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Analytical Pricing of An Insurance Embedded Option: Alternative Formulas and Gaussian Approximation

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Abstract. Analytical pricing of a double-trigger option with the *Black-Scholes-Vasicek* (*BSV*) state price deflator is considered. In the context of market-consistent valuation of insurance liabilities, the option appears as embedded option of index-linked endowment policies that provide combined protection against inflation and a minimum interest rate guarantee by death. A first analytical pricing formula in terms of the standard bivariate normal distribution is derived. Then, using alternatively the canonical BSV deflator, a second integral representation is derived. Based on an elementary Gaussian integral in three variables a third integral decomposition is obtained and approximated by closed-form Gaussian expressions using a simple approximation by Lin of the normal tail probability integral. Similarly to the invariance of the Black-Scholes and Margrabe formulas with respect to the market prices of the risk factors, two of the alternative double-trigger option pricing formulas and the proposed Gaussian approximation also share this property. A numerical example rounds up the analysis by showing accuracy of the Gaussian approximation within some few negligible basis points.

1. **Introduction**

The first rigorous mathematical derivation of the Black-Scholes formula by Merton (1973) relies on a dynamic delta-hedge portfolio and a risk-free argument of no-arbitrage. Later on Duffie (1992) introduced state-price deflators, which led to the insight that deflator based market valuation using the real-world probability measure is equivalent to market valuation based on a risk-neutral martingale measure.

The *Black-Scholes-Vasicek* (*BSV*) *deflator* associated to a multiple risk economy has been introduced in H rlimann (2011a). Besides a new elementary proof of the (slightly extended) Black-Scholes formula the BSV deflator provides a validation of it in a financial market with multiple economic risks. The same holds true for

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Margrabe's formula for a European option to exchange one risky asset for another one.

We pursue the previous analysis and offer as a new application the analytical pricing of a double-trigger option, which appears in H rlimann (2011b) as embedded option in the context of replication of life insurance liabilities. This option provides a dual guarantee in the sense of an inflation indexed protection and a minimum interest rate guarantee by death. The theoretical foundation of insurance replication is rooted in the topic of market-consistent actuarial valuation, for which we refer to W thrich *et al.* (2010) for a thorough introduction. We also like to mention that this actuarial financial subject is basic to modern risk management and has relevance in the regulatory compliance of solvency systems.

A more detailed account follows. Section 2 recalls the modelling framework of a multiple risk economy, the Black-Scholes-Vasicek (BSV) deflator and another alternative canonical version of it. Section 3 derives an analytical pricing formula for the considered insurance embedded option. Similarly to the Black-Scholes and Margrabe formulas, the double-trigger option pricing formula is shown to be invariant with respect to the market prices of the risk factors. This property should hold and expresses the fact that prices should be equal, whether one works with state-price deflators or an equivalent risk-neutral measure. Based on the canonical BSV deflator, we derive in Section 4 an alternative integral representation for pricing the considered double-trigger option. Section 5 offers a more detailed analysis of the latter and proposes an analytical Gaussian approximation of the involved normal tail probability integrals, which is based on a simple approximation by Lin (1989) of the normal tail probability function. Finally, Section 6 is devoted to numerical computation. The analytical formula of Section 3 involves bivariate normal tail probability integrals, which can be evaluated using software based numerical integration and/or appropriate convergent infinite series expansion. Software based numerical integration can also be used to compute the alternative formula in Section 4. A numerical example shows that the approximate Gaussian prices are accurate within some few negligible basis points. A numerical example rounds up the analysis and explains the business use of the considered insurance embedded option.

2. **The Black-Scholes-Vasicek deflator for multiple economic risks**

Recall the construction of *Black-Scholes-Vasicek* (*BSV*) *deflator* introduced in H rlimann (2011a). Consider a multiple risk economy with $m \geq 1$ risky assets, whose real-world prices follow lognormal distributions. Given the current prices at time $s \geq 0$ the future prices of these risky assets at time $t > s$ are described by

$$S_t^{(k)} = S_s^{(k)} \exp\left\{(m_k(s,t) - \tfrac{1}{2}\sigma_k^2)(t-s) + v_k\sqrt{t-s}\cdot W_{t-s}^{(k)}\right\},$$
$$0 \leq s < t,\ k = 1,\dots,m, \quad (2.1)$$

where the $W_{t-s}^{(k)}$'s are *correlated* standard Wiener processes such that $E[dW_{t-s}^{(i)} dW_{t-s}^{(j)}] = \rho_{ij}\, dt$. For simplicity we assume that the correlation matrix $C = (\rho_{ij})$ is positive semi-definite with non-vanishing determinant. The quantities $m_k(s,t)$ and $v_k(s,t)$ are interpreted as *mean* and *standard deviation* per time unit of the return differences on these risky assets, and σ_k is a *volatility* parameter. The representation (2.1) includes two of the most popular return models:

Black-Scholes return model. $dr_t^{(k)} = \mu_k\, dt + \sigma_k dW_t^{(k)}$

$$m_k(s,t) = \mu_k, \quad v_k(s,t) = \sigma_k, \qquad 0 \le s < t. \tag{2.2}$$

Vasicek (Ornstein-Uhlenbeck) return model. $dr_t^{(k)} = a_k\left(b_k - r_t^{(k)}\right) dt + \sigma_k dW_t^{(k)}$

$$m_k(s,t) = \frac{(b_k - r_s^{(k)})(1 - e^{-a_k(t-s)})}{t-s}, \quad v_k(s,t) = \sigma_k \sqrt{\frac{1 - e^{-2a_k(t-s)}}{2a_k(t-s)}}. \tag{2.3}$$

The economic model contains also a deterministic money market account with value

$$M_t = M_s \exp\left((t-s)R(s,t)\right), \quad 0 \le s < 1, \tag{2.4}$$

where $R(s,t)$, $0 \le s < t$, are the deterministic continuously-compounded spot rates. The related price at time s of a zero-coupon bond paying one unit of money at time t is denoted by

$$P(s,t) = \exp(-(t-s)R(s,t)), \quad 0 \le s < t. \tag{2.5}$$

The *BSV deflator* has the same form as the price processes in (2.1), i.e.

$$D_t^{(m)} = D_s^{(m)} \exp\left\{\alpha^{(m)}(s,t)(t-s) - \beta^{(m)}(s,t)^T \sqrt{t-s} \cdot W_{t-s}\right\}, \quad 0 \le s < t, \tag{2.6}$$

for some parametric function $\alpha^{(m)}(s,t)$ and vectors $\beta^{(m)}(s,t) = (\beta_1^{(m)}(s,t), \ldots, \beta_m^{(m)}(s-t))^T, W_{t-s} = (W_{t-s}^{(l)}, \ldots, W_{t-s}^{(m)})^T$. To define a state-price deflator the stochastic processes (2.1) and (2.6) must satisfy martingale conditions

$$\begin{aligned} &E_s[D_t^{(m)}] = D_s^{(m)} P(s,t) = D_s^{(m)} e^{-(t-s)R(s,t)}, \\ &E_s[D_t^{(m)} S_t^{(k)}] = D_s^{(m)} S_s^{(k)}, \quad 0 \le s < 1, k = 1, \ldots, m. \end{aligned} \tag{2.7}$$

Theorem 2.1 (BSV deflator). *Given is a financial market with a risk-free money market account and m risky assets that have log-normal real-world prices* (2.1). *Assume a non-singular positive semi-definite correlation matrix C. Then, the BSV deflator* (2.6) *is determined by*

$$\begin{aligned} D_t^{(m)} = D_s^{(m)} \exp\Bigg\{ &-R(s,t)(t-s) - \tfrac{1}{2}\sum_{j=1}^{m} \beta_j^{(m)}(s,t)^2 (t-s) \\ &- \sum_{1 \le i < j \le m} \rho_{ij} \beta_i^{(m)}(s,t) \beta_j^{(m)}(s,t)(t-s) - \sum_{j=1}^{(m)} \beta_j^{(m)}(s,t) \sqrt{t-s} \cdot W_{t-s}^{(j)} \Bigg\}, \end{aligned} \tag{2.8}$$

$$\beta_j(m)(s,t) = \det(C)^{-1} \cdot \sum_{i=1}^{m} (-1)^{i+j} \det(C_j^{(i)}) \cdot \lambda_i(s,t),$$

$$\lambda_i(s,t) = \frac{m_i(s,t) - R(s,t) - \frac{1}{2}(\sigma_i^2 - \nu_i^2(s,t))}{\nu_i(s,t)}, \quad 0 \le s < t, \qquad (2.9)$$

where $C_j^{(i)}$ *is the matrix formed by deleting the i-th row and j-th column of* C*. The quantity* $\lambda_i(s,t)$ *is called* market price of the i-th risky asset.

