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Numerical Solution of Hunter-Saxton Equation by Using Iterative Methods

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Abstract. In this paper, a Hunter-Saxton equation is solved by using the *Adomian's decomposition method* (ADM), *modified Adomian's decomposition method* (MADM), *variational iteration method* (VIM), *modified variational iteration method* (MVIM) and *homotopy analysis method* (HAM). The approximation solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

1. Introduction

The Hunter-Saxton equation

$$u_{txx} = -2u_x u_{xx} - u u_{xxx}, \quad t > 0. \tag{1}$$

models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field, x being the space variable in a reference frame moving with the unperturbed wave speed and t being a slow time variable [1]. In recent years, some works have been done in order to find the numerical solution of this equation. For example [2-7]. In this work, we develop the ADM, MADM, VIM, MVIM and HAM to solve the Eq. (1) with the initial conditions as follows:

$$\begin{aligned} &u(x,0) = f(x),\\ &u_{tx}(a,t) = g(t),\\ &u_t(a,t) = g_1(t). \end{aligned} \tag{2}$$

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq. (1). Also, the existence and uniqueness

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of the solution and convergence of the proposed method are proved in section 3. A example is presented in section 4 to illustrate the accuracy of these methods.

To obtain the approximate solution of Eq. (1), by integrating three times from Eq. (1) with respect to x, t and using the initial conditions we obtain,

$$u(x,t) = F(x,t) - 2\int_0^t \int_a^x (x-t)\, F_1(u(x,t))\, dt\, dx \\ - \int_0^t \int_a^x (x-t) F_2(u(x,t))\, dt\, dx, \quad (3)$$

where,

$$D^i(u(x,t)) = \frac{\partial^i u(x,t)}{\partial x^i}, \quad i = 1,2,3,$$
$$F(x,t) = f(x) + (x-a)\int_0^t (g_1(t) + g(t))\, dt,$$
$$F_1(u(x,t)) = D(u(x,t))D^2(u(x,t)),$$
$$F_2(u(x,t)) = u(x,t)D^3(u(x,t)).$$

In Eq. (3), we assume $F(x,t)$ is bounded for all t in $J = [0,T]$ and x in $[a,b]$ $(T,a,b \in \mathbb{R})$.

The terms $F_1(u(x,t)), F_2(u(x,t))$ are Lipschitz continuous with $|F_i(u) - F_i(u^*)| \leq L_i |u - u^*|$ $(i = 1,2)$, and

$$|x - t| \leq M.$$
$$\alpha = T(b-a)M(2L_1 + L_2),$$
$$\beta = 1 - T(1-\alpha).$$

2. **The iterative methods**

2.1. *Description of the MADM and ADM*

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = g_1(x,t), \quad (4)$$

where $u(x,t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq. (4), and using the given conditions we obtain

$$u(x,t) = f_1(x) - L^{-1}(Ru) - L^{-1}(Nu), \quad (5)$$

where the function $f_1(x)$ represents the terms arising from integrating the source term $g_1(x,t)$. The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\right]_{\lambda=0}. \tag{7}$$

Adomian polynomials were introduced in [8-10] as

$$\begin{aligned} A_0 &= G_1(u_0),\\ A_1 &= u_1 G_1'(u_0),\\ A_2 &= u_2 G_1'(u_0) + \frac{1}{2!}u_1^2 G_1''(u_0),\\ A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!}u_1^3 G_1'''(u_0), \ldots\ . \end{aligned} \tag{8}$$

2.1.1. *Adomian decomposition method.*

The standard decomposition technique represents the solution of $u(x,t)$ in (4) as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{9}$$

where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$$\begin{aligned} u_0 &= F(x,t)\\ u_1 &= -2\int_0^t\int_a^x (x-t)A_0(x,t)\,dt\,ds - \int_0^t\int_a^x (x-t)B_0(x,t)\,dt\,dx,\\ &\ \vdots\\ u_{n+1} &= -2\int_0^t\int_a^x (x-t)A_n(x,t)\,dt\,ds - \int_0^t\int_a^x (x-t)B_n(x,t)\,dt\,dx, \quad n \geq 0. \end{aligned} \tag{10}$$

Substituting (8) into (10) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in a series form defined by (9) follows immediately.

