{k\geq 0} \binom{n-mk}{k}, \quad n \geq 0,\ m \geq 1, \tag{3.7}$$

$$f_n^{(m)} = \begin{cases} \displaystyle\sum_{k\geq 0} \binom{n+\frac{m+1}{2}-mk}{k}, & \text{if } m \text{ is odd},\\ \displaystyle\sum_{k\geq 0} \binom{n+\frac{m}{2}-(m-1)k}{k}, & \text{if } m \text{ is even}. \end{cases} \tag{3.8}$$

In particular, for odd m we have

$$f_n^{(m)} = F_{n+\frac{m+1}{2}}^{(m)}. \tag{3.9}$$

Proof. Proof of the (3.7)-(3.8) is over by the same scheme. Therefore, we prove (3.7) only. Denote the right part of (3.7) via $\Phi_n^{(m)}$. We have

$$\begin{aligned}\Phi_{n-1}^{(m)} + \Phi_{n-m-1}^{(m)} &= \sum_{k\geq 0}\binom{n-mk-1}{k} + \sum_{k\geq 0}\binom{n-m(k+1)-1}{k}\\ &= \sum_{k\geq 0}\binom{n-mk-1}{k} + \sum_{k\geq 1}\binom{n-mk-1}{k-1}\\ &= \sum_{k\geq 0}\binom{n-mk}{k} = \Phi_n^{(m)}.\end{aligned}$$

Besides, for $k \geq 1$, $m \leq n$, we have $n-mk \leq m-mk \leq 0$.

Hence,

$$\sum_{k\geq 0}\binom{n-mk}{k} = \binom{n-mk}{k} = 1, \quad \text{if } k=0.$$

Thus, for numbers $\Phi_n^{(m)}$ and $F_n^{(m)}$, formulas (3.1)-(3.2) are valid and, consequently, $\Phi_n^{(m)} = F_n^{(m)}$, $n \geq 0$. □

Lemma 4. *Let $h_k \equiv k \pmod 2$, $h_k = 0$ or 1, and $\gamma_m \equiv m-1 \pmod 2$, $\gamma_m = 0$ or 1. Then*

$$\sum_{k\geq 1}\binom{\left\lfloor \frac{n-mk-h_k}{2}\right\rfloor}{\left\lfloor \frac{k}{2}\right\rfloor} = f^{(m)}_{\left\lfloor \frac{n-\gamma_m}{2}\right\rfloor} - 1. \tag{3.10}$$

Proof. Proof is the same for all assumptions regarding the parity of m and n. Therefore, we consider only case when m and n are even. Then $\gamma = 1$ and

$$\left\lfloor \frac{n-mk-h_k}{2} \right\rfloor = \frac{n-mk}{2} - h_k.$$

Therefore, it is enough to prove that

$$\sum_{k\ge 0} \binom{\frac{1}{2}(n-mk)-h_k}{\lfloor \frac{k}{2} \rfloor} = f^{(m)}_{\frac{n}{2}-1}. \tag{3.11}$$

Denote the left part of (3.11) via $G^{(m)}_{\frac{n}{2}}$. First of all, let us show that $G^{(m)}_{\frac{n}{2}}$ satisfies to (3.4). We have

$$\begin{aligned}
G^{(m)}_{\frac{n}{2}-1} + G^{(m)}_{\frac{n}{2}-m-1} &= \sum_{k\ge 0} \binom{\frac{n}{2} - \frac{mk+2h_k}{2} - 1}{\lfloor \frac{k}{2} \rfloor} + \sum_{k\ge 0} \binom{\frac{n}{2} - \frac{m(k+2)+2h_{k+2}}{2} - 1}{\lfloor \frac{k}{2} \rfloor} \\
&= \sum_{k\ge 0} \binom{\frac{n}{2} - \frac{mk+2h_k}{2} - 1}{\lfloor \frac{k}{2} \rfloor} + \sum_{k\ge 2} \binom{\frac{n}{2} - \frac{mt+2h_t}{2} - 1}{\lfloor \frac{t}{2} \rfloor - 1} \\
&= 2 + \sum_{k\ge 2} \binom{\frac{n}{2} - \frac{mk+2h_k}{2}}{\lfloor \frac{k}{2} \rfloor} \\
&= \sum_{k\ge 0} \binom{\frac{n}{2} - \frac{mk+2h_k}{2}}{\lfloor \frac{k}{2} \rfloor} \\
&= G^{(m)}_{\frac{n}{2}}.
\end{aligned}$$

