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Multiple Objective Fractional Subset Programming Based on Generalized (ρ,η,A)-Invex Functions

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Abstract. Motivated by the recent investigations [7, 8, 10–12], a general framework for a class of (ρ,η,A)-invex n-set functions is introduced, and then some results on the optimality conditions for multiple objective fractional subset programming on the generalized (ρ,η,A)-invexity are explored. The obtained results are general in nature and seem to be application-oriented to other results on fractional subset programming in literature.

1. **Introduction**

Motivated by the investigations [1–8, 10–12], some results on primal problems based on a generalized invex n-set functions are established. More importantly, generalized convexity of n-set functions is introduced, that can be applied to explore results on parametric semi-parametric sufficient efficiency conditions for multiobjective fractional subset programming problems. Recently, Mishra *et al.* [7] published some results on optimality conditions for multiple objective fractional subset programming with invex and related non-convex functions based on a class of generalized convex n-set functions introduced by Zalmai [11] to the context of parametric and semi-parametric sufficient efficiency conditions for a multiobjective fractional subset programming problem. We present using the generalized (ρ,η,A)-invexity of non-differentiable functions, the following multiple objective fractional subset programming problem to the context of generalized (ρ,η,A)-invex functions:

$$\text{(P)}\qquad \text{Minimize}\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\dots,\frac{F_p(S)}{G_p(S)}\right)$$
$$\text{subject to } H_j(S)\le 0 \text{ for } j\in\{1,\dots,m\}, S\in\Lambda^n,$$

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where Λ^n is the n-fold product of σ-algebra Λ of subsets of a given set X and, F_i, G_i, $i \in \{1,\ldots,p\}$ and $H_j(S) \leq 0$ for $j \in \{1,\ldots,m\}$ are real-valued functions defined on Λ^n, and $G_i(S) > 0$ for each $i \in \{1,\ldots,p\}$ and for all $S \in \Lambda^n$. Mishra *et al.* [7] investigated several parametric and semi-parametric sufficient conditions for the multiobjective fractional subset programming problems based on generalized invexity assumptions. Moreover, these results are also applicable to other classes of problems with multiple, fractional, and conventional objective functions.

Among other results, the obtained results generalize the recent results on generalized invexity (including [7, 10]) to the case of the generalized (ρ,η,A)-*invexity* relating to the case of semi-parametric sufficient efficiency conditions for the multiobjective fractional subset programming problem. The obtained results not only generalize the existing results but also unify other results on the fractional programming. For more details, we refer the reader [1–14].

2. Generalized (ρ,η,A)-Invexities

In this section, we develop some concepts and notations for the problem on hand. Let (X,Λ,μ) be a finite atomless measure space with $L_1(X,\Lambda,\mu)$ separable, and let d denote the pseudometric on Λ defined by

$$d(R,S) = \left[\sum_{i=1}^{n}\mu^2(R_i \Delta S_i)\right]^{1/2} \text{ for } R = (R_1,\ldots,R_n),\, S = (S_1,\ldots,S_n) \in \Lambda^n,$$

where Δ denotes the symmetric difference. Thus, (Λ^n, d) is a pseudo-metric space. Suppose that $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ be a vector-valued function. Here the following basic definitions are upgraded based on Caiping and Xinmin [2].

Definition 2.1. A function $F : \Lambda \to R$ is said to be differentiable at S^* if there is a $DF(S^*) \in L_1(X,\Lambda,\mu)$, the derivative of F at S^*, such that for each $S \in \Lambda$

$$F(S) = F(S^*) + \langle DF(S^*), \eta(S,S^*)\rangle + V_F(S,S^*),$$

where $V_F(S,S^*)$ is $o(d(S,S^*))$, i.e, $\lim_{d(S,S^*)\to 0} \frac{V_F(S,S^*)}{d(S,S^*)} = 0$.

Definition 2.2. A function $G : \Lambda \to R$ is said to have partial derivatives at $S^* = (S_1^*,\ldots,S_n^*) \in \Lambda^n$ with respect to ith argument if the function

$$F(S_i) = G(S_1^*,\ldots,S_{i-1},S_i,S_{i+1},\ldots,S_n^*)$$

has the derivative $DF(S_i^*)$ for $i \in \{1,\ldots,n\}$, and in this case, the ith derivative of G at S^* is defined by $D_iG(S^*) = DF(S_i^*)$, $i \in \{1,\ldots,n\}$.

Definition 2.3. A function $G : \Lambda^n \to R$ is called differentiable at S^* if all the derivatives $D_iG(S^*)$, $i \in \{1,\ldots,n\}$ exist and

$$G(S) = G(S^*) + \sum_{i=1}^{n}\langle DG_i(S^*), \eta(S_i,S_i^*)\rangle + W_G(S,S^*),$$

where $W_G(S,S^*)$ is $o(d(S,S^*))$ for all $S \in \Lambda^n$.

Next, we develop the notion of the generalized (ρ,η)-invexity based on Caiping and Xinmin [2]. Let $S, S^* \in \Lambda^n$, let the function $F : \Lambda^n \to R$ with components F_i for $i \in \{1,\dots,n\}$, be differentiable at S^*.