2.1.2. *The modified Adomian decomposition method.*

The modified decomposition method was introduced by Wazwaz [11]. The modified forms was established based on the assumption that the function $F(x,t)$

can be divided into two parts, namely $F_1(x,t)$ and $F_2(x,t)$. Under this assumption we set

$$F(x,t) = F_1(x,t) + F_2(x,t). \tag{11}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part F_1 be assigned to the zeroth component u_0, whereas the remaining part F_2 be combined with the other terms given in (11) to define u_1. Consequently, the modified recursive relation

$$\begin{aligned} u_0 &= -F_1(x,t), \\ u_1 &= -F_2(x,t) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &\vdots \\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{aligned} \tag{12}$$

was developed.

To obtain the approximation solution of Eq. (1), according to the MADM, we can write the iterative formula (12) as follows:

$$\begin{aligned} u_0 &= F_1(x,t), \\ u_1 &= F_2(x,t) - 2\int_0^t \int_a^x (x-t)A_0(x,t)\,dt\,dx - \int_0^t \int_a^x (x-t)B_0(x,t)\,dt\,dx, \\ &\vdots \\ u_{n+1} &= -2\int_0^t \int_a^x (x-t)A_n(x,t)\,dt\,dx - \int_0^t \int_a^x (x-t)B_n(x,t)\,dt\,dx. \end{aligned} \tag{13}$$

The operators $F_1(u)$, $F_2(u)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$\begin{aligned} F_1(u) &= \sum_{i=0}^{\infty} A_i, \\ F_2(u) &= \sum_{i=0}^{\infty} B_i. \end{aligned}$$

where A_i and B_i are the Adomian polynomials.

Also, we can use the following formula for the Adomian polynomials [12]:

$$\begin{aligned} A_n &= F_1(s_n) - \sum_{i=0}^{n-1} A_i, \\ B_n &= F_2(s_n) - \sum_{i=0}^{n-1} B_i. \end{aligned} \tag{14}$$

Where the partial sum is $s_n = \sum_{i=0}^{n} u_i(x,t)$.

2.2. *Description of the VIM and MVIM*

In the VIM [13-17, 28], we consider the following nonlinear differential equation:

$$L(u(x,t)) + N(u(x,t)) = g_1(x,t), \tag{15}$$

where L is a linear operator,N is a nonlinear operator and $g_1(x,t)$ is a known analytical function. In this case, a correction functional can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\tau)\{L(u_n(x,\tau)) + N(u_n(x,\tau)) - g_1(x,\tau)\}d\tau, \quad n \geq 0, \tag{16}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $u_n(x,\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(x,t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{17}$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq. (1), according to the VIM, we can write iteration formula (16) as follows:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + L_t^{-1}(\lambda[u_n(x,t) - F(x,t) \\ + 2\int_0^t \int_a^x (x-t)F_1(u_n(x,t))\, dt\, dx \\ + \int_0^t \int_a^x (x-t)F_2(u_n(x,t))\, dt\, dx]), \quad n \geq 0. \end{aligned} \tag{18}$$

Where,

$$L_t^{-1}(\cdot) = \int_0^t (\cdot)\, d\tau\,.$$

To find the optimal λ, we proceed as

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta L_t^{-1}\Bigg(\lambda\Bigg[u_n(x,t)+F(x,t) + 2k\int_0^t D(u_n(x,t))\,dt+a\int_0^t F_1(u_n(x,t))\,dt - 2\int_0^t F_2(u_n(x,t))\,dt-\int_0^t F_3(u_n(x,t))\,dt\Bigg]\Bigg). \tag{19}$$

From Eq. (19), the stationary conditions can be obtained as follows:

$$\lambda'=0 \quad \text{and} \quad 1+\lambda'=0.$$

Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in (18), the following iteration formula is obtained.