Proof. The martingale conditions (2.7) are equivalent with a system of linear equations, which is solved using Cramer's rule (see H rlimann (2011a), Proposition 2). □

Example 2.1. For a single economic risk one has $\beta_1^{(1)}(s,t) = \lambda_1(s,t)$. For $m = 2$ one has

$$\beta_1^{(2)}(s,t) = \frac{\lambda_1(s,t) - \rho_{12}\lambda_2(s,t)}{1-\rho_{12}^2}, \quad \beta_2^{(2)}(s,t) = \frac{\lambda_2(s,t) - \rho_{12}\lambda_1(s,t)}{1-\rho_{12}^2}. \quad (2.10)$$

In some calculations it appears useful to switch to the "diagonal version" of the representation (2.1). For this let be given the spectral decomposition of the correlation matrix $C = B \cdot B^T$ with B an $m \times m$ matrix. If $b_i = (b_{i1}, \dots, b_{im})$ denotes the i-th row vector of B, then the decomposition reads $b_i \cdot b_j^T = \rho_{ij}$. Now, instead (2.1) it might be convenient to implement the equivalent diagonal representation

$$S_t^{(k)} = S_s^{(k)} \exp\left\{ \left(m_k(s,t) - \tfrac{1}{2}\sigma_k^2\right)(t-s) + \nu_k(s,t)\sqrt{t-s} \cdot b_k \cdot W_{t-s}^{\perp} \right\},$$
$$0 \le s < t,\ k = 1, \dots, m, \quad (2.11)$$

with a vector $W_{t-s}^{\perp} = (W_{t-s}^{(1)}, \dots, W_{t-s}^{(m)})^T$ of *independent* standard Wiener processes. The resulting *canonical BSV deflator* has the form (2.6) with independent standard Wiener processes.

Theorem 2.2 (Canonical BSV deflator). *Given is a financial market with a risk-free money market account and m risky assets that have log-normal real-world prices* (2.11). *Assume a non-singular positive semi-definite correlation matrix* $C = B \cdot B^T$*. Then, the canonical BSV deflator is determined by*

$$D_t^{(m)} = D_s^{(m)} \exp\left\{ -R(s,t)(t-s) + \tfrac{1}{2}\beta^{(m)}(s,t)^T \beta^{(m)}(s,t) - \beta^{(m)}(s,t)^T \sqrt{t-s} \cdot W_{t-s}^{\perp} \right\},$$
$$0 \le s < t, \quad (2.12)$$

$$\beta_j^{(m)}(s,t) = \det(B)^{-1} \cdot \sum_{i=1}^{m} (-1)^{i+1} \det(B_j^{(i)}) \cdot \lambda_i(s,t),$$

$$\lambda_i(s,t) = \frac{m_i(s,t) - R(s,t) - \frac{1}{2}(\sigma_i^2 - \nu_i^2(s,t))}{\nu_i(s,t)}, \quad 0 \le s < t, \qquad (2.13)$$

where $B_j^{(i)}$ *is the matrix formed by deleting the i-th row and j-th column of B.*

Proof. The martingale conditions (2.7) are equivalent with the system of equations

$$R(s,t)+\alpha^{(m)}(s,t)+\tfrac{1}{2}\sum_{j=1}^{m}\beta_j^{(m)}(s,t)^2=0, \quad 0\le s<t, \tag{2.14}$$

$$\alpha^{(m)}(s,t)+m_k(s,t)-\tfrac{1}{2}\sigma_k^2+\tfrac{1}{2}\sum_{j=1}^{m}(\nu_k(s,t)b_{kj}-\beta_j^{(m)}(s,t))^2=0,$$
$$0\le s<t, k=1,\dots,m. \tag{2.15}$$

Insert (2.14) into (2.15) using the definition of $\lambda_i(s,t)$ and the property $\sum_{j=1}^{m} b_{kj}^2=1$ to obtain the matrix equation $B\cdot\beta^{(m)}(s,t)=\lambda(s,t)$, where $\lambda(s,t)=(\lambda_1(s,t),\dots,\lambda_m(s,t))^T$. The formula in (2.13) follows by Cramer's rule. The expression $\alpha^{(m)}(s,t)$ follows from (2.14). □

Example 2.2. The bivariate case $m=2$ is relevant in Section 4.
Let $\Lambda=\begin{pmatrix}1+\rho & 0\\ 0 & 1-\rho\end{pmatrix}$ be the matrix of eigenvalues of the correlation matrix $C=\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$, and let $S=\begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{pmatrix}$ be the corresponding matrix of eigenvectors. The matrix $B=S\cdot\sqrt{\Lambda}\begin{pmatrix}\sqrt{\frac{1+\rho}{2}} & \sqrt{\frac{1-\rho}{2}}\\ \sqrt{\frac{1+\rho}{2}} & -\sqrt{\frac{1-\rho}{2}}\end{pmatrix}$ satisfies the property $C=B\cdot B^T$. One obtains the canonical BSV deflator

$$\begin{aligned}D_t^{(2)}=D_s^{(2)}\exp\{&-R(s,t)(t-s)-\tfrac{1}{2}(\beta_1^{(2)}(s,t)^2+\beta_2^{(2)}(s,t)^2)(t-s)\\&-\beta_1^{(2)}(s,t)\sqrt{t-s}\cdot W_{t-s}^{(1)}-\beta_2^{(2)}(s,t)\sqrt{t-s}\cdot W_{t-s}^{(2)}\}, \quad 0\le s<t,\end{aligned}$$

$$\beta_1^{(2)}(s,t)=\frac{\lambda_1(s,t)+\lambda_2(s,t)}{\sqrt{2(1+\rho)}}, \quad \beta_2^{(2)}(s,t)=\frac{\lambda_1(s,t)-\lambda_2(s,t)}{\sqrt{2(1-\rho)}}. \tag{2.16}$$

3. A bivariate normal representation

Consider an equity-linked endowment policy with a minimum interest guarantee by death. Such a guarantee can be replicated using European put options on the equity index (e.g. (W thrich *et al.* (2010), Example 3.7)). Let n be the contract term, $S_t^{(1)}$ the price of one unit of investment in the equity index, whose initial value is $S_0^{(1)}$, $r^k=(1+i)^k$, $k=1,\dots,n$, with i the minimum guaranteed interest rate. Upon death at time k the benefit payment decomposes as $\max(S_k^{(1)},r^k)=S_k^{(1)}+(r^k-S_k^{(1)})_+$. Therefore replication of the minimum interest rate guarantee requires a series of put options with strike time k and strike price r^k. Similarly, for an equity linked inflation protected endowment policy, the death benefit at time k decomposes as $\max(S_k^{(1)},S_k^{(2)})=S_k^{(1)}+(S_k^{(2)}-S_k^{(1)})_+$, where $S_t^{(2)}$ represents the inflation index with initial value $S_0^{(2)}=1$. In this situation, the embedded option is a European exchange option with strike time

k to exchange the inflation index for the equity index (see H rlimann (2011b) for further details). Combine now the inflation protection and a guaranteed minimum death benefit, say T_k at time k (generalizing $T_k = r^k$). The required double-triggered insurance embedded option at time k has the contingent financial payoff $(\max\{S_k^{(2)}, T_k\} - S_k^{(1)})_+$.