It is left to verify the coincidence of the initial conditions (3.5) for $f^{(m)}_{i-1}$ and for

$$G^{(m)}_i = \sum_{k\ge 0} \binom{i - \frac{mk}{2} - h_k}{\lfloor \frac{k}{2} \rfloor}, \quad i = 1, 2, \ldots, m. \tag{3.12}$$

Notice that, the summands in (3.12) equal to 0 for those and only those k for which the following inequality is satisfied:

$$i - \frac{mk}{2} \ge \begin{cases} \frac{k}{2}, & \text{if } k \text{ is even,} \\ \frac{k+1}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

or, the same, for

$$k \ge \begin{cases} \frac{2i}{m+1}, & \text{if } k \text{ is even,} \\ \frac{2i-1}{m+1}, & \text{if } k \text{ is odd.} \end{cases}$$

Thus, for $i = 1, 2, \ldots, \frac{m}{2}$, we have $k = 0$. Thus, by (3.12),

$$G^{(m)}_i = \binom{i}{0} = 1$$

and, for $i = \frac{m}{2} + 1, \frac{m}{2} + 2, \ldots, m$ we have $k = 0$ or $k = 1$.

Therefore, by (3.12),

$$G_i^{(m)} = \binom{i}{0} + \binom{i - \frac{m}{2} - 1}{0} = 2$$

and the lemma follows. □

Remark 2. Let show that there does not exist an extension of a definition of $\{f_n^{(m)}\}$ for $m = 0$ such that the equality (3.10) holds.

Indeed, for $m = 0$, the left part of (3.10) equals to

$$\sum_{k=2,4,\ldots} \binom{\lfloor \frac{n}{2} \rfloor}{\frac{k}{2}} + \sum_{k=1,3,\ldots} \binom{\lfloor \frac{n-1}{2} \rfloor}{\frac{k-1}{2}},$$

while the right part equals to "$f^{(n)}_{\lfloor \frac{n-1}{2} \rfloor} - 1$". This means that there must exist a function, say, $g(x)$ such that

$$\sum_{k=2,4,\ldots} \binom{\lfloor \frac{n}{2} \rfloor}{\frac{k}{2}} = g\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right),$$

or

$$g\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) = \begin{cases} 2^{\frac{n}{2}} - 1, & \text{if } n \text{ is even}, \\ 2^{\frac{n-1}{2}} - 1, & \text{if } n \text{ is odd}. \end{cases} \tag{3.13}$$

For $n = 3$ and $n = 4$, we simultaneously have $g(1) = 1$ and $g(1) = 3$.

Lemma 5. *The following formula holds*

$$\sum_{i=1}^{\lfloor \frac{n}{m+1} \rfloor} \frac{1}{k} \sum_{d|(n,k)} \varphi(d) \binom{\frac{n-mk}{d} - 1}{\frac{k}{d} - 1} = \frac{1}{n} \sum_{\substack{d|n \\ d \le \lfloor \frac{n}{m+1} \rfloor}} \varphi(d) \left((m+1) F^{(m)}_{\frac{n}{d}} - m F^{(m)}_{\frac{n}{d}} - 1 \right). \tag{3.14}$$