$$\begin{aligned} u_0(x,t)&=F(x,t),\\ u_{n+1}(x,t)&=u_n(x,t)-L_t^{-1}\\ &\times\Bigg(u_n(x,t)-F(x,t)+2\int_0^t\int_a^x(x-t)F_1(u_n(x,t))\,dt\,dx\\ &+\int_0^t\int_a^x(x-t)F_2(u_n(x,t))\,dt\,dx\Bigg),\quad n\geq 0. \end{aligned} \tag{20}$$

To obtain the approximation solution of Eq. (1), based on the MVIM [18,19], we can write the following iteration formula:

$$\begin{aligned} u_0(x,t)&=F(x,t),\\ u_{n+1}(x,t)&=u_n(x,t)-L_t^{-1}\Bigg(-2\int_0^t\int_a^x(x-t)F_1(u_n(x,t)-u_{n-1}(x,t))\,dt\,dx\\ &-\int_0^t\int_a^x(x-t)F_2(u_n(x,t)-u_{n-1}(x,t))\,dt\,dx\Bigg),\quad n\geq 0. \end{aligned} \tag{21}$$

Relations (20) and (21) will enable us to determine the components $u_n(x,t)$ recursively for $n\geq 0$.

2.3. *Description of the HAM*

Consider

$$N[u]=0,$$

where N is a nonlinear operator, $u(x,t)$ is unknown function and x is an independent variable. let $u_0(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h\neq 0$ an auxiliary parameter, $H_1(x,t)\neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$\begin{aligned}&(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)] \\ &\quad = \hat{H}[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q]. \end{aligned} \tag{22}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$.

Enforcing the homotopy (22) to be zero, i.e.,

$$\hat{H_1}[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q] = 0, \tag{23}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \tag{24}$$

When $q = 0$, the zero-order deformation Eq. (24) becomes

$$\phi(x;0) = u_0(x,t), \tag{25}$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq. (24) is equivalent to

$$\phi(x,t;1) = u(x,t). \tag{26}$$

Thus, according to (25) and (26), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [20-22, 24-27].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{27}$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0}.$$

Let the initial guess $u_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series (27) of $\phi(x,t;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{28}$$

From Eq. (27), we can write Eq. (24) as follows

$$(1-q)L[\phi(x,t,q)-u_0(x,t)] = (1-q)L\left[\sum_{m=1}^{\infty} u_m(x,t)\, q^m\right]$$
$$= q\, h\, H_1(x,t)N[\phi(x,t,q)]$$
$$\Rightarrow \quad L\left[\sum_{m=1}^{\infty} u_m(x,t)\, q^m\right] - q\, L\left[\sum_{m=1}^{\infty} u_m(x,t) q^m\right] = q\, h\, H_1(x,t)N[\phi(x,t,q)] \tag{29}$$

By differentiating (29) m times with respect to q, we obtain

$$\left\{L\left[\sum_{m=1}^{\infty} u_m(x,t)\, q^m\right] - q\, L\left[\sum_{m=1}^{\infty} u_m(x,t) q^m\right]\right\}^{(m)}$$
$$= \{q\, h\, H_1(x,t)N[\phi(x,t,q)]\}^{(m)}$$
$$= m!\; L[u_m(x,t) - u_{m-1}(x,t)]$$
$$= h\, H_1(x,t)\; m\; \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}.$$

Therefore,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t)\Re_m(u_{m-1}(x,t)), \tag{30}$$

where,

$$\Re_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{31}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq. (30) is governing the linear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by (31) for any nonlinear operator N.