From now on we make use of simplifying notations. We fix the time frame $0 \le s < t$ and omit dependence upon s, t where appropriate. Let $m = 2$ in the Example 2.1. We set $\beta_i = -\beta_i^{(2)}(s,t)\sqrt{t-s}$, $\upsilon_i = v_i(s,t)\sqrt{t-s}$, $\lambda_i = \lambda_i(s,t)\sqrt{t-s}$, $i = 1,2$, $r = R(s,t)(t-s)$, $\rho = \rho_{12}$, $\beta^2 = \beta_1^2 + 2\rho\beta_1\beta_2 + \beta_2^{(2)}$, $\mu_i = \ln S_s^{(i)} + r + \lambda_i\upsilon_i - \frac{1}{2}\upsilon_i^{(2)}$, $i = 1,2$. As usual $\Phi(x)$ is the the standard normal distribution, $\overline{\Phi} = 1 - \Phi(x)$ and $\varphi(x) = \Phi'(x)$ is the density. Similarly, the bivariate standard normal density is defined and denoted by

$$\varphi_2(x,y;\rho) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{ -\frac{1}{2(1-\rho^2)}(x^2 - 2pxy + y^2)\right\}.$$

The bivariate standard normal distribution and the survival function are defined and denoted by

$$\Phi_2(x,y;\rho) = \int_{-\infty}^{x}\int_{-\infty}^{y} \varphi_2(u,v;\rho)dudv, \quad \overline{\Phi}_2(x,y;\rho) = \int_{x}^{\infty}\int_{y}^{\infty} \varphi_2(u,v;\rho)dudv.$$

From (2.1)and Theorem 2.1 one knows that the standardized random vector of real-world prices

$$(Z_1, Z_2) = \left(\frac{\ln S_t^{(1)} - \mu_i}{\upsilon_1}, \frac{\ln S_t^{(2)} - \mu_2}{\upsilon_2}\right)$$

has a standard bivariate normal distribution, and the BSV deflator reads (in simplified notation)

$$D_t^{(2)} = D_s^{(2)} \exp\{-r - \tfrac{1}{2}\beta^2 + \beta_1 Z_1 + \beta_2 Z_2\}, \quad 0 \le s < t.$$

Theorem 3.1. *Given is the bivariate BSV deflator of* Example 2.1. *The market value at time $s \ge 0$ of an European double-triggered embedded option on the risky assets $S_t^{(i)}$, $i = 1,2$, with strike time $t > s$ and financial payoff $G_t^{(1,2)} = (\max\{S_t^{(1)}, K\} - S_t^{(2)})_+$ is given by*

$$\begin{aligned} E_s[D_t^{(2)} G_t^{(1,2)}] = D_s^{(2)} \cdot \{ & S_s^{(1)} \cdot R_1 - S_s^{(2)} \cdot R_2 - K \cdot P(S,T) \cdot R_3 \\ & + K \cdot P(s,t) \cdot \overline{\Phi}(d_2^{(2)}(s,t)) - S_s^{(2)} \cdot \overline{\Phi}(d_1^{(2)}(s,t))\} \end{aligned} \tag{3.1}$$

with the analytical formulas

$$\begin{aligned} R_1 &= \overline{\Phi}(-d_1^{(1)}(s,t)) - \overline{\Phi}_2(d(s,t), -d_1^{(1)}(s,t); (\rho v_2(s,t) - v_1(s,t))/v(s,t)), \\ R_2 &= \overline{\Phi}_2(-d_2^{(1)}(s,t) - \rho v_2(s,t)\sqrt{t-s}, d_1^{(2)}(s,t); \rho) \\ &\quad - \overline{\Phi}_2(-d(s,t), -d_2^{(1)}(s,t) - \rho v_2(s,t)\sqrt{t-s}; (\rho v_2(s,t) - v_1(s,t))/v(s,t)), \\ R_3 &= \overline{\Phi}_2(d_2^{(2)}(S,t) - d_2^{(1)}(s,t); -\rho) \end{aligned} \tag{3.2}$$

and the notations

$$
\begin{aligned}
d_1^{(k)}(s,t) &= \frac{\ln(S_s^{(k)}/K) + (R(s,t) + \frac{1}{2}v_k^2(s,t))(t-s)}{v_k(s,t)\sqrt{t-s}}, \\
d_2^{(k)}(s,t) &= d_1^{(k)}(s,t) - v_k(s,t)\sqrt{t-s}, \\
d(s,t) &= \frac{\ln(S_s^{(1)}/S_s^{(2)}) + \frac{1}{2}v^2(s,t)(t-s)}{v(s,t)\sqrt{t-s}}, \\
v(s,t)^2 &= v_1(s,t)^2 - 2\rho v_1(s,t)v_2(s,t) + v_2(s,t)^2 .
\end{aligned} \tag{3.3}
$$

Remark 3.1. The defining parameter and functions in (3.2)-(3.3) immediately imply that the double-trigger option pricing formula is invariant with respect to the market prices in (2.9).

Proof. First of all, one notes that the financial payoff can be rewritten as

$$
\begin{aligned}
G_t^{(1,2)} &= (S_t^{(1)} - S_t^{(2)})_+ \cdot 1\{S_t^{(1)} > K\} + (K - S_t^{(2)})_+ \cdot 1\{S_t^{(1)} \le K\} \\
&= [(S_t^{(1)} - S_t^{(2)})_+ - (K - S_t^{(2)})_+] \cdot 1\{S_t^{(1)} > K\} + (K - S_t^{(2)})_+ \\
&= (S_t^{(1)} - S_t^{(2)})_+ \cdot 1\{S_t^{(1)} > K\} \cdot 1\{S_t^{(2)} > K\} + (S_t^{(1)} - K) \cdot 1\{S_t^{(1)} > K\} \\
&\quad \cdot 1\{S_t^{(2)} \le K\} + (K - S_t^{(2)})_+ .
\end{aligned}
$$

Clearly, the last term is the financial payoff of a classical put option, whose price in the multiple risk economy is described by the Black-Scholes formula (Theorem 1 in H rlimann (2011a)). It remains to calculate the expected values

$$
\begin{aligned}
E_1 &= 1/D_s^{(2)} \cdot E_s[D_t^{(2)} \cdot (S_t^{(1)} - S_t^{(2)})_+ \cdot 1\{S_t^{(1)} > K\} \cdot 1\{S_t^{(2)} > K\}], \\
E_2 &= 1/D_s^{(2)} \cdot E_s[D_t^{(2)} \cdot (S_t^{(1)} - K) \cdot 1\{S_t^{(1)} > K\} \cdot 1\{S_t^{(2)} \le K\}] .
\end{aligned}
$$

Making use of the further decompositions

$$
\begin{aligned}
E_1 &= I_1 - I_2 , \\
I_k &= 1/D_s^{(2)} \cdot E_s[D_t^{(2)} \cdot S_t^{(k)} \cdot 1\{S_t^{(1)} > K\} \cdot 1\{S_t^{(2)} > K\} \cdot 1\{S_t^{(1)} \ge S_t^{(2)}\}], \quad k = 1,2, \\
E_2 &= M_1 - M_2 , \\
M_1 &= 1/D_s^{(2)} \cdot E_s[D_t^{(2)} \cdot S_t^{(1)} \cdot 1\{S_t^{(1)} > K\} \cdot 1\{S_t^{(2)} \le K\}], \\
M_2 &= 1/D_s^{(2)} \cdot E_s[D_t^{(2)} \cdot K \cdot 1\{S_t^{(1)} > K\} \cdot 1\{S_t^{(2)} \le K\}],
\end{aligned}
$$

the evaluation is reduced to the calculation in three steps of the terms $I_1 + M_1$, I_2, M_2. Throughout we use that

$$
\begin{aligned}
S_t^{(k)} > K \qquad &\Leftrightarrow \qquad Z_k > c_k = \frac{\ln K - \mu_k}{\upsilon_k}, \quad k = 1,2, \\
S_t^{(1)} \ge S_t^{(2)} \qquad &\Leftrightarrow \qquad Z_1 \ge a + bZ_2, \qquad a = \frac{\mu_2 - \mu_1}{\upsilon_1},\ b = \frac{\upsilon_2}{\upsilon_1},
\end{aligned}
$$

and set for simplification $d_1^{(k)} = d_1^{(k)}(s,t)$, $d_2^{(k)} = d_2^{(k)}(s,t)$, $k = 1,2$, $d = d(s,t)$, $\upsilon^2 = v(s,t)^2(t-s)$.