Proof. Put $k = dd_1$. Then we have

$$\begin{aligned} \sum_{k=1}^{\lfloor \frac{n}{m+1} \rfloor} \frac{1}{k} \sum_{d|(n,k)} \varphi(d) \binom{\frac{n-mk}{d} - 1}{\frac{k}{d} - 1} &= \sum_{\substack{dd_1 \le \lfloor \frac{n}{m+1} \rfloor \\ d|n}} \frac{1}{dd_1} \varphi(d) \binom{\frac{n}{d} - md_1 - 1}{d_1 - 1} \\ &= \sum_{\substack{d \le \lfloor \frac{n}{m+1} \rfloor \\ d|n}} \frac{\varphi(d)}{d} \sum_{d_1=1}^{\lfloor \frac{n}{(m+1)d} \rfloor} \frac{1}{d_1} \binom{\frac{n}{d} - md_1 - 1}{d_1 - 1}. \end{aligned} \tag{3.15}$$

Put $h = \frac{n}{d}$. Notice that $d \le \frac{n}{m+1}$, therefore $t \ge m + 1$.

Furthermore,

$$\sum_{d_1\ge 1}\frac{h}{d_1}\binom{h-md_1-1}{d_1-1}=\sum_{d_1\ge 1}\frac{h-md_1}{d_1}\binom{h-md_1-1}{d_1-1}$$
$$+m\sum_{d_1\ge 1}\binom{h-md_1-1}{d_1-1}$$
$$=\sum_{d_1\ge 1}\binom{h-md_1}{d_1}+m\sum_{k\ge 0}\binom{(h-m-1)-mk}{k}.$$

Thus, by Lemma 3

$$\sum_{d_1\ge 1}\frac{1}{d_1}\binom{\frac{n}{d}-md_1-1}{d_1-1}=\frac{d}{n}\left(F^{(m)}_{\frac{n}{d}}+mF^{(m)}_{\frac{n}{d}-m-1}-1\right).$$

Substituting this to (3.15) and taking in to account that

$$F_{\frac{n}{d}-m-1}=F_{\frac{n}{d}}-F_{\frac{n}{d}-1},$$

we obtain (3.14). □

4. **Variations 5-8**

Let now consider two-color bracelets with k isolated black beads such that between every two black beads there exist at least m white ones. Let us consider an aggregate, denoting it *black**, which contains a black bead and the following in succession after it in a fixed direct the m 0's. This gives a one-to-one correspondence between the considered set of all two-color bracelets with k isolated black beads such that between every two black beads there exist at least m white ones and the set of all two-color bracelets of $n-mk$ beads containing k $black^*$ beads. Therefore, it follows from Theorem 1 that there are $R(n-mk,\ k)$ incongruent configurations. Notice that always $n-mk\ge k$, or $k\le\lfloor\frac{n}{m+1}\rfloor$.

Summing $R(n-mk,\ k)$ by k from 1 to $\lfloor\frac{n}{m+1}\rfloor$ we find the number $\alpha_n^{(m)}$ of all two-color incongruent bracelets with isolated black beads such that between every two black beads there exist at least m white ones.

$$\alpha_n^{(m)}=\frac{1}{2}\sum_{k=1}^{\lfloor\frac{n}{m+1}\rfloor}\left(\frac{1}{k}\sum_{d|(k,n)}\varphi(d)\binom{\frac{n-mk}{d}-1}{\frac{k}{d}-1}+\binom{\lfloor\frac{n-mk-h_k}{2}\rfloor}{\lfloor\frac{k}{2}\rfloor}\right),\tag{4.1}$$

where $h_k\equiv k \pmod 2$, $h_k=0$ or 1, $m\ge 1$.

By (4.1) and Lemmas 4, 5 we have

$$\alpha_n^{(m)}=\frac{1}{2}f^{(m)}_{\lfloor\frac{n-\gamma_m}{2}\rfloor}-\frac{1}{2}+\frac{1}{2n}\sum_{d|n,d\le\lfloor\frac{n}{m+1}\rfloor}\varphi(d)\left((m+1)F^{(m)}_{\frac{n}{d}}-mF^{(m)}_{\frac{n}{d}}-1\right),\tag{4.2}$$

where $\gamma_m\equiv m-1 \pmod 2$, $\gamma_m=0$ or 1.