To obtain the approximation solution of Eq. (1), according to HAM, let

$$N[u(x,t)] = u(x,t) - F(x,t) + 2\int_0^t \int_a^x (x-t)F_1(u(x,t))\, dt\, dx$$
$$+ \int_0^t \int_a^x (x-t)F_2(u(x,t))\, dt\, dx,$$

so,

$$\Re_m(u_{m-1}(x,t)) = u_{m-1}(x,t) - F(x,t) + 2\int_0^t \int_a^x (x-t)F_1(u(x,t))\, dt\, dx$$
$$+ \int_0^t \int_a^x (x-t)F_2(u(x,t))\, dt\, dx, \tag{32}$$

Substituting (32) into (30)

$$
\begin{aligned}
&L[u_m(x,t)-\chi_m u_{m-1}(x,t)]\\
&\quad = hH_1(x,t)[u_{m-1}(x,t)+2\int_0^t\int_a^x (x-t)F_1(u(x,t))\,dt\,dx\\
&\qquad +\int_0^t\int_a^x (x-t)F_2(u(x,t))\,dt\,dx+(1-\chi_m)F(x,t)]. \qquad (33)
\end{aligned}
$$

We take an initial guess $u_0(x,t)=F(x,t)$, an auxiliary linear operator $Lu=u$, a nonzero auxiliary parameter $h=-1$, and auxiliary function $H_1(x,t)=1$. This is substituted into (33) to give the recurrence relation

$$
\begin{aligned}
u_0(x,t) &= F(x,t),\\
u_{n+1}(x,t) &= -2\int_0^t\int_a^x (x-t)F_1(u_n(x,t))\,dt\,dx\\
&\quad -\int_0^t\int_a^x (x-t)F_2(u_n(x,t))\,dt\,dx, \quad n\geq 1. \qquad (34)
\end{aligned}
$$

If $|u_n(x,t)|<1$, then the series solution (34) convergence uniformly.

3. **Existence and convergency of iterative methods**

Theorem 3.1. *Let* $0<\alpha<1$*, then equation* (1)*, has a unique solution.*

Proof. Let u and u^* be two different solutions of (3) then

$$
\begin{aligned}
|u-u^*| &= |-2\int_0^t\int_a^x (x-t)[F_1(u(x,t))-F_1(u^*(x,t))]\,dt\,dx\\
&\quad -\int_0^t\int_a^x (x-t)[F_2(u(x,t))-F_2(u^*(x,t))]\,dt\,dx|\\
&\leq \int_0^t\int_a^x |x-t|\,|2F_1(u(x,t))-F_1(u^*(x,t))|\,dt\,dx\\
&\quad +\int_0^t\int_a^x |x-t|\,|F_2(u(x,t))-F_2(u^*(x,t))|\,dt\,dx\\
&\leq TM(b-a)(2L_1+L_2)\,|u-u^*|\\
&= \alpha|u-u^*|.
\end{aligned}
$$

From which we get $(1-\alpha)|u-u^*|\leq 0$. Since $0<\alpha<1$, then $|u-u^*|=0$. Implies $u=u^*$ and completes the proof. □

Theorem 3.2. *The series solution* $u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$ *of problem* (1) *using MADM convergence when* $0<\alpha<1$*,* $|u_1(x,t)|<\infty$*.*

Proof. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\|s_n - s_m\| &= \max_{\forall x,t} |s_n - s_m| \\
&= \max_{\forall x,t} \left| \sum_{i=m+1}^{n} u_i(x,t) \right| \\
&= \max_{\forall x,t} \left| \sum_{i=m+1}^{n} \left(-2 \int_0^t \int_a^x (x-t) A_{i-1} dt\ dx - \int_0^t \int_a^x (x-t) B_{i-1} dt\ dx \right) \right| \\
&= \max_{\forall x,t} \left| -2 \int_0^t \int_a^x (x-t) \left(\sum_{i=m}^{n-1} A_i \right) dt\ dx - \int_0^t \int_a^x \left(\sum_{i=m}^{n-1} B_i \right) dt\ dx \right|.
\end{aligned}$$

From [12], we have

$$\sum_{i=m}^{n-1} A_i = F_1(s_{n-1}) - F_1(s_{m-1}),$$

$$\sum_{i=m}^{n-1} B_i = F_2(s_{n-1}) - F_2(s_{m-1}).$$

So,

$$\begin{aligned}
\|s_n - s_m\| = \max_{\forall x,t} \bigg| &-2 \int_0^t \int_a^x (x-t)[F_1(s_{n-1}) - F_1(s_{m-1})]\ dt\ dx \\
&- \int_0^t \int_a^x (x-t)[F_2(s_{n-1}) - F_2(s_{m-1})]\ dt\ dx \bigg| \\
\leq\ & 2 \int_0^t \int_a^x (x-t)|F_1(s_{n-1}) - F_1(s_{m-1})|\ dt\ dx \\
&+ \int_0^t \int_a^x (x-t)|F_2(s_{n-1}) - F_2(s_{m-1})|\ dt\ dx \\
\leq\ & \alpha \|s_n - s_m\|.
\end{aligned}$$

