Journal of Informatics and Mathematical Sciences
Vol. 10, No. 3, pp. 399–409, 2018
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
http://www.rgnpublications.com
DOI: 10.26713/jims.v10i3.487

Research Article

The Monopoly in the Join of Graphs

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Abstract. In a graph $G=(V,E)$, a set $M \subseteq V(G)$ is said to be a monopoly set of G if every vertex $v \in V-M$ has, at least, $\frac{d(v)}{2}$ neighbors in M. The monopoly size $mo(G)$ of G is the minimum cardinality of a monopoly set among all monopoly sets of G. A join graph is the complete union of two arbitrary graphs. In this paper, we investigate the monopoly set in the join of graphs. As consequences the monopoly size of the join of graphs is obtained. Upper and lower bound of the monopoly size of join graphs are obtained. The exact values of monopoly size for the join of some standard graphs with others are obtained.

Keywords. Monopoly set; Monopoly size; Join of graphs; Monopoly size of join of graphs

MSC. 05C07; 05C69

Received: October 2, 2016 **Accepted:** April 20, 2017

1. Introduction

The concept of monopoly in a graph was introduced in (2013) by Khoshkhak *et al.* [6] and defined as, a set $M \subseteq V(G)$ is called a monopoly set of G if for every vertex $v \in V(G)-M$ has at least $\frac{d(v)}{2}$ neighbors in M. The monopoly size of G, denoted by $mo(G)$, is the minimum cardinality of a monopoly set in G. Some mathematical properties of monopoly in graphs have studied in [11], Other types of monopoly in graphs have been subsequently proposed by Naji and Soner in [8]-[14].In particular, the monopoly in graphs is a dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg [15]. For more details in monopoly and dynamos in graphs, we refer the reader to

[1,2,4,7,16]. In this paper, we study the monopoly set of join graph. Upper and lower bound of monopoly size of join graph are obtained. The exact values of monopoly size for a join graph of some standard graphs are obtained.

We begin by stating the terminology and notations used through this article. A graph $G=(V,E)$ is a simple graph, that is finite, having no loops no multiple and directed edges. An edge $\{x,y\}$ is said to join the vertices x and y and is denoted by xy. Thus, vertices x and y are the end vertices of the edge xy. As usual, we denote by $n=|V|$ to the number of vertices in a graph G. For a vertex $v\in V(G)$, the open neighborhood of v in a graph G, denoted $N_G(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N_G[v]=N_G(v)\cup\{v\}$. The degree of vertex v in G is $d_G(v)=|N_G(v)|$, and the degree of a vertex v with respect to a subset $S\subset V(G)$ is $d_S(v)=|N_G(v)\cap S|$. We denote by $\Delta(G)$ and $\delta(G)$ to maximum and minimum degree among the vertices of G, respectively. An isolated vertex in G is a vertex with degree zero. As usual, $\overline{G}$ denotes the complement of G, for a subset $S\subseteq V$, $\overline{S}=V-S$ and kG denotes the k disjoint copies of G. A complete graph K_n is a graph which in every two vertices are adjacent, while a total disconnected (or an empty) graph, denoted $\overline{K_n}$, has order n and no edges. The graph K_1 is said to be trivial graph. Two graphs are isomorphic if there is a correspondence between their vertex sets that preserves adjacency. Thus $G=(V,E)$ is isomorphic to $G'=(V',E')$ if there is a bijection¢ $f:V\to V'$ such that $xy\in E$ if and only if $f(x)f(y)\in E'$. Clearly, isomorphic graphs have the same order and size and degrees. In accordance with this convention, if G and H are isomorphic graphs, then we write either $G\cong H$ or simply $G=H$. The join graph $G+H$ is the complete union of two graphs G and H, in other word, is the graph with vertex set

$$V(G+H)=V(G)\cup V(H)$$

and edge set

$$E(G+H)=E(G)\cup E(H)\cup\{uv: u\in V(G), v\in V(H)\}.$$

Note that, For any two graphs G_1 and G_2, $G_1+G_2=G_2+G_1$ and if $n_i=|V(G_i)|$, $\Delta_i=\Delta(G_i)$ and $\delta_i=\delta(G_i)$, for $i\in\{1,2\}$, then $n=n_1+n_2$, $\Delta(G_1+G_2)=\Delta=\max\{\Delta_1+n_2,\Delta_2+n_1\}$ and $\delta(G_1+G_2)=\delta=\min\{\delta_1+n_2,\delta_2+n_1\}$. $\lfloor x\rfloor$ ($\lceil x\rceil$) denotes the greatest (smallest) integer number less (greater) than or equal to x.