***Step* 1:** Calculation of $I_1 + M_1$.

For a non-zero contribution to I_1, resp. M_1, one requires $Z_1 > c_1$, $c_2 < Z_2 \le (Z_1 - a)/b$, resp. $Z_1 > c_1$, $Z_2 \le c_2$. It follows that

$$I_1 + M_1 = S_s^{(1)} \cdot \int_{c_1}^{\infty} \int_{-\infty}^{(x-a)/b} \exp\big(\lambda_1 \upsilon_1 - \tfrac{1}{2}\upsilon_1^2 - \tfrac{1}{2}\beta^2 + (\beta_1 + \upsilon_1)x + \beta_2 y\big) \cdot \varphi_2(x, y; \rho)\, dy\, dx$$

Making use of Lemma A.1 in the Appendix one obtains that

$$I_1 + M_1 = S_s^{(1)} \cdot \int_{c_1}^{\infty} \exp(\lambda_1 \upsilon_1 - \tfrac{1}{2}\upsilon_1^2 - \tfrac{1}{2}\beta_2^2 + \tfrac{1}{2}\beta_2^2 + (\beta_1 + \upsilon_1)x) \cdot \varphi(x - \rho\beta_2) \cdot \Phi\left(\frac{\frac{(x-a)}{b} - \rho x - (1-\rho^2)\beta_2}{\sqrt{1-\rho^2}}\right) dx\,.$$

Now, note that

$$\begin{aligned} &\sqrt{2\pi} \cdot \exp((\beta_1 + \upsilon_1)x) \cdot \varphi(x - \rho\beta_2) \\ &\quad = \exp\big(-\tfrac{1}{2}(x - \upsilon_1 - \beta_1 - \rho\beta_2)^2 + \tfrac{1}{2}(\beta_1 + \upsilon_1)^2 + \rho(\beta_1 + \upsilon_1)\beta_2\big) \end{aligned}$$

and use that $\beta_1 + \rho\beta_2 = -\lambda_1$ and $\beta^2 = \beta_1^2 + 2\rho\beta_1\beta_2 + \beta_2^2$ to see that

$$I_1 + M_1 = S_s^{(1)} \cdot \int_{c_1}^{\infty} \varphi(x - \upsilon_1 + \lambda_1) \cdot \Phi\left(\frac{\frac{(1-\rho b)x - a}{b} - (1-\rho^2)\beta_2}{\sqrt{1-\rho^2}}\right) dx\,.$$

Making the change of variable $t = x - \upsilon_1 + \lambda_1$ and using the relationships

$$\begin{aligned} &c_1 - \upsilon_l + \lambda_1 = -d_1^{(1)}, \quad (1-\rho^2)\beta_2 = \rho\lambda_1 - \lambda_2, \quad 1/b = \upsilon_1/\upsilon_2, \\ &a\upsilon_1 = \mu_2 - \mu_1 = \ln(S_s^{(1)}/S_s^{(2)}) + \lambda_2\upsilon_2 - \lambda_1\upsilon_1 - \tfrac{1}{2}(\upsilon_2^2 - \upsilon_1^2) \end{aligned}$$

one sees further that

$$I_1 + M_1 = S_s^{(1)} \cdot \int_{-d_1^{(1)}}^{\infty} \varphi(t) \cdot \Phi\left(\frac{(\upsilon_1 - \rho\upsilon_2)t + d\upsilon}{\upsilon_2\sqrt{1-\rho^2}}\right) dt\,.$$

An application of formula (A.2) in the Appendix shows the desired result $I_1 + M_1 = S_s^{(1)} \cdot R_1$.

***Step* 2:** Calculation of I_2.

Proceeding similarly to Step 1 one sees that

$$I_2 = S_s^{(2)} \cdot \int_{c_1}^{\infty} \int_{c_2}^{(x-a)/b} \exp\big(\lambda_2\upsilon_2 - \tfrac{1}{2}\upsilon_2^2 - \tfrac{1}{2}\beta^2 + \beta_1 x + (\beta_2 + \upsilon_2)y\big)\varphi_2(x, y; \rho) dy\, dx.$$

It appears useful to calculate the double integral as difference of two double integrals $\int_{c_1}^{\infty}\int_{c_2}^{\infty}$ and $\int_{c_1}^{\infty}\int_{(x-a)/b}^{\infty}$ leading to the corresponding difference denoted

$I_2 = J_1 - J_2$. To obtain an expression for J_1 one notes that (use the relationships $\beta_1 + \rho\beta_2 = -\lambda_1$, $\rho\beta_1 + \beta_2 = -\lambda_2$)

$$\sqrt{2\pi(1-\rho^2)} \cdot \exp(\beta_1 x + (\beta_2 + v_2)y) \cdot \varphi_2(x, y; \rho)$$
$$= \exp\left(-\frac{1}{2(1-\rho^2)} \left[(x - \rho y - (1-\rho^2)\beta_1)^2 + (1-\rho^2)(y - v_2 + \lambda_2)^2 \right] + \tfrac{1}{2}(v_2^2 + \beta^2 - 2\lambda_2 v_2) \right).$$

With the transformation of variables $s = y - v_2 + \lambda_2$, $t = x - \rho v_2 + \lambda_1$, and the relationships $c_1 - \rho v_2 + \lambda_1 = -d_2^{(1)} - \rho v_2$, $c_2 - \rho v_2 + \lambda_2 = -d_1^{(2)}$, one obtains

$$J_1 = S_s^{(2)} \cdot \int_{-d_2^{(1)}-\rho}^{\infty} \int_{v_2 - d_1^{(2)}}^{\infty} \varphi_2(t.s; \rho) ds\, dt = \overline{\Phi}_2(-d_2^{(1)} - \rho v_2, -d_1^{(2)}; \rho).$$

To calculate J_2 one proceeds similarly to *Step* 1. An application of Lemma A.1 yields

$$J_2 = S_s^{(2)} \cdot \int_{c_1}^{\infty} \left\{ \exp(\lambda_2 v_2 - \tfrac{1}{2}v_2^2 - \tfrac{1}{2}\beta^2 + \tfrac{1}{2}(\beta_2 + v_2)^2 + \beta_1 x) \cdot \varphi(x - \rho(\beta_2 + v_2)) \cdot \overline{\Phi}\left(\frac{\frac{x-a}{b} - \rho x - (1-\rho^2)(\beta_2 + v_2)}{\sqrt{1-\rho^2}} \right) \right\} dx.$$

Noting further that

$$\sqrt{2\pi} \cdot \exp(\beta_1 x) \cdot \varphi(x - \rho(\beta_2 + v_2)) = \exp(-\tfrac{1}{2}(x - \rho(\beta_2 + v_2) - \beta_1)^2 + \tfrac{1}{2}\beta_1^2 + \rho(\beta_2 + v_2)\beta_1)$$

and using that $\beta_1 + \rho\beta_2 = -\lambda_1$, $\rho\beta_1 + \beta_2 = -\lambda_2$ and $\beta^2 = \beta_1^2 + 2\rho\beta_1\beta_2 + \beta_2^2$ one obtains

$$J_2 = S_s^{(2)} \cdot \int_{c_1 - \rho v_2 + \lambda_1}^{\infty} \varphi(x - \rho v_2 + \lambda_1) \cdot \overline{\Phi}\left(\frac{\frac{(1-\rho b)x - a}{b} - (1-\rho^2)(\beta_2 + v_2)}{\sqrt{1-\rho^2}} \right) dx.$$

With the change of variable $t = x - \rho v_2 + \lambda_1$, the relationships

$$c_1 - \rho v_2 + \lambda_1 = -d_2^{(1)} - \rho v_2, \quad (1-\rho^2)\beta_2 = \rho\lambda_1 - \lambda_2, \quad 1/b = v_1/v_2,$$
$$a v_1 = \mu_2 - \mu_1 = \ln(S_s^{(1)}/S_s^{(2)}) + \lambda_2 v_2 - \lambda_1 v_1 - \tfrac{1}{2}(v_2^2 - v_1^2)$$

and an application of formula (A.2), one gets

$$J_2 = S_s^{(2)} \cdot \int_{-d_2^{(1)} - \rho v_2}^{\infty} \varphi(t) \cdot \overline{\Phi}\left(\frac{(v_1 - \rho v_2)t - dv}{v_2\sqrt{1-\rho^2}} \right) dt$$
$$= \overline{\Phi}_2(-d, -d_2^{(1)} - \rho v_2; (\rho v_2 - v_1)/v).$$

Together this yields the result $I_2 = J_1 - J_2 = S_s^{(2)} \cdot R_2$.