Let $n = m+1$, then

$$\|s_n - s_m\| \leq \alpha \|s_m - s_{m-1}\| \leq \alpha^2 \|s_{m-1} - s_{m-2}\| \leq \ldots \leq \alpha^m \|s_1 - s_0\|.$$

We have,

$$\begin{aligned}
\|s_n - s_m\| &\leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \ldots + \|s_n - s_{n-1}\| \\
&\leq [\alpha^m + \alpha^{m+1} + \ldots + \alpha^{n-1}] \|s_1 - s_0\| \\
&\leq \alpha^m [1 + \alpha + \alpha^2 + \ldots + \alpha^{n-m-1}] \|s_1 - s_0\| \\
&\leq \alpha^m \left[\frac{1 - \alpha^{n-m}}{1 - \alpha} \right] \|u_1(x,t)\|.
\end{aligned}$$

Since $0 < \alpha < 1$, we have $(1-\alpha^{n-m}) < 1$, then

$$\|s_n - s_m\| \le \frac{\alpha^m}{1-\alpha} \max_{\forall t} |u_1(x,t)|. \tag{35}$$

But $|u_1(x,t)| < \infty$, so, as $m \to \infty$, then $\|s_n - s_m\| \to 0$. We conclude that s_n is a Cauchy sequence, therefore the series is convergence and the proof is complete. □

Theorem 3.3. *The series solution* $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ *of problem* (1) *using VIM converges when* $0 < \alpha < 1$, $0 < \beta < 1$.

Proof.

$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}\Big(\Big[u_n(x,t) - F(x,t) + 2\int_0^t \int_a^x (x-t)F_1(u_n(x,t))dt\,dx + \int_0^t \int_a^x (x-t)F_2(u_n(x,t))\,dt\,dx\Big]\Big) \tag{36}$$

$$u(x,t) = u(x,t) - L_t^{-1}\Big(\Big[u(x,t) - F(x,t) + 2\int_0^t \int_a^x (x-t)F_1(u(x,t))\,dt\,dx + \int_0^t \int_a^x (x-t)F_2(u(x,t))\,dt\,dx\Big]\Big). \tag{37}$$

By subtracting relation (36) from (37),

$$\begin{aligned} u_{n+1}(x,t) - u(x,t) = u_n(x,t) - u(x,t) - L_t^{-1}\Big(&u_n(x,t) - u(x,t) \\ &+ 2\int_0^t \int_a^x (x-t)[F_1(u_n(x,t)) - F_1(u(x,t))]dt\,dx \\ &+ \int_0^t \int_a^x (x-t)[F_2(u_n(x,t))\,dt - F_2(u(x,t))]\,dt\,dx\Big), \end{aligned}$$

if we set, $e_{n+1}(x,t) = u_{n+1}(x,t) - u_n(x,t)$, $e_n(x,t) = u_n(x,t) - u(x,t)$, $|e_n(x,t^*)| = \max_t |e_n(x,t)|$ then since e_n is a decreasing function with respect to t from the mean value theorem we can write,

$$\begin{aligned} e_{n+1}(x,t) = e_n(x,t) + L_t^{-1} \\ \times \Big(e_n(x,t) + 2\int_0^t \int_a^x (x-t)[F_1(u_n(x,t)) - F_1(u(x,t))]\,dt\,dx \\ + \int_0^t \int_a^x (x-t)[F_2(u_n(x,t)) - F_2(u(x,t))]\,dt\,dx\Big) \\ \le e_n(x,t) + L_t^{-1}[e_n(x,t) + L_t^{-1}|e_n(x,t)|TM(b-a)(2L_1+L_2)] \\ \le e_n(x,t) - Te_n(x,\eta) + TM(b-a)(2L_1+L_2)L_t^{-1}L_t^{-1}|e_n(x,t)| \\ \le (1 - T(1-\alpha)|e_n(x,t^*)|, \end{aligned}$$

where $0 \leq \eta \leq t$. Hence, $e_{n+1}(x,t) \leq \beta |e_n(x,t^*)|$.