For terminologies and notations in graph theory not defined here, we refer the reader to the books [3,5].

The following are some fundamental results which will be required for many of our arguments in this paper:

Theorem 1.1 ([6]). *Let G be a graph on n vertices with m edges whose maximum degree is $\Delta(G)$. Then*

$$\frac{2m}{3\Delta(G)}\le mo(G)\le\frac{n}{2}.$$

Theorem 1.2 ([11]). *Let G be a graph of order n and minimum degree $\delta\ge 1$. Then*

$$\frac{\delta}{2}\le mo(G)\le n-\frac{\delta+2}{2}.$$

Theorem 1.3 ([11]). *For any graph G of order n, $mo(G) = 1$ if and only if G has a vertex v of degree $n-1$ and $G - v = sK_1 \cup tK_2$, for $0 \le s,\ t \le n-1$.*

2. The Monopoly Set in the Join of Graphs

In this section, we investigate monopoly set of the join of two graphs.

Theorem 2.1. *Let G_1 and G_2 be two graphs and let M be a monopoly set of $G_1 + G_2$ such that $|N_{G_i}(v) \cap (M \cap V(G_i))| \ge \frac{d_{G_i}(v)}{2}$, for every $v \in V(G_i) - M$ and $i \in \{1,2\}$. Then M is a monopoly set of both G_1 and G_2.*

Proof. Let M be a monopoly set of $G_1 + G_2$. Set $M_1 = M \cap V(G_1)$. Since, $|N_{G_1}(v) \cap M_1| = |N_{G_1}(v) \cap (M \cap V(G_1))| \ge \frac{d_{G_1}(v)}{2}$, for every $v \in V(G_1) - M_1$, it follows that M_1 is a monopoly set of G_1 and since $M_1 \subseteq M$ then M is a monopoly set of G_1. Similarly, M is a monopoly set of G_2. □

The converse of Theorem 2.1, is not true in general. For example, let $G_1 = P_4$ with vertex set v_1, v_2, v_3, v_4 and $G_2 = K_{1,3}$ with vertex set $\{u_0, u_1, u_2, u_3\}$, where u_0 is the central vertex and let we take $M = \{v_2, v_3, u_0\}$. Then $M \cap V(G_1)$ is a monopoly set of G_1 and $M \cap V(G_2)$ is also a monopoly set of G_2. But M is not a monopoly set of $G_1 + G_2$, because $|N_{G_1+G_2}(u_1) \cap M| = 1 < \frac{5}{2} = \frac{d_{G_1+G_2}(u_1)}{2}$.

Theorem 2.2. *Let G be a connected graph of order $n_1 \ge 2$ and let M be a minimum monopoly set of G. Then for any graph H of order n_2, M is a monopoly set of $G + H$, if and only if the following conditions are holding*

(a) $|M| = \frac{n_1}{2}$.

(b) *H is totally disconnected.*

(c) *$n_2 \le 2|N_G(v) \cap M| - d_G(v)$, where v is the vertex of minimum degree in $V(G) - M$.*

Proof. Let G be a connected graph of order $n_1 \ge 2$, H be a graph of order n_2 and let M be a minimum monopoly set of G. Assume that M is a monopoly set of $G + H$. Since G is a connected graph then by Theorem 1.1, $|M| \le \frac{n_1}{2}$. If $|M| \ne \frac{n_1}{2}$, then

$$|N_{G+H}(v) \cap M| \le |M| < \frac{n_1}{2} \le \frac{n_1 + d_G(v)}{2} = \frac{d_{G+H}(v)}{2}, \quad \text{for every } v \in (V(H) - M).$$

Hence, M is not a monopoly set of $G + H$, a contradiction. Thus, the condition (a) must hold.