***Step* 3:** Calculation of M_2.

A non-zero contribution to M_2 is given when $Z_1 > c_1$, $Z_2 \le c_2$, hence

$$M_2 = Ke^{-r} \int_{c_1}^{\infty} \int_{-\infty}^{c_2} \exp(-\tfrac{1}{2}\beta^2 + \beta_1 x + \beta_2 y) \cdot \varphi_2(x, y; \rho)\, dy\, dx$$

and, by a further application of Lemma A.1, one obtains

$$M_2 = Ke^{-r} \cdot \int_{c_1}^{\infty} \exp\big(-\tfrac{1}{2}\beta^2 + \tfrac{1}{2}\beta_2^2 + \beta_1 x\big) \cdot \varphi(x-\rho\beta_2) \cdot \Phi\left(\frac{c_2 - \rho x - (1-\rho^2)\beta_2}{\sqrt{1-\rho^2}}\right) dx\,.$$

Noting that

$$\sqrt{2\pi} \cdot \exp(\beta_1 x) \cdot \varphi(x-\rho\beta_2) = \exp\big(-\tfrac{1}{2}(x-\beta_1-\rho\beta_2)^2 + \tfrac{1}{2}\beta_1^2 + \rho\beta_1\beta_2\big)$$

and using that $\beta_1 + \rho\beta_2 = -\lambda_1$, $(1-\rho^2)\beta_2 = \rho\lambda_1 - \lambda_2$ and $\beta^2 = \beta^2 + 2\rho\beta_1\beta_2 + \beta_2^2$ one gets

$$\begin{aligned} M_2 &= Ke^{-r} \cdot \int_{c_1}^{\infty} \varphi(x+\lambda_1) \cdot \Phi\left(\frac{c_2 - \rho x - \rho\lambda_1 + \lambda_2}{\sqrt{1-\rho^2}}\right) dx \\ &= Ke^{-r} \cdot \int_{c_1+\lambda_1}^{\infty} \varphi(t) \cdot \overline{\Phi}\left(\frac{\rho t - c_2 - \lambda_2}{\sqrt{1-\rho^2}}\right) dt\,. \end{aligned}$$

But one has $c_1 + \lambda_1 = -d_2^{(1)}$, $c_2 + \lambda = -d_2^{(2)}$, and with formula (A.2) one obtains that $M_2 = Ke^{-r} \cdot \overline{\Phi}_2(d_2^{(2)}, -d_2^{(1)}; -\rho) = K \cdot P(s,t) \cdot R_3$. The proof of Theorem 3.1 is complete. □

4. **Alternative analytical integral representation**

What is the impact on calculation of choosing instead the canonical BSV deflator? Similarly to Section 3 we make use of the following simplifying notations and assumptions. We fix the time frame $0 \le s < t$ and omit dependence upon s, t. The elements of the matrix B in Example 2.2 are denoted b_{ij}, $i = 1,2$, and we set $\beta_{ij} = v_i(s,t)\sqrt{t-s} \cdot b_{ij}$, $i, j = 1,2$, $\Delta\beta = \det(\beta_{ij}) = -\sqrt{1-\rho^2} \cdot v_1(s,t)v_2(s,t) \cdot (t-s)$. We assume that $\beta_{21} > \beta_{11}$, and note that otherwise, a similar result can be derived. Set further

$$\begin{aligned} &\beta_i = -\beta_t^{(2)}(s,t)\sqrt{t-s}\,, \\ &\mu_i = \ln S_s^{(i)} + (m_i(s,t) - \tfrac{1}{2}\sigma_i^2)(t-s) - R(s,t)(t-s) - \tfrac{1}{2}(\beta_1^2 + \beta_2^2), \quad i = 1,2, \\ &\upsilon_{ij} = \beta_j + \beta_{ij}, \;\; i,j = 1,2, \quad \Delta\upsilon = \upsilon_{11}\upsilon_{22} - \upsilon_{12}\upsilon_{21}. \end{aligned}$$

Theorem 4.1. *Given is the bivariate canonical BSV deflator of* Example 2.2. *The market value at time $s \ge 0$ of a European double-triggered embedded option on the financial instruments I_1, I_2 with strike time $t > s$ and financial payoff $G_t^{(1,2)} = (\max\{S_t^{(1)}, K\} - S_t^{(2)})$ is given by*

$$\begin{aligned} E[D_t^{(2)} G_t^{(1,2)}] = D_s^{(2)} \cdot \{&S_s^{(1)} \cdot R_1 - S_s^{(2)} \cdot R_2 - K \cdot P(s,t) \cdot R_3 + K \cdot P(s,t) \\ &\cdot \overline{\Phi}(d_2^{(2)}(s,t)) - S_s^{(2)} \cdot \overline{\Phi}(d_1^{(2)}(s,t))\} \end{aligned} \tag{4.1}$$

with

$$
\begin{aligned}
d_1^{(2)}(s,t) &= \frac{\ln(S_s^{(2)}/K) + (R(s,t) + \frac{1}{2}v_2^2(s,t))(s,t)}{v_2(s,t)\sqrt{t-s}}, \\
d_2^{(2)}(s,t) &= d_1^{(2)}(s,t) - v_2(s,t)\sqrt{t-s},
\end{aligned} \tag{4.2}
$$

and the analytical integral representation formula

$$
\begin{aligned}
R_i &= \int_L^{\infty} \varphi(y - \upsilon_{i2}) \cdot [\Phi(\upsilon_{i1} - c_i(y)) - \Phi(\upsilon_{i1} - d(y))]\, dt, \quad i = 1,2, \\
R_3 &= \int_L^{\infty} \varphi(y - \beta_2) \cdot [\Phi(\beta_1 - c_1(y)) - \Phi(\beta_1 - c_2(y))]\, dy, \\
L &= \frac{\begin{pmatrix} [\ln(K/S_s^{(1)}) - (m_1(s,t) - \frac{1}{2}\sigma_1^2)(t-s)] \cdot \beta_{21} \\ -[\ln(K/S_s^{(2)}) - (m_2(s,t) - \frac{1}{2}\sigma_2^2)(t-s)] \cdot \beta_{11} \end{pmatrix}}{|\Delta\beta|}, \\
c_i(y) &= \frac{\ln(K/S_s^{(i)}) - (m_i(s,t) - \frac{1}{2}\sigma_1^2)(t-s) - \beta_{i2} \cdot y}{\beta_{i1}}, \\
d(y) &= \frac{\mu_1 - \mu_2 + (\beta_{12} - \beta_{22}) \cdot y}{\beta_{21} - \beta_{11}}.
\end{aligned} \tag{4.3}
$$

Proof. Following the proof of Theorem 3.1 one has to calculate the expected values

$$
\begin{aligned}
I1 &= 1/D_s^{(2)} \cdot E_s[D_t^{(2)} \cdot (S_t^{(1)} - S_t^{(2)})_+ \cdot 1\{S_t^{(1)} > K\} \cdot 1\{S_t^{(2)} > K\}], \\
I2 &= 1/D_s^{(2)} \cdot E_s[D_t^{(2)} \cdot (S_t^{(1)} - K) \cdot 1\{S_t^{(1)} > K\} \cdot 1\{S_t^{(2)} \le K\}].
\end{aligned}
$$

Set $Z_i = W_{t-s}^{(i)}$, $i = 1,2$, for the independent standard Wiener processes in (2.11)-(2.12). Using the made notations one has (use the assumption $\beta_{21} > \beta_{11} > 0$)

$$
\begin{aligned}
S_t^{(i)} > K &\quad \Leftrightarrow \quad Z_1 > c_i(Z_2), \quad i = 1,2, \\
S_t^{(1)} \ge S_t^{(2)} &\quad \Leftrightarrow \quad Z_1 \le d(Z_2).
\end{aligned}
$$