Now (as in proof of (2.2)) we notice that the binomial coefficient $\binom{\lfloor \frac{n-mk-h_k}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor}$ gives the number of those two-color bracelets with k black beads of the considered type that are symmetric respectively any diameter. Thus, by Lemma 4, the number $\beta_n^{(m)}$ of all these bracelets equals to

$$\beta_n^{(m)} = f^{(m)}_{\lfloor \frac{n-\gamma_m}{2} \rfloor} - 1, \quad \gamma_m \equiv m-1 \pmod 2, \quad \gamma_m = 0 \text{ or } 1. \tag{4.3}$$

Now, using the scheme of the proof of Theorem 2, we obtain the following results.

Theorem 3. *The following formulas solve Variations* 5-8*:*

$$N_{n,m} = -\sum_{\delta | n, \delta \geq 2} \mu(\delta) \alpha^{(m)}_{\frac{n}{\delta}}, \tag{4.4}$$

where $\alpha_n^{(m)}$ is defined by (4.2)*;*

$$S_{n,m} = -1 - \sum_{\delta | n, \delta \geq 2} \mu(d) \beta^{(m)}_{\frac{n}{d}}, \tag{4.5}$$

where $\beta_n^{(m)}$ is defined by (4.3)*;*

$$N^{(k)}_{n,m} = -\sum_{d | (n,k), d \geq 2} \mu(d) R\left(\frac{n-mk}{d}, \frac{k}{d}\right); \tag{4.6}$$

$$S^{(k)}_{n,m} = -\sum_{d | (n,k), d \geq 2} \mu(d) \binom{\left\lfloor \frac{1}{2}\left(\frac{n-mk}{d} - h_{\frac{k}{d}}\right) \right\rfloor}{\left\lfloor \frac{k}{2d} \right\rfloor}. \tag{4.7}$$

5. **An example**

In the case of $m = 1$, by (3.3) and (3.6), we have

$$F_n^{(1)} = F_n, \quad f_n^{(1)} = F_{n+1},$$

where

$$F_0 = 1,\ F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5,\ F_5 = 8,\ F_6 = 13, \ldots$$

are the Fibonacci numbers.

Thus, by (4.2)-(4.4),

$$\alpha_n^{(1)} = \frac{1}{2n} \sum_{d | n, d \leq \lfloor \frac{n}{2} \rfloor} \varphi(d) \left(2F_{\frac{n}{d}} - F_{\frac{n}{d}} - 1\right) + \frac{1}{2} F_{\lfloor \frac{n}{2} \rfloor + 1} - \frac{1}{2},$$

$$\beta_n^{(1)} = F_{\lfloor \frac{n}{2} \rfloor + 1} - 1.$$

Therefore, by Theorem 3, we find

$$N_{24,1} = \alpha^{(1)}_{12} + \alpha^{(1)}_{8} - \alpha^{(1)}_{4} = 25 + 7 - 2 = 30,$$

$$S_{24,1} = -1 + F_{\frac{24}{2 \cdot 2}+1} + F_{\frac{24}{2 \cdot 3}+1} - F_{\frac{24}{2 \cdot 6}+1} = -1 + 21 + 8 - 3 = 25.$$

Thus among 30 incongruent two-color SR-bracelets of 24 beads with isolated black beads only five ones do not have a diameter of symmetry.