Since, M is a monopoly set of $G + H$ and $|M| = \frac{n_1}{2}$, it follows that

$$|M| = \frac{n_1}{2} \ge |N_{G+H}(v) \cap M| \ge \frac{d_{G+H}(v)}{2} = \frac{d_H(v) + n_1}{2}, \quad \text{for every } v \in (V(H) - M).$$

Hence $d_H(v) = 0$, for every $v \in V(H) - M$ and since $M \subset V(G)$ then $d_H(v) = 0$ for every $v \in V(H)$. Thus H is totally disconnected. Now, since M is a monopoly set of $G + H$ then $|N_{G+H}(v) \cap M| \ge \frac{n_2 + d_G(v)}{2}$, for every $v \in V(G) - M$. Since $M \subset V(G)$, it follows that $|N_G(v) \cap M| = |N_{G+H}(v) \cap M|$, for every $v \in V(G) - M$. Hence $n_2 \le 2|N_G(v) \cap M| - d_G(v)$, for v is the vertex of minimum degree in $V(G) - M$.

Conversely, let G be a connected graph of order $n_1 \geq 2$, H be a graph of order n_2 and let M be a monopoly set of G. Suppose that the three conditions are holding. Since $M \subset V(G)$, then $M \cap V(H) = \phi$ and since $N_{G+H}(v) = N_G(v) \cup V(H)$, for every $v \in V(G)$, it follows that

$$N_{G+H}(v) \cap M = (N_G(v) \cap M) \cup V(H) \cap M = N_G(v) \cap M \cup \phi = N_G(v) \cap M.$$

By the condition (c), this implies, for every $v \in V(G) - M$

$$d_{G+H}(v) = d_G(v) + n_2 \leq 2|N_G(v) \cap M| = 2|N_{G+H}(v) \cap M|. \tag{2.1}$$

By condition (b), H is totally disconnected, then $d_{G+H}(v) = n_1$, for every $v \in V(H)$. By this and using condition (a), for every $v \in V(H)$

$$|N_{G+H}(v) \cap M| = |N_G(v) \cap M| = |M| = \frac{n_1}{2} = \frac{d_{G+H}(v)}{2}. \tag{2.2}$$

Hence, by equations (2.1) and (2.2), M is monopoly set of $G + H$. □

Corollary 2.3. *For any two connected nontrivial graphs G_1 and G_2. If M is a minimum monopoly set of G_1 or G_2, then M is not a monopoly set of $G_1 + G_2$.*

Theorem 2.4. *Let G_1 and G_2 be two graphs of orders n_1 and n_2, respectively and let M_i be a monopoly set of G_i, for every $i \in \{1,2\}$. If $|M_i| \geq \frac{n_i}{2}$, for every $i = 1,2$, then $M_1 \cup M_2$ is a monopoly set of $G_1 + G_2$.*

Proof. Let M_1 and M_2 be monopoly sets of G_1 and G_2, respectively such that $M_1 \geq \frac{n_1}{2}$ and $|M_2| \geq \frac{n_2}{2}$. Since, for every $v \in V(G_1) - M_1$

$$\begin{aligned} |N_{G_1+G_2}(v) \cap (M_1 \cup M_2)| &= |N_{G_1}(v) \cap M_1| + |N_{G_2}(v) \cap M_2| \\ &\geq \frac{d_{G_1}(v)}{2} + |M_2| \\ &\geq \frac{d_{G_1}(v)}{2} + \frac{n_2}{2} \\ &= \frac{d_{G_1+G_2}(v)}{2} \end{aligned} \tag{2.3}$$

and similarly, for every $v \in V(G_2) - M_2$

$$|N_{G_1+G_2}(v) \cap (M_1 \cup M_2)| \geq \frac{d_{G_1+G_2}(v)}{2}. \tag{2.4}$$

Hence, by equations (2.3) and (2.4),

$$|N_{G_1+G_2}(v) \cap (M_1 \cup M_2)| \geq \frac{d_{G_1+G_2}(v)}{2}, \quad \text{for every } v \in V(G_1 + G_2) - (M_1 \cup M_2).$$

Therefore, $M_1 \cup M_2$ is a monopoly set of $G_1 + G_2$. □

The converse of Theorem 2.4, in general, is not true. For example, in this situation. Let $G_1 = P_3$ with vertex set v_1, v_2, v_3 and let $G_2 = K_{1,4}$ with vertex set u_0, u_1, u_2, u_3, u_4, where u_0 is the central vertex. Take $M_1 = \{v_2\}$ and $M_2 = \{u_0, u_1\}$. Clearly, M_1 and M_2 are a monopoly sets of G_1 and G_2, respectively and $M_1 \cup M_2$ is a monopoly set of $G_1 + G_2$. However, $|M_1| = 1 < \frac{n_1}{2}$ and also $|M_2| = 2 < \frac{n_2}{2}$.