Therefore, for a non-zero contribution to $I1$ one requires $d(Z_2) > c(Z_2) = \max\{c_1(Z_2), c_2(Z_2)\}$. It follows that

$$
\begin{aligned}
I1 &= \int_{-\infty}^{\infty} [J_1(y) - J_2(y)] \cdot 1\{d(y) > c(y)\} \cdot \varphi(y)\, dy, \\
J_i(y) &= \int_{c(y)}^{d(y)} \exp(\mu_i + \upsilon_{i1}x + \upsilon_{i2}y) \cdot \varphi(x)\, dx \\
&= \exp(\mu_i + \upsilon_{i2}y + \tfrac{1}{2}\upsilon_{i1}^2) \cdot [\Phi(\upsilon_{i1} - c(y)) - \Phi(\upsilon_{i1} - d(y))], \quad i = 1,2,
\end{aligned}
$$

where use has been made of the identity

$$
\int_a^{\infty} \exp(bx) \cdot \varphi(x)\, dx = \exp(\tfrac{1}{2}b^2) \cdot \Phi(b - a).
$$

On the other side, one has the relationships $\exp(\mu_i + \frac{1}{2}\upsilon_{i1}^2 + \frac{1}{2}\upsilon_{i2}^2) = S_s^{(i)}$, $i = 1, 2$, which imply the integral difference representation

$$I1 = S_s^{(1)} \cdot R1^{(1)} - S_s^{(2)} \cdot R1^{(2)}$$
$$R1^{(i)} = \int_{-\infty}^{\infty} 1\{d(y) > c(y)\} \cdot \varphi(y - \upsilon_{i2})$$
$$\cdot [\Phi(\upsilon_{i1} - c(y)) - \Phi(\upsilon_{i1} - d(y))]\, dy, \quad i = 1, 2.$$

This formula can be made more explicit. Since $\beta_{12} > 0, \beta_{22} < 0$, the difference $c_2(y) - c_1(y)$ is strictly monotone increasing, and one shows that $c_2(L) = c_1(L)$, which implies that

$$c(y) = \max\{c_1(y), c_2(y)\} = \begin{cases} c_1(y) & y \le L, \\ c_2(Y) & y \ge L. \end{cases}$$

Another calculation shows that

$$d(y) - c_i(y) = \frac{|\Delta\beta| \cdot (y - L)}{\beta_{i1} \cdot (\beta_{21} - \beta_{11})}, \quad i = 1, 2.$$

Together, this implies that $d(y) - c(y) > 0 \Leftrightarrow y > L$ and $c(y) = c_2(y)$. It follows that

$$R1^{(i)} = \int_{L}^{\infty} \varphi(y - \upsilon_{i1}) \cdot [\Phi(\upsilon_{i1} - c_2(y)) - \Phi(\upsilon_{i1} - d(y))]\, dy, \quad i = 1, 2.$$

To calculate the expected value $I2$ one proceeds similarly. One has

$$S_t^{(1)} > K \quad \Leftrightarrow \quad Z_1 > c_1(Z_2),$$
$$S_s^{(2)} \le K \quad \Leftrightarrow \quad Z_1 \le c_2(Z_2).$$

Therefore, for a non-zero contribution to $I2$ one requires $c_2(Z_2) > c_1(Z_2)$, hence $Z_2 > L$ by the preceding considerations. With similar integral evaluations one obtains

$$I2 = \int_{L}^{\infty} \varphi(y)\, dy \int_{c_1(y)}^{c_2(y)} \big[\exp(\mu_1 + \upsilon_{1I}x + \upsilon_{12}y) - K \cdot P(s,t)$$
$$\cdot \exp(-\tfrac{1}{2}(\beta_1^2 + \beta_2^2) + \beta_1 x + \beta_2 y) \big] \cdot \varphi(x)\, dx$$
$$= S_s^{(1)} \cdot \int_{L}^{\infty} \varphi(y - \upsilon_{12}) \cdot [\Phi(\upsilon_{11} - c_1(y)) - \Phi(\upsilon_{11} - c_2(y))]\, dy$$
$$- K \cdot P(s,t) \cdot \int_{L}^{\infty} \varphi(y - \beta_2) \cdot [\Phi(\beta_1 - c_1(y)) - \Phi(\beta_1 - c_2(y))]\, dy\,.$$

Finally, through addition one obtains

$$I1 + I2 = S_s^{(1)} \cdot R1^{(1)} - S_s^{(2)} \cdot R1^{(2)} + I2 = S_s^{(1)} \cdot R_1 - S_s^{(2)} \cdot R_2 - K \cdot P(s,t) \cdot R_3,$$

which shows the desired analytical integral representation formula. $\square$

A first look at the defining parameter and functions in (4.3) does not immediately imply that the option pricing formula is invariant with respect to the market prices of risk. This property is obtained for the third alternative representation in Section 5.

5. **Analytical Gaussian approximation**

Unfortunately no closed-form analytical expressions are available for infinite integrals of the type R_i, $i = 1,2,3$ in Theorem 4.1. To evaluate them numerically, it is possible to use numerical integration, which is offered by many computer algebra systems (e.g. MATHCAD). Alternatively, for a direct spreadsheet calculation (e.g. EXCEL) we propose to use the simple but accurate analytical approximation by Lin (1989) of the standard normal tail probability function given by

$$\overline{\Phi}(x) \approx \tfrac{1}{2}\exp(-\alpha x - \gamma x^2), \quad x \geq 0,\ \alpha = 0.717,\ \gamma = 0.416. \tag{5.1}$$

To do so, we will decompose the three infinite integrals using an elementary Gaussian integral of *Lin type* in three variables, which contain a term of the form $\overline{\Phi}(c \cdot x)$, $x \geq 0$ with non-negative coefficient $c > 0$ and is defined by

$$J(x,y,z) = \int_z^{\infty} \varphi(t+x) \cdot \overline{\Phi}(y \cdot t)\, dt, \quad y > 0, z \geq 0. \tag{5.2}$$

Lemma 5.1. *The Lin type integral* (5.2) *satisfies the following approximation*

$$J(x,y,z) \approx J^*(x,y,z) = \frac{1}{2\sqrt{F(y)}} \cdot \exp\left(-\tfrac{1}{2}x^2 + \tfrac{1}{2}\frac{G(x,y)^2}{F(y)} \right) \cdot \overline{\Phi}\left(\sqrt{F(y)} \cdot z + \frac{G(x,y)}{\sqrt{F(y)}} \right), \quad y > 0, z \geq 0, \tag{5.3}$$

with $F(y) = 1 + 2 \cdot \gamma \cdot y^2$, $G(x,y) = x + \alpha \cdot y$.

Proof. Inserting (5.1) into (5.2) and completing squares in the usual way, one obtains (5.3). □

The mentioned decomposition, which also proves invariance with respect to the market prices of risk, is stated in the following result. The notations of the preceding Section 4 hold.

Proposition 5.1. *The infinite integrals R_i, $i = 1,2,3$ of* Theorem 4.1 *can be expressed as functions of the Lin type integral* (5.2) *as follows:*

$$\begin{aligned} R_1 = {} & \\ & \Phi(d(\rho,s,t)) - J(-d_1^{(1)}(\rho,s,t), f(\rho), \tfrac{1}{2}[d_2^{(2)}(\rho,s,t) + d_1^{(1)}(\rho,s,t) + \rho \cdot v_1(\rho,s,t)]) \\ & - J(-d(\rho,s,t), e(s,t)f(\rho), 0) + J(d(\rho,s,t), e(s,t)f(\rho), 0) \\ & - J(d(\rho,s,t), e(s,t)f(\rho), \tfrac{1}{2}[d_1^{(1)}(\rho,s,t) - \rho \cdot v_1(\rho,s,t) - d_2^{(2)}(\rho,s,t)] - d(\rho,s,t)) \end{aligned} \tag{5.4}$$

$$
\begin{aligned}
R_2 = {} & \\
& \Phi(d_1^{(2)}(\rho,s,t)) - \Phi(d(\rho,s,t)) - J(-d_1^{(2)}(\rho,s,t), f(\rho), 0) \\
& + J(-d_1^{(2)}(\rho,s,t), f(\rho), \tfrac{1}{2}[d_1^{(2)}(\rho,s,t) + d_2^{(1)}(\rho,s,t) + \rho\cdot v_2(\rho,s,t)]) \\
& + J(d_1^{(2)}(\rho,s,t), f(\rho), 0) - J(d(\rho,s,t), e(s,t)f(\rho), 0) + J(-d(\rho,s,t), e(s,t)f(\rho), 0) \\
& - J(-d(\rho,s,t), e(s,t)f(\rho), d(\rho,s,t) + \tfrac{1}{2}[d_2^{(1)}(\rho,s,t) + \rho\cdot v_2(\rho,s,t) - d_1^{(2)}(\rho,s,t)])
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
R_3 = {} & \\
& \overline{\Phi}(d_2^{(2)}(\rho,s,t)) - J(-d_2^{(1)}(\rho,s,t), f(\rho), \tfrac{1}{2}[d_2^{(1)}(\rho,s,t) + d_2^{(2)}(\rho,s,t)]) \\
& + J(-d_2^{(2)}(\rho,s,t), f(\rho), 0) - J(-d_2^{(2)}(\rho,s,t), f(\rho), \tfrac{1}{2}[d_2^{(1)}(\rho,s,t) + d_2^{(2)}(\rho,s,t)]) \\
& - J(d_2^{(2)}(\rho,s,t), f(\rho), 0)
\end{aligned}
\tag{5.6}
$$