Further, by Theorem 3, we find

$$N^{(6)}_{24,1} = -\sum_{d=2,3,6} \mu(d) R\left(\frac{18}{d}, \frac{6}{d}\right) = 9,$$

$$S^{(6)}_{24,1} = -\sum_{d=2,3,6} \mu(d) \binom{\left\lfloor \frac{1}{2}\left(\frac{18}{d} - h_{\frac{6}{d}}\right) \right\rfloor}{\left\lfloor \frac{3}{d} \right\rfloor} = 6,$$

$$N^{(8)}_{24,1} = -\sum_{d=2,4,8} \mu(d) R\left(\frac{16}{d}, \frac{8}{d}\right) = 8,$$

$$S^{(8)}_{24,1} = -\sum_{d=2,4,8} \mu(d) \binom{\left\lfloor \frac{1}{2}\left(\frac{16}{d} - h_{\frac{8}{d}}\right) \right\rfloor}{\left\lfloor \frac{4}{d} \right\rfloor} = 6.$$

Since

$$(N^{(6)}_{24,1} - S^{(6)}_{24,1}) + (N^{(8)}_{24,1} - S^{(8)}_{24,1}) = 5,$$

then only in cases $k = 6$ and $k = 8$ there are correspondingly 3 and 2 two-color SR-bracelets of 24 beads with isolated black beads which do not have a diameter of symmetry; in all other cases they do have a diameter of symmetry.

In Table 1 we show all 30 incongruent two-color SR-bracelets of 24 beads with isolated black beads. 0's denote white beads and 1's denote black ones.

Note that in Table 1 only bracelets with the ordinal numbers 12, 13, 16, 19 and 22 have no a diameter of symmetry; it is interesting that the bracelets with the ordinal numbers 5, 7, 10, 20 do not have a diameter of symmetry that connects any two beads. More exactly, a diameter of symmetry of these bracelets has the following endpoints: the midpoint between positions 24, 1 and the midpoint between positions 12, 13. A diameter of symmetry of other bracelets connects beads 1 and 13.

6. **Other variations**

We start with the two open questions arising as observations of Table 5.

Variation 9. To enumerate incongruent two-color SR-bracelets with a diameter of symmetry connecting any two beads.

Notice, furthermore, that among 21 such bracelets in Table 5 only 3 ones have a diameter of symmetry connecting beads of different colors (see bracelets with ordinal numbers 2, 25, 26). The following enumerating problem arises naturally.

Variation 10. To enumerate incongruent two-color SR-bracelets with a diameter of symmetry connecting any two beads of different colors.

In conclusion, we give several cases of enumeration of the incongruent t-color bracelets ($t \geq 3$) containing beads of every of t colors.

Table 1.

ordinal number	two-color *SR*-bracelets	number of black beads
1	1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0	2
2	1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0	3
3	0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0	4
4	0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0	4
5	0 1 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1 0	4
6	0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0	4
7	0 0 0 1 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 1 0 0 0	4
8	0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0	6
9	0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0	6
10	0 1 0 0 0 0 1 0 0 1 0 0 0 0 1 0 0 1 0 0 0 0 1 0	6
11	0 0 0 0 1 0 1 0 1 0 0 0 0 0 0 0 1 0 1 0 1 0 0 0	6
12	1 0 0 0 0 0 0 1 0 0 1 0 1 0 0 0 0 0 0 1 0 0 1 0	6
13	1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 0 0 1 0 0 0 1 0	6
14	0 0 0 1 0 0 1 0 0 1 0 0 0 0 0 1 0 0 1 0 0 1 0 0	6
15	1 0 0 0 0 1 0 1 0 0 0 0 1 0 0 0 0 1 0 1 0 0 0 0	6
16	1 0 0 0 0 1 0 0 0 1 0 0 1 0 0 0 0 1 0 0 0 1 0 0	6
17	1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0	8
18	0 0 0 1 0 1 0 1 0 1 0 0 0 0 0 1 0 1 0 1 0 1 0 0	8
19	1 0 0 0 0 1 0 0 1 0 1 0 1 0 0 0 0 1 0 0 1 0 1 0	8
20	0 0 1 0 1 0 0 1 0 1 0 0 0 0 1 0 1 0 0 1 0 1 0 0	8
21	1 0 0 0 1 0 1 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 0 0	8
22	1 0 0 0 1 0 0 1 0 0 1 0 1 0 0 0 1 0 0 1 0 0 1 0	8
23	0 0 1 0 0 1 0 1 0 0 1 0 0 0 1 0 0 1 0 1 0 0 1 0	8
24	0 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0	8
25	1 0 1 0 0 0 1 0 1 0 1 0 0 0 1 0 1 0 1 0 0 0 1 0	9
26	1 0 0 1 0 1 0 0 1 0 0 1 0 1 0 0 1 0 0 1 0 1 0 0	9
27	1 0 1 0 1 0 0 0 1 0 1 0 1 0 1 0 1 0 0 0 1 0 1 0	10
28	1 0 0 1 0 1 0 1 0 1 0 0 1 0 0 1 0 1 0 1 0 1 0 0	10
29	1 0 1 0 0 1 0 1 0 0 1 0 1 0 1 0 0 1 0 1 0 0 1 0	10
30	1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0	12