Proposition 2.5. *For any two graphs G_1 and G_2 of orders n_1 and n_2, respectively. If $n_1 < n_2$ and M is a monopoly set of G_2, then $V(G_1) \cup M$ is a monopoly set of $G_1 + G_2$.*

Proof. Let G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, such that $n_1 < n_2$ and let M be a monopoly set of G_2. since, for every $v \in V(G_1+G_2) - (V(G_1) \cup M) = V(G_2) - M$,

$$\begin{aligned}|N_{G_1+G_2}(v) \cap (V(G_1) \cup M)| &= |V(G_1)| + |N_{G_2}(v) \cap M| \\ &\geq n_1 + \frac{d_{G_2}(v)}{2} \\ &\geq \frac{d_{G_1+G_2}(v)}{2}.\end{aligned}$$

Then $V(G_1) \cup M$ is a monopoly set of $G_1 + G_2$. □

3. Monopoly Size of the Join of Graphs

Since the join of any two graphs G_1 and G_2 is connected then by Theorem 1.1, the proof of the following result is straightforward.

Observation 3.1. For any two graphs G_1 and G_2 of orders n_1 and n_2, respectively.

$$1 \leq mo(G_1 + G_2) \leq \frac{n_1 + n_2}{2}.$$

The following result characterize all two graphs with monopoly size of join its is one.

Theorem 3.2. *For any two graphs $G_1$and G_2, $mo(G_1 + G_2) = 1$ if and only if $G_1 = K_1$ and $G_2 = sK_1 \cup tK_2$.*

Proof. The proof is immediate consequences of the definition of the join of graphs and Theorem 1.3. □

Corollary 3.3. *Let G_1 and G_2 be connected nontrivial graphs of orders n_1 and n_2, respectively. Then*

$$2 \leq mo(G_1 + G_2) \leq \frac{n_1 + n_2}{2}.$$

Observation 3.4. For any two graphs G_1 and G_2 of orders n_1 and n_2 and minimum degree δ_1 and δ_2, respectively.

$$mo(G_1 + G_2) \geq \min\left\{\frac{\delta_1 + n_2}{2}, \frac{\delta_2 + n_1}{2}\right\}.$$

Corollary 3.5. *For any two graphs G_1 and G_2 of orders n_1 and n_2 and minimum degree δ_1 and δ_2, respectively. If $n_1 \leq n_2$ and $\delta_2 \leq \delta_1$, then*

$$\frac{n_1 + \delta_2}{2} \leq mo(G_1 + G_2) \leq n_2.$$

Proof. Since $n_1 \leq n_2$ and $\delta_2 \leq \delta_1$ it follows that $\min\left\{\frac{\delta_1+n_2}{2}, \frac{\delta_2+n_1}{2}\right\} = \frac{n_1+\delta_2}{2}$ and hence by Observation 3.4, the lower bound is holding. For the upper bound, since $n_1 \leq n_2$ and by

Observation 3.1, $mo(G_1+G_2) \leq \frac{n_1+n_2}{2} \leq \frac{n_2+n_2}{2} = n_2$. □

Proposition 3.6. *For any connected graphs G_1 and G_2 of orders n_1 and n_2 and minimum degree δ_1 and δ_2, respectively.*

$$mo(G_1+G_2) \geq \min\left\{\left\lceil \frac{\delta_2}{2} \right\rceil + mo(G_1), \left\lceil \frac{\delta_1}{2} \right\rceil + mo(G_2)\right\}.$$

Proof. The proof is immediate consequences of Theorem 1.1 and Theorem 3.4. □

By Proposition 3.6, for any two graphs G_1 and G_2, $mo(G_1+G_2) \geq \min\{mo(G_1), mo(G_2)\}$, but not need $mo(G_1+G_2) \geq mo(G_i)$, for every $i \in \{1,2\}$. That means, in general, if M is a monopoly set of G_1+G_2 then not need M is a monopoly set of both G_1 and G_2. For example, in this situation. Let $G_1 = K_1$ and $G_2 = mK_2$, for $m \geq 2$. Then, by Theorem 3.2, $mo(G_1+G_2) = 1$. However, $mo(G_2) = m \geq 2$.