Therefore,

$$\|e_{n+1}\| = \max_{\forall t \in J} |e_{n+1}| \leq \beta \max_{\forall t \in J} |e_n| \leq \beta \|e_n\|.$$

Since $0 < \beta < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete. □

Theorem 3.4. *If the series solution* (34) *of problem* (1) *using HAM convergent then it converges to the exact solution of the problem* (1).

Proof. We assume:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t),$$

$$\widehat{F_1}(u(x,t)) = \sum_{m=0}^{\infty} F_1(u_m(x,t)),$$

$$\widehat{F_2}(u(x,t)) = \sum_{m=0}^{\infty} F_2(u_m(x,t)).$$

where,

$$\lim_{m \to \infty} u_m(x,t) = 0.$$

We can write,

$$\sum_{m=1}^{n} [u_m(x,t) - \chi_m u_{m-1}(x,t)] = u_1 + (u_2 - u_1) + \ldots + (u_n - u_{n-1}) = u_n(x,t). \tag{38}$$

Hence, from (38),

$$\lim_{n \to \infty} u_n(x,t) = 0. \tag{39}$$

So, using (39) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = L\left[\sum_{m=1}^{\infty} [u_m(x,t) - \chi_m u_{m-1}(x,t)]\right] = 0.$$

therefore from (30), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t) \sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H_1(x,t) \neq 0$, we have

$$\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x,t)) = 0. \tag{40}$$

By substituting $\mathfrak{R}_{m-1}(u_{m-1}(x,t))$ into the relation (40) and simplifying it, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\mathfrak{R}_{m-1}(u_{m-1}(x,t)) &= \sum_{m=1}^{\infty}\Big[u_{m-1}(x,t)+2\int_0^t\int_a^x(x-t)F_1(u_{m-1}(x,t))\,dt\,dx\\ &\quad+\int_0^t\int_a^x(x-t)F_2(u_{m-1}(x,t))\,dt\,dx+(1-\chi_m)F(x,t)\Big]\\ &= u(x,t)-F(x,t)+2\int_0^t\int_a^x(x-t)\widehat{F_1}(u(x,t))\,dt\,dx\\ &\quad+\int_0^t\int_a^x(x-t)\widehat{F_2}(u(x,t))\,dt\,dx. \end{aligned} \tag{41}$$

From (40) and (41), we have

$$\begin{aligned}u(x,t)=F(x,t)&-2\int_0^t\int_a^x(x-t)\widehat{F_1}(u(x,t))\,dt\,dx\\ &-\int_0^t\int_a^x(x-t)\widehat{F_2}(u(x,t))\,dt\,dx,\end{aligned}$$

therefore, $u(x,t)$ must be the exact solution. □

4. Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM and HAM. The program has been provided with Mathematica 6 according to the following algorithm. In this algorithm ε is a given positive value.

Algorithm

Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relation (10) for ADM, (13) for MADM and (34) for HAM.
Step 3. If $|u_{n+1}-u_n|<\varepsilon$ then go to *Step* 4,
else $n \leftarrow n+1$ and go to *Step* 2.
Step 4. Print $u(x,t)=\sum_{i=0}^{n}u_i(x,t)$ as the approximate of the exact solution.

Algorithm 2

Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations (20) for VIM and (21) for MVIM.
Step 3. If $|u_{n+1}-u_n|<\varepsilon$ then go to *Step* 4,
else $n \leftarrow n+1$ and go to *Step* 2.
Step 4. Print $u_n(x,t)$ as the approximate of the exact solution.