Variation 11. To find the number $N_{n,1}^{(0,1,2)}$ incongruent three-color (with colors $\{0,1,2\}$) *SR*-bracelets with isolated beads of colors 1 and 2.

Theorem 4.

$$N_{n,1}^{(0,1,2)} = \sum_{\substack{k\geq 2\\(k,n)>1}}^{\lfloor \frac{n}{2} \rfloor} N_{n,1}^{(k)} N_{(k,n)}. \tag{6.1}$$

Proof. We generate the required bracelets from $N_{n,1}^{(k)}$ two-color *SR*-bracelets with colors 0 and 1, k, $2 \leq k \leq \lfloor (\frac{n}{2}) \rfloor$. Removing beads of color 0, we obtain k places for generating two-color subbracelet of colors 1 and 2. Since we consider *SR*-bracelets, then, in view of subgroup of subgroup of a cyclic grope is a subgroup, we should have $(l,k)|(k,n)|n$. Thus

$$N_{n,1}^{(0,1,2)} = \sum_{\substack{k\geq 2\\(k,n)>1}}^{\lfloor\frac{n}{2}\rfloor} N_{n,1}^{(k)} \sum_{\substack{l\geq 2\\1<(l,k)|(k,n)}}^{\lfloor\frac{(k,n)}{2}\rfloor} N_{(k,n)}^{(l)}. \tag{6.2}$$

Since

$$\sum_{\substack{l\geq 2\\(l,n)>1}}^{\lfloor\frac{n}{2}\rfloor} N_n^{(l)} = N_n$$

(and always $(l,n)|n$), then from (6.2) we obtain (6.1). □

Thus the required enumeration we get from formulas (6.1), (2.1), (4.4) and (4.6).

Let us calculate, e.g., $N_{12,1}^{(0,1,2)}$. By (6.1), we have

$$N_{12,1}^{(0,1,2)} = N_{12,1}^{(2)}N_2 + N_{12,1}^{(3)}N_3 + N_{12,1}^{(4)}N_4 + N_{12,1}^{(6)}N_6\,.$$

Since, for prime $p, N_p = 0$, then

$$N_{12,1}^{(0,1,2)} = N_{12,1}^{(4)}N_4 + N_{12,1}^{(6)}N_6\,. \tag{6.3}$$

According to (2.5)-(2.6), we find $\alpha_i = i+1, i = 1,2,3$. Thus, by (2.1), we have

$$N_4 = -2+\alpha_2 = 1, \quad N_6 = -2+\alpha_3+\alpha_2-\alpha_1 = 3\,.$$

Furthermore, by (4.6) and (1.1), we have

$$N_{12,1}^{(4)} = R(4,2) = 2$$

and

$$N_{12,1}^{(6)} = R(3,3)+R(2,2)-R(1,1).$$

Since, by (1.1),

$$R(n,n) = \frac{1}{2}(1+\frac{1}{n}\sum_{d|n}\varphi(d)) = 1,$$

then $N_{12,1}^{(6)} = 1$ and, by (6.3), finally we find

$$N_{12,1}^{(0,1,2)} = 2\cdot 1 + 1\cdot 3 = 5\,.$$

In Table 6 we demonstrate all 5 these three-color bracelets of 12 beads.