Remark 3.7. For any two graphs G and H, the summation of the monopoly sizes $mo(G)$ of a graph G with the monopoly size $mo(H)$ of a graph H and the monopoly size $mo(G+H)$ of the join $G+H$ are not comparable. For examples:

- $mo(K_2+C_n) = 2 < 1 + \lceil \frac{n}{3} \rceil = mo(K_2) + mo(C_n)$, for every $n \geq 4$.
- $mo(P_3+P_5) = 3 = 1+2 = mo(P_3) + mo(P_5)$.
- $mo(K_{1,5}+K_{1,5}) = 6 > 2 = mo(K_{1,5}) + mo(K_{1,5})$

For the details of the above examples, see the next results.

Proposition 3.8. *Let G_1 and G_2 be graphs of orders n_1 and n_2 respectively, such that $n_1 < n_2$. Then*

$$mo(G_1+G_2) \leq n_1 + mo(G_2).$$

The bound is sharp, The graphs K_2 and $K_{1,n}$, for $n \geq 3$ attending it.

Proof. The proof is immediately consequences of Proposition 2.5. □

Theorem 3.9. *For any two graphs G_1 and G_2 of orders n_1 and n_2, respectively. If $n_1 < n_2$ and $\Delta_2 \leq n_1$, then*

$$mo(G_1+G_2) = n_1.$$

Proof. Let G_1 and G_2 be two graphs of orders n_1 and n_2 respectively, such that $n_1 < n_2$ and $\Delta_2 \leq n_1$. Clearly, that $V(G_1+G_2) - V(G_1) = V(G_2)$. Set $M = V(G_1)$ and $V(G_1+G_2) = V$. Hence, for every $v \in V(G_2)$, $v \in V - M$. Since, $N_{G_2}(v) \cap M = \phi$ for every $v \in V - M$, it follows that, for every $v \in V - M$

$$\begin{aligned}|N_{G_1+G_2}(v) \cap M| &= |N_{G_1}(v) \cap M| + |N_{G_2}(v) \cap M| \\ &= |M| + 0 = n_1\end{aligned}$$

$$= \frac{n_1 + n_1}{2} \geq \frac{n_1 + \Delta_2}{2}$$
$$\geq \frac{n_1 + d_{G_2}(v)}{2} = \frac{d_{G_1+G_2}(v)}{2}.$$

Hence, M is a monopoly set of $G_1 + G_2$. Therefore,

$$mo(G_1 + G_2) \leq n_1. \tag{3.1}$$

Conversely, Let M be a monopoly set of $G_1 + G_2$. Suppose, on the contrary, that $|M| < n_1$. We consider the following cases:

Case 1: If $M \cap V(G_2) = \phi$, then $M \subset V(G_1)$ and hence there exists at least a vertex $v \in V(G_1) - M$ and since $n_1 < n_2$, it follows that

$$|N_{G_1+G_2}(v) \cap M| = |N_{G_1}(v) \cap M| \leq d_{G_1}(v)$$
$$= \frac{d_{G_1}(v) + d_{G_1}(v)}{2} \leq \frac{d_{G_1}(v) + n_1}{2}$$
$$< \frac{d_{G_1}(v) + n_2}{2} = \frac{d_{G_1+G_2}(v)}{2}.$$

Hence, M is not a monopoly set of $G_1 + G_2$, a contradiction.

Case 2: If $M \cap V(G_1) = \phi$, then $M \subset V(G_2)$ and hence there exists at least a vertex $v \in V(G_2) - M$ and since

$$|N_{G_1+G_2}(v) \cap \overline{M}| \geq n_1 > |M| \geq |N_{G_1+G_2}(v) \cap M|,$$

it follows that

$$d_{G_1+G_2}(v) = |N_{G_1+G_2}(v) \cap M| + |N_{G_1+G_2}(v) \cap \overline{M}| > 2|N_{G_1+G_2}(v) \cap M|.$$

Hence, M is not a monopoly set of $G_1 + G_2$, a contradiction.

Case 3: If $M \cap V(G_1) \neq \phi$ and $M \cap V(G_2) \neq \phi$, then there exist at least a vertex $v \in V(G_1) - M$ and a vertex $u \in V(G_2) - M$. Thus, M must contain at lest $\lfloor \frac{n_1}{2} \rfloor$ vertices from $V(G_1)$ and $\lfloor \frac{n_2}{2} \rfloor$ vertices from $V(G-2)$ and hence $|M| \geq \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor > 2\lfloor \frac{n_1}{2} \rfloor \geq n_1 - 1$, a contradiction of our supposition.

Accordingly, the three cases above, any subset $M \subseteq V(G_1 + G_2)$ with $|M| < n_1$ is not a monopoly set of $G_1 + G_2$.