with the following parametric functions, valid for $0 \le s < t$,

$$
\begin{aligned}
& d_1^{(i)}(\rho,s,t) = \frac{\ln(S_s^{(i)}/K) + (R(s,t) + \frac{1}{2}v_i^2(s,t))(t-s)}{v_i(s,t)\sqrt{t-s}} \cdot \sqrt{\frac{2}{1-\rho}}, \quad i = 1,2, \\
& d_2^{(i)}(\rho,s,t) = d_1^{(i)}(s,t) - v_i(\rho,s,t)\sqrt{t-s}, \quad v_i(\rho,s,t) = v_i(s,t)\cdot\sqrt{\frac{2}{1-\rho}}, \quad i = 1,2, \\
& d(\rho,s,t) = \frac{1}{2}\frac{v(\rho,s,t)^2\sqrt{t-s}}{v_1(s,t)+v_2(s,t)} \cdot \sqrt{\frac{2}{1-\rho}}, \\
& v(\rho,s,t)^2 = v_1(s,t)^2 + v_2(s,t)^2 - 2\rho v_1(s,t)v_2(s,t), \\
& f(\rho) = \sqrt{\frac{1-\rho}{1+\rho}}, \quad e(s,t) = \frac{v_1(s,t)+v_2(s,t)}{v_2(s,t)-v_1(s,t)}.
\end{aligned}
\tag{5.7}
$$

Proof. Consider first the following linear changes of variables:

$$
\begin{aligned}
& \upsilon_{11} - d(y) = a\cdot(D_1 - y), \quad D_1 > L, \quad \upsilon_{21} - d(y) = a\cdot(D_2 - y), \quad D_2 > L, \\
& \upsilon_{11} - c_1(y) = b\cdot(y - C_1), \quad C_1 < L, \quad \beta_1 - c_1(y) = b\cdot(y - E_1), \quad E_1 < L, \\
& \upsilon_{21} - c_2(y) = c\cdot(C_2 - y), \quad C_2 > L, \quad \beta_1 - c_2(y) = c\cdot(E_2 - y), \quad E_2 > L,
\end{aligned}
\tag{5.8}
$$

with the defining constants

$$
\begin{aligned}
& a = \frac{\beta_{12}-\beta_{22}}{\beta_{21}-\beta_{11}}, \quad b = \frac{\beta_{12}}{\beta_{11}}, \quad c = \frac{-\beta_{22}}{\beta_{21}}, \\
& C_i = \frac{\ln(K/S_s^{(i)}) - (m_i(s,t) - \frac{1}{2}\sigma_1^2)(t-s) - \beta_{i1}\cdot\upsilon_{i1}}{\beta_{i2}}, \quad i = 1,2, \\
& D_i = \frac{\mu_2 - \mu_1 + (\beta_{21}-\beta_{11})\cdot\upsilon_{i1}}{\beta_{12}-\beta_{22}}, \quad i = 1,2, \\
& E_i = \frac{\ln(K/S_s^{(i)}) - (m_i(s,t) - \frac{1}{2}\sigma_i^2)(t-s) - \beta_{i1}\cdot\beta_1}{\beta_{i2}}, \quad i = 1,2.
\end{aligned}
\tag{5.9}
$$

Through straightforward algebraic manipulation one obtains the following decompositions:

$$
\begin{aligned}
R_1 = {} & \overline{\Phi}(D_1 - \upsilon_{12}) - J(C_1 - \upsilon_{12}, b, L - C_1) + J(\upsilon_{12} - D_1, a, 0) \\
& - J(\upsilon_{12} - D_1, a, D_1 - L) - J(D_1 - \upsilon_{12}, a, 0), \qquad (5.10)
\end{aligned}
$$

$$
\begin{aligned}
R_2 = {} & \Phi(C_2 - \upsilon_{22}) - \Phi(D_2 - \upsilon_{22}) - J(\upsilon_{22} - C_2, c, 0) \\
& + J(\upsilon_{22} - C_2, c, C_2 - L) + J(C_2 - \upsilon_{22}, c, 0) + J(\upsilon_{22} - D_2, a, 0) \\
& - J(\upsilon_{22} - D_2 - L) - J(D_2 - \upsilon_{22}, a, 0), \qquad (5.11)
\end{aligned}
$$

$$
\begin{aligned}
R_3 = {} & \overline{\Phi}(E_2 - \beta_2) - J(E_1 - \beta_2, b, L - E_1) + J(\beta_2 - E_2, c, 0) \\
& - J(\beta_2 - E_2, c, E_2 - L) - J(E_2 - \beta_2, c, 0). \qquad (5.12)
\end{aligned}
$$

Further calculations, which makes use of the defined quantities (5.7), the notations preceding Theorem 4.1, the formulas (2.16) for $\beta_i = \beta_i^{(2)}(s,t)\sqrt{t-s}$, $i = 1,2$, in terms of the market prices of risk, and the definition (2.13) of the latter quantities, lead to the following relationships

$$
\begin{aligned}
& a = e(s,t)f(\rho), \qquad b = c = f(\rho), \qquad D_1 - \upsilon_{12} = -d(\rho,s,t), \\
& C_1 - \upsilon_{12} = -d_1^{(1)}(\rho,s,t), \\
& L - \upsilon_{12} = \tfrac{1}{2}[d_2^{(2)}(\rho,s,t) - d_1^{(1)}(\rho,s,t) + \rho \cdot v_1(\rho,s,t)], \\
& L - C_1 = (L - \upsilon_{12}) - (C_1 - \upsilon_{12}) \\
& \qquad = \tfrac{1}{2}[d_2^{(2)}(\rho,s,t) + d_1^{(1)}(\rho,s,t) + \rho \cdot v_1(\rho,s,t)], \\
& D_1 - L = (D_1 - \upsilon_{12}) - (L - \upsilon_{12}) \\
& \qquad = \tfrac{1}{2}[d_1^{(1)}(\rho,s,t) - \rho \cdot v_1(\rho,s,t) - d_2^{(2)}(\rho,s,t)] - d(\rho,s,t), \\
& D_2 - \upsilon_{22} = d(\rho,s,t), \qquad C_2 - \upsilon_{22} = d_1^{(2)}(\rho,s,t), \\
& L - \upsilon_{22} = \tfrac{1}{2}[d_1^{(2)}(\rho,s,t) - d_2^{(1)}(\rho,s,t) - \rho \cdot v_2(\rho,s,t)], \\
& C_2 - L = (C_2 - \upsilon_{22}) - (L - \upsilon_{22}) \\
& \qquad = \tfrac{1}{2}[d_1^{(2)}(\rho,s,t) + d_2^{(1)}(\rho,s,t) + \rho \cdot v_2(\rho,s,t)], \\
& D_2 - L = (D_2 - \upsilon_{22}) - (L - \upsilon_{22}) \\
& \qquad = d(\rho,s,t) + \tfrac{1}{2}[d_2^{(1)}(\rho,s,t) + \rho \cdot v_2(\rho,s,t) - d_1^{(2)}(\rho,s,t)], \\
& E_2 - \beta_2 = d_2^{(2)}(\rho,s,t), \qquad E_1 - \beta_2 = -d_2^{(1)}(\rho,s,t), \\
& L - \beta_2 = \tfrac{1}{2}[d_2^{(2)}(\rho,s,t) - d_2^{(1)}(\rho,s,t)], \\
& L - E_1 = (L - \beta_2) - (E_1 - \beta_2) = \tfrac{1}{2}[d_2^{(1)}(\rho,s,t) + d_2^{(2)}(\rho,s,t)], \\
& E_2 - L = (E_2 - \beta_2) - (L - \beta_2) = \tfrac{1}{2}[d_2^{(1)}(\rho,s,t) + d_2^{(2)}(\rho,s,t)]
\end{aligned}
$$

Inserting the latter into (5.10)-(5.12) shows the representations (5.4)-(5.6). □

6. **A numerical example**

Similarly to the expressions of Theorem 4.1 no closed-form analytical expressions are available for the bivariate normal survival functions involved in the expressions R_i, $i = 1,2,3$ of Theorem 3.1. Again, one must rely on software based numerical integration. Usually, the numerical implementation of the bivariate normal distribution is done with the tetrachoric series, an infinite bivariate expansion based on the Hermite polynomials. However, this expansion converges only slightly faster than a geometric series with quotient ρ. Vasicek (1998) has improved on this numerical evaluation and has derived another series that converges approximately as a geometric series with quotient $1-\rho^2$. On the other side it is theoretically possible to proceed as in Section 5 and develop a Gaussian approximation based on decomposition into Lin type integrals.