Lemma 4.1. *The computational complexity of the ADM is $O(n^3)$, MADM is $O(n^3)$, VIM is $O(13n)$, MVIM is $O(10n)$ and HAM is $O(8n)$.*

Proof. The number of computations including division, production, sum and subtraction.

ADM:

In *Step* 2,
$A_n, B_n : 2n^2 + 10n + 8.$
$u_0 : 4.$
$u_1 : 24.$
$\vdots$
$u_{n+1} : 24.$

In *Step* 4, the total number of the computations is equal to $\sum_{i=0}^{n+1} u_i(x,t) = O(n^3)$.

MADM:

In *Step* 2,
$A_n, B_n : 2n^2 + 10n + 8.$
$u_0 : 4.$
$u_1 : 25.$
$u_2 : 24.$
$\vdots$
$u_{n+1} : 24.$

In *Step* 4, the total number of the computations is equal to $u_0 + u_1 + \sum_{i=2}^{n+1} u_i(x,t) = O(n^3)$.

VIM:

In *Step* 2,
$u_0 : 4.$
$u_1 : 13.$
$\vdots$
$u_{n+1} : 13.$

In *Step* 4, the total number of the computations is equal to $\sum_{i=0}^{n+1} u_i(x,t) = O(13n)$.

MVIM:

In *Step* 2,
$u_0 : 4.$
$u_1 : 10.$

$\vdots$

$u_{n+1}: 10.$

In *Step* 4, the total number of the computations is equal to $\sum_{i=0}^{n+1} u_i(x,t) = O(10n)$.

HAM:

In *Step* 2,

$u_0: 4.$

$u_1: 8.$

$\vdots$

$u_{n+1}: 8.$

In *Step* 4, the total number of the computations is equal to $\sum_{i=0}^{n+1} u_i(x,t) = O(8n)$. $\square$

By comparing the results of computational complexity, we see that the number of computations in HAM is less than the number of computations in ADM, MADM, VIM and MVIM.

Example 4.2. Consider the hunter-Saxeton equation as follows:

$$u_t + 2u_x - u_{xxt} + uu_x = 2u_x u_x x + uu_x xx\,.$$

With the exact solution is $u(x,t) = e^{x-3t}$ and $\epsilon = 10^{-3}$.

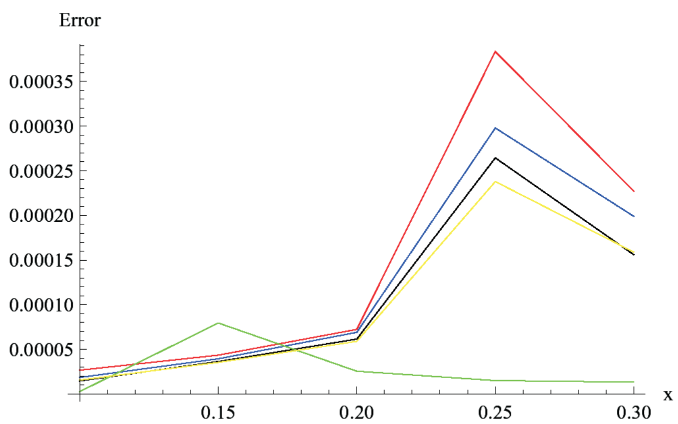

Figure 1. The comparison between the results of the methods in the example for $t = 0.35$
(The comparison between the results of the methods in the Example 4.1, Red = Error ADM($n = 22$)4, Blue = Error MADM($n = 19$), Black = Error VIM($n = 15$), Yellow = Error MVIM($n = 12$), Green = Error HAM($n = 9$))

Figure 1, shows that, approximate solution of the Hunter-Saxton equation is convergence with 7 iterations by using the HAM. By comparing the results of

Figure 1, we can observe that the HAM is more rapid convergence than the ADM, MADM, VIM and MVIM.

5. **Conclusion**

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate analytical solution of the Hunter-Saxton equation. For this purpose, we showed that the HAM is more rapid convergence than the ADM, MADM, VIM and MVIM. Also, the number of computations in HAM is less than the number of computations in ADM, MADM, VIM and MVIM.

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