Hence,

$$mo(G_1 + G_2) \geq n_1. \tag{3.2}$$

Therefore, by equations (3.1) and (3.2), $mo(G_1 + G_2) = n_1$. □

Theorem 3.10. *For any connected graph G,*

$$mo(G) \leq mo(K_1 + G) \leq mo(G) + 1.$$

The bound are sharp, the complete graph K_n for n is even numbers join with C_4 attending the lower bound and the complete graph K_n, for n is odd join with P_n for $n \equiv 0 \pmod 3$ attending the upper bound.

4. Monopoly Size of the Join of the Isomorphic Graphs

In this section, we are interesting in the study of a monopoly size in the join of the isomorphic graphs.

By the properties of the isomorphic graphs and the results in Section 2, the proof of the following results are straightforward.

Proposition 4.1. *Let G and H be two isomorphic graphs with $|V(G)| = |V(H)| = n$. Then*

(1) $mo(G+H) = 1$ *if and only if* $G = H = K_1$.

(2) $mo(G+H) \geq \frac{\delta+n}{2}$, *the bound is sharp, P_3 attending it.*

Theorem 4.2. *Let G be a graph of order $n \geq 2$. Then $mo(G+G) \geq mo(G)$, with the equality holds if and only if G is totally disconnected.*

Proof. Let G be a graph of order n and let M be a monopoly set of G with $|M| = mo(G)$. Then, we consider the following cases

Case 1: If G is a connected graph, then by Corollary 2.3, $mo(G+G) \geq mo(G)$.

Case 2: If G is a disconnected graph with at lest one edge, then

$$mo(G) = |M| \leq n-1.$$

Thus by Proposition 3.6, $mo(G+G) \geq n-1 \geq mo(G)$.

Case 3: If G is totally disconnected, then $mo(G) = n$. Since, $G+G$ is a complete bipartite graph and for any complete bipartite graph $K_{r,s}$, $mo(K_{r,s}) = \min\{r,s\}$, it follows that $mo(G+G) = mo(G) = n$.

The three previous cases lead to the proof of the second part of the theorem. □

Theorem 4.3. *For any two isomorphic connected nontrivial graphs G and H,*

$$mo(G+H) \geq mo(G) + mo(H).$$

The bound is sharp, the join of a graph P_3 with itself attending it.

Proof. Let G and H be two connected nontrivial graphs such that $G \cong H$ and let M_1 and M_2 be monopoly sets of G and H, respectively with $|M_1| = mo(G)$ and $|M_2| = mo(H)$. Since G and H are connected graphs then by Theorem 1.1, $|M_1| \leq \frac{n}{2}$ and $|M_2| \leq \frac{n}{2}$. Hence, by Theorem 2.4, $M_1 \cup M_2$ is a monopoly set of $G+H$ if and only if $|M_1| = \frac{n}{2}$ and $|M_2| = \frac{n}{2}$. Therefore, $mo(G+H) \geq |M_1| + |M_2| = mo(G) + mo(H)$. □

Corollary 4.4. *For any connected graph G, $mo(G+G) \geq 2mo(G)$.*

Corollary 4.5. *Let G be a graph of order n and minimum degree δ. Then for the integer $k \geq 2$,*

$$\frac{\delta+(k-1)n}{2} \leq mo\left(\sum_{i=1}^{k} G\right) \leq \frac{kn}{2}.$$

Corollary 4.6. *For any connected nontrivial graph G and for integer $k \geq 2$,*

$$mo\left(\sum_{i=1}^{k} G\right) \geq kmo(G).$$

5. Monopoly Size of the Join of Some Standard Graphs

In this section, we compute the exact values of the size monopoly of the join of some standard graph as the join of trivial graph K_1, path P_n, cycle C_n, complete graph K_n and star graph $K_{1,n}$ with others. By the results in Sections 2 and 3, the proof of the following results are straightforward.