Unfortunately and in contrast to the approximation made in Section 5, our Gaussian approximation of Theorem 3.1 has been rather inaccurate and thus impractical.

We illustrate calculations for a double-trigger option, which is embedded in a n-year equity-linked endowment policy and combines an inflation protection with a guaranteed minimum death benefit of amount $T_k = SI \cdot (1+i)^k$ at the end of the year of death at possible times $k = 1\ldots,n$ (SI, i are the sum insured respectively the guaranteed interest rate). This kind of embedded options is encountered in some pension insurance contracts. Setting $SI = 1$ without loss of generality, the double-trigger option at time k has the contingent financial payoff

$$(I_k - S_k)\cdot 1\{I_k > T_k\} + (T_k - S_k)\cdot 1\{I_k \le T_k\}, \tag{6.1}$$

where S_t, $t = 0,1,\ldots,n$, with initial value $S_0 = 1$, and I_t, $t = 0,1,\ldots,n$, with initial value $I_0 = 1$, represent the equity index respectively the inflation index. We suppose them of the form

$$\begin{aligned} S_t &= S_s \exp\{(\mu_s - \tfrac{1}{2}\sigma_5^2)(t-s) + \sigma_s\sqrt{t-s}\cdot W_{t-s}^S\}, \\ I_t &= I_s \exp\{(m_I(s,t) - \tfrac{1}{2}\sigma_I^2)(t-s) + \nu_I\sqrt{t-s}\cdot W_{t-s}^I\}, \quad 0 \le s < t, \end{aligned} \tag{6.2}$$

where W_{t-s}^S, W_{t-s}^I are correlated standard Wiener processes such that $E[dW_{t-s}^S dW_{t-s}^I] = \rho\, dt$,, and the parameters associated to the inflation index are given by

$$m_I(s,t) = \frac{(b_I - q_s)(1-e^{-a_I(t-s)})}{t-s}, \quad \nu_I(s,t) = \sigma_I 1\sqrt{\frac{1-e^{-2a_I(t-s)}}{2a_I(t-s)}}, \tag{6.3}$$

with q_s the (continuous) force of inflation at time s. Implicitly, we assume that the inflation index satisfies the stochastic differential equation

$$dI_t = a_I(b_I - q_t)\,dt + \sigma_I I_t dW_t^I, \tag{6.4}$$

where the force of inflation q_t is assumed to follow the Ornstein-Uhlenbeck process

$$dq_t = a_I(b_I - q_t)\,dt + \sigma_I dW_t^I. \tag{6.5}$$

The parameters are $n = 5$, $i = 2\%$, $\sigma_s = 15\%$, $a_I = 0.3382$, $b_I = 0.02327$, $\sigma_I = 0.03059$, $\rho = -0.2$. The inflation index parameters are those of the Swiss CPI (see H rlimann (2011b) for inflation parameter estimation, in particular comments on the choice of $\rho = -0.2$). Note that the assumption $\beta_{21} > \beta_{11}$, made to derive Theorem 4.1 and Proposition 5.1, is fulfilled because $\sigma_S > v_I(s,t)$ for all $0 \le s < t \le n$. The required zero-coupon bond prices are chosen as follows:

k	0	1	2	3	4	5
$P(0,k)$	1	0.96686	0.93202	0.89951	0.86727	0.83527

The obtained (exact) double-triggered option prices and the Gaussian approximation of Proposition 5.1 are listed in Table 6.1. For better understanding of the results it appears useful to compare them with the prices for embedded options with a minimum interest guarantee only (European put options on the equity index) and an inflation protection only (European exchange options) that have been listed in H rlimann (2011b), Tables 4.4 and 4.5. Though the inflation protection alone is more expensive than the guaranteed interest rate protection alone (additional involved inflation volatility), the double-triggered embedded option is only slightly more expensive than the latter. The accuracy of the Gaussian approximation is excellent. It lies within two basis points and is on the safe side.

Table 6.1. Comparisons of percentage option prices and Lin type Gaussian approximation

strike time	vanilla put option: minimum interest guarantee	exchange option: equity-linked inflation protection	double-triggered embedded option		
			numerical integral	analytical approximation	appr. error (bps)
1	5.273	6.269	6.433	6.450	1.8
2	6.890	8.790	8.879	8.898	1.9
3	7.990	10.695	10.737	10.753	1.7
4	8.750	12.286	12.301	12.316	1.4
5	9.278	13.679	13.683	13.696	1.3

Appendix: Multivariate normal integral identities

For their own interest, the crucial identities used in the proof of Theorem 3.1 are stated separately and derived for the sake of completeness.

Lemma A.1. *For any constant b and any real function of one variable $c(x)$ one has the identity*

$$\int_{-\infty}^{c(x)} e^{by} \cdot \varphi_2(x,y;\rho)\,dy$$
$$= e^{\frac{1}{2}b^2} \cdot \varphi(x-\rho b) \cdot \Phi\left(\frac{c(x)-\rho x-(1-\rho^2)b}{\sqrt{1-\rho^2}}\right). \qquad \text{(A.1)}$$

Proof. Complete squares in the numerator of the expression in curly brackets

$$\sqrt{2\pi(1-\rho^2)} \cdot e^{by} \cdot \varphi_2(x,y;\rho)$$
$$= \exp\left\{-\frac{(x^2-2\rho xy+y^2)-2b(1-\rho^2)y}{2(1-\rho^2)}\right\}$$

to get the identity

$$y^2-2\rho xy+x^2-2b(1-\rho^2)y$$
$$=(y-\rho x-b(1-\rho^2))^2+(1-\rho^2)(x-b\rho)^2-(1-\rho^2)b^2,$$

which immediately implies the relationship (A.1). □

Lemma A.2. *For any constants a,b,c one has the identities*

$$\int_c^{\infty} \varphi(x) \cdot \overline{\Phi}(a+bx)\,dx = \overline{\Phi}_2\left(\frac{a}{\sqrt{1+b^2}}, c; \frac{-b}{\sqrt{1+b^2}}\right), \qquad \text{(A.2)}$$

$$\int_{-\infty}^{\infty} \varphi(x) \cdot \overline{\Phi}(a+bx)\,dx = \overline{\Phi}\left(\frac{a}{\sqrt{1+b^2}}\right). \qquad \text{(A.3)}$$

Proof. Rewrite the integral in (A.2) as

$$I = \int_c^{\infty} \varphi(x) \cdot \overline{\Phi}(a+bx)\,dx = \int_c^{\infty} \varphi(x) \int_{a+bx}^{\infty} \varphi(y)\,dy\,dx.$$

With the change of variable $z = y - bx$ one obtains

$$I = \int_a^{\infty}\int_c^{\infty} \varphi(x)\varphi(z+bx)\,dx\,dz\,.$$

A further transformation yields the identity $\varphi(x)\varphi(z+bx) = \varphi_2(X,Z;\rho)$ with $X = x$, $Z = \frac{z}{\sqrt{1+b^2}}$, $\rho = \frac{-b}{\sqrt{1+b^2}}\cdot$ The result follows by definition of the standard bivariate normal survival function $\overline{\Phi}_2(x,y;\rho)$. The limit of (A.2) when $c \to -\infty$ identifies with (A.3). □

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