Table 2.

ordinal number	3-color bracelets
1	1 0 0 2 0 0 1 0 0 2 0 0
2	1 0 1 0 2 0 1 0 1 0 2 0
3	2 0 2 0 1 0 2 0 2 0 1 0
4	1 0 2 0 1 0 2 0 1 0 2 0
5	1 0 2 0 0 0 1 0 2 0 0 0

Variation 12. To find the number $S_{n,1}^{(0,1,2)}$ incongruent three-color (with colors $\{0,1,2\}$) *SR*-bracelets with a diameter of symmetry having isolated beads of colors 1 and 2.

A solution is given by the following similar to (6.1) formula.

Theorem 5.

$$S_{n,1}^{(0,1,2)} = \sum_{\substack{k=2\\(k,n)>1}}^{\lfloor \frac{n}{2} \rfloor} S_{n,1}^{(k)} S_{(k,n)}. \tag{6.4}$$

Proof of Theorem 5 is based on quite similar arguments that proof of Theorem 4. Thus the required enumeration we obtain from formulas (6.2), (2.2), (4.5) and (4.7). So, e.g., we find that $S_{12,1}^{(0,1,2)} = 4$. Thus there is only (up to the congruence) three-color *SR*-bracelet in Table 6 that has not a diameter of symmetry. It is easy to see that it is the last bracelet in this table.

Finally note that the idea of construction formulas (6.1), (6.4) is easily generalized. E.g., if $N_n^{(0,1,\ldots,t)}(S_n^{(0,1,\ldots,t)})$ denotes the number of incongruent $(t+1)$-color *SR*-bracelets (having a diameter of symmetry), $N_{n,1}^{(0,1,\ldots,t)}(S_{n,1}^{(0,1,\ldots,t)})$ denotes the number of incongruent $(t+1)$-color *SR*-bracelets with isolated beads of colors $1,\ldots,t$ (having a diameter of symmetry), etc., then we have the following recursion formulas.

Theorem 6.

$$N_n^{(0,1,\ldots,t)} = \sum_{\substack{k=2\\(k,n)>1}}^{\lfloor \frac{n}{2} \rfloor} N_n^{(k)} N_{(k,n)}^{(0,1,\ldots,t-1)}, \tag{6.5}$$

$$S_n^{(0,1,\ldots,t)} = \sum_{\substack{k=2\\(k,n)>1}}^{\lfloor \frac{n}{2} \rfloor} S_n^{(k)} S_{(k,n)}^{(0,1,\ldots,t-1)}, \tag{6.6}$$

$$N_{n,1}^{(0,1,\ldots,t)} = \sum_{\substack{k=2\\(k,n)>1}}^{\lfloor \frac{n}{2} \rfloor} N_{n,1}^{(k)} N_{(k,n)}^{(0,1,\ldots,t-1)}, \tag{6.7}$$

$$S_{n,1}^{(0,1,\ldots,t)} = \sum_{\substack{k=2\\(k,n)>1}}^{\lfloor \frac{n}{2} \rfloor} S_{n,1}^{(k)} S_{(k,n)}^{(0,1,\ldots,t-1)} \tag{6.8}$$

etc.

References

[1] H. Gupta, Enumeration of incongruent cyclic k-gons, *Indian J. Pure Appl. Math.* **10**(8) (1979), 964–999.

[2] M. Hall, *Combinatorial Theory*, Blaisdell, 1967.

[3] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, 1982.

[4] V. Shevelev, Necklaces and convex k-gons, *Indian J. Pure Appl. Math.* **35**(5) (2004), 629–638.

[5] V. Shevelev (2011), *Spectrum of permanent's values and its extremal magnitudes in* Λ_n^3 *and* $\Lambda_n(\alpha, \beta, \gamma)$, `http://arxiv.org./abs/1104.4051`.

[6] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`.

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