Proposition 5.1. *For the join of trivial graph K_1,*

(1) *For $n \geq 1$, $mo(K_1 + P_n) = \lfloor \frac{n}{3} \rfloor + 1$;*

(2) *For $n \geq 3$, $mo(K_1 + C_n) = \begin{cases} 2, & \text{if } n = 4; \\ \lceil \frac{n}{3} \rceil + 1, & \text{otherwise.} \end{cases}$*

(3) *For $n \geq 1$, $mo(K_1 + K_n) = \lceil \frac{n}{2} \rceil$;*

(4) *For $n \geq 2$, $mo(K_1 + K_{1,n}) = 2$;*

(5) *For $1 \leq n \leq m$, $mo(K_1 + K_{n,m}) = \begin{cases} n, & \text{if } n = m; \\ \min\{n,m\} + 1, & \text{otherwise.} \end{cases}$*

Proposition 5.2. *For the join of the path P_n, $n \geq 2$,*

(1) *For $m \geq 2$, $mo(P_n + P_m) = \begin{cases} 2\lfloor \frac{n}{2} \rfloor, & \text{if } n = m; \\ \min\{n,m\}, & \text{otherwise.} \end{cases}$*

(2) *For $m \geq 3$, $mo(P_n + C_m) = \begin{cases} n-1, & \text{if } n = m \geq 6 \text{ and } n \text{ is even}; \\ \min\{n,m\}, & \text{otherwise.} \end{cases}$*

(3) *For $m \geq 2$, $mo(P_n + K_m) = \begin{cases} m, & \text{if } n \geq m; \\ \lfloor \frac{n+m}{2} \rfloor, & \text{otherwise.} \end{cases}$*

(4) *For $m \geq 3$, $mo(P_n + K_{1,m}) = \begin{cases} m+1, & \text{if } n > m+1; \\ n+1, & \text{if } n < m; \\ n, & \text{if } n = m; \\ n-1, & \text{if } n = m+1 \text{ and } n \text{ is odd}; \\ n, & \text{if } n = m+1 \text{ and } n \text{ is even}. \end{cases}$*

Proposition 5.3. *For the join of the cycle C_n, $n \geq 3$,*

(1) *For $m \geq 3$, $mo(C_n + C_m) = \begin{cases} n-1, & \text{if } n = m \geq 6 \text{ and } n \text{ is even}; \\ \min\{n,m\}, & \text{otherwise.} \end{cases}$*

(2) *For $m \geq 2$, $mo(C_n + K_m) = \begin{cases} m, & \text{if } n \geq m; \\ \lfloor \frac{n+m}{2} \rfloor, & \text{otherwise.} \end{cases}$*

(3) *For* $m \geq 3$, $mo(C_n + K_{1,m}) = \begin{cases} m+1, & \text{if } n > m+1; \\ n+1, & \text{if } n < m; \\ n, & \text{if } n = m; \\ n, & \text{if } n = m+1 \text{ and } n \text{ is odd;} \\ n-1, & \text{if } n = m+1 \text{ and } n \text{ is even.} \end{cases}$

Proposition 5.4. *For the join of complete graph* K_n, $n \geq 2$,

(1) *For* $m \geq 2$, $mo(K_n + K_m) = \lceil \frac{n+m}{2} \rceil$;

(2) *For* $m \geq 3$, $mo(K_n + K_{1,m}) = \begin{cases} m+1, & \text{if } n < m; \\ n, & \text{if } n = m; \\ n, & \text{if } n = m+1; \\ \lceil \frac{n+m}{2} \rceil, & \text{if } n > m+1. \end{cases}$

Proposition 5.5. *For the join of the star graph* $K_{1,n}$, $n \geq 2$,

$$\text{For } m \geq 2,\ mo(K_{1,n} + K_{1,m}) = \begin{cases} n, & \text{if } n = m \text{ and } n \text{ is even;} \\ n+1, & \text{if } n = m \text{ and } n \text{ is odd;} \\ \lceil \frac{n-2}{2} \rceil + \lceil \frac{m-2}{2} \rceil + 2, & \text{otherwise.} \end{cases}$$

6. Conclusion

In this paper, we initiated the study of the monopoly in the join of graphs. We discussed the properties of the monopoly set in the join of graphs. The monopoly size $mo(G+H)$ of the join of two graphs G and H is presented and also some upper and lower bound of the monopoly size of join graphs are obtained. However, there are a lot of problems in this concepts for future study, we mention some of them as follows:

(1) Generalize all or some results therein this paper for more than two graphs.

(2) Classification all two graphs G and H such that $mo(G+H) = mo(G) + mo(H)$.

(3) Classification all graphs G with n vertices such that $mo(G+G) = \frac{\delta(G)+n}{2}$.

(4) Calculate the monopoly size for the join of others graph families.

(5) Calculate the monopoly size for others graph operations.

Acknowledgement

We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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