Definition 2.4. Let $A : \Lambda^n \to R$ be a function on Λ^n. A differentiable function $F : \Lambda \to R$ is said to be (ρ,η,A)-pseudo-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S^* \in \Lambda^n$, and $\rho > 0$,

$$\langle \Sigma_{i=1}^p F_i'(S^*), \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 \geq 0 \Rightarrow \Sigma_{i=1}^p F_i(S) \geq \Sigma_{i=1}^p F_i(S^*).$$

Definition 2.5. Let $A : \Lambda^n \to R$ be a function on Λ^n. A differentiable function $F : \Lambda \to R$ is said to be (ρ,η,A)-strictly-pseudo-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S^* \in \Lambda^n$, and $\rho > 0$,

$$\langle \Sigma_{i=1}^p F_i'(S^*), \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 \geq 0 \Rightarrow \Sigma_{i=1}^p F_i(S) > \Sigma_{i=1}^p F_i(S^*).$$

Definition 2.6. Let $A : \Lambda^n \to R$ be a function on Λ^n. A differentiable function $F : \Lambda \to R$ is said to be (ρ,η,A)-prestrictly-pseudo-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S^* \in \Lambda^n$, and $\rho > 0$,

$$\langle \Sigma_{i=1}^p F_i'(S^*), \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 > 0 \Rightarrow \Sigma_{i=1}^p F_i(S) \geq \Sigma_{i=1}^p F_i(S^*).$$

Definition 2.7. Let $A : \Lambda^n \to R$ be a function on Λ^n. A differentiable function $F : \Lambda \to R$ is said to be (ρ,η,A)-quasi-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S^* \in \Lambda^n$, and $\rho > 0$,

$$\Sigma_{i=1}^p F_i(S) \leq \Sigma_{i=1}^p F_i(S^*) \Rightarrow \langle \Sigma_{i=1}^p F_i'(S^*), \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 \leq 0.$$

Definition 2.8. Let $A : \Lambda^n \to R$ be a function on Λ^n. A differentiable function $F : \Lambda \to R$ is said to be (ρ,η,A)-strictly-quasi-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S^* \in \Lambda^n$, and $\rho > 0$,

$$\Sigma_{i=1}^p F_i(S) \leq \Sigma_{i=1}^p F_i(S^*) \Rightarrow \langle \Sigma_{i=1}^p F_i'(S^*), \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 < 0.$$

Definition 2.9. Let $A : \Lambda^n \to R$ be a function on Λ^n. A differentiable function $F : \Lambda \to R$ is said to be (ρ,η,A)-prestrictly-quasi-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S^* \in \Lambda^n$, and $\rho > 0$,

$$\Sigma_{i=1}^p F_i(S) < \Sigma_{i=1}^p F_i(S^*) \Rightarrow \langle \Sigma_{i=1}^p F_i'(S^*), \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 \leq 0.$$

Definition 2.10. A differentiable function $F : \Lambda \to R$ is said to be pseudo-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S^* \in \Lambda^n$,

$$\langle \Sigma_{i=1}^p F_i'(S^*), \eta(S,S^*)\rangle \geq 0 \Rightarrow \Sigma_{i=1}^p F_i(S) \geq \Sigma_{i=1}^p F_i(S^*).$$

Definition 2.11. A differentiable function $F : \Lambda \to R$ is said to be strictly-pseudo-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S^* \in \Lambda^n$,

$$\langle \Sigma_{i=1}^p F_i'(S^*), \eta(S,S^*)\rangle \geq 0 \Rightarrow \Sigma_{i=1}^p F_i(S) > \Sigma_{i=1}^p F_i(S^*)$$

Definition 2.12. A differentiable function $F : \Lambda \to R$ is said to be prestrictly-pseudo-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L^n_\infty$ such that for each $S^* \in \Lambda^n$,

$$\langle \Sigma_{i=1}^p F_i'(S^*), \eta(S, S^*)\rangle > 0 \Rightarrow \Sigma_{i=1}^p F_i(S) \geq \Sigma_{i=1}^p F_i(S^*).$$

Definition 2.13. A differentiable function $F : \Lambda \to R$ is said to be quasi-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L^n_\infty$ such that for each $S^* \in \Lambda^n$,

$$\Sigma_{i=1}^p F_i(S) \leq \Sigma_{i=1}^p F_i(S^*) \Rightarrow \langle \Sigma_{i=1}^p F_i'(S^*), \eta(S, S^*)\rangle \leq 0.$$

Definition 2.14. A differentiable function $F : \Lambda \to R$ is said to be strictly-quasi-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L^n_\infty$ such that for each $S^* \in \Lambda^n$,

$$\Sigma_{i=1}^p F_i(S) \leq \Sigma_{i=1}^p F_i(S^*) \Rightarrow \langle \Sigma_{i=1}^p F_i'(S^*), \eta(S, S^*)\rangle < 0.$$

Definition 2.15. A differentiable function $F : \Lambda \to R$ is said to be prestrictly-quasi-invex at S^* if there exists a vector-valued function $\eta : \Lambda^n \times \Lambda^n \to L^n_\infty$ such that for each $S^* \in \Lambda^n$, and $\rho > 0$,

$$\Sigma_{i=1}^p F_i(S) < \Sigma_{i=1}^p F_i(S^*) \Rightarrow \langle \Sigma_{i=1}^p F_i'(S^*), \eta(S, S^*)\rangle \leq 0.$$

Note that $S^* \in \Xi$ is an efficient solution to (P) if there exists no $S \in \Xi$ such that

$$\left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \dots, \frac{F_p(S)}{G_p(S)}\right) \leq \left(\frac{F_1(S^*)}{G_1(S^*)}, \frac{F_2(S^*)}{G_2(S^*)}, \dots, \frac{F_p(S^*)}{G_p(S^*)}\right).$$

To this context, based on Mishra *et al.* [7], we consider the following auxiliary problem:

(Pλ) $\quad \underset{S\in\Xi}{\text{Minimize}}(F_1(S) - \lambda_1 G_1(S), \dots, F_p(S) - \lambda_p G_p(S)),$

where λ_i for $i \in \{1, \dots, p\}$ are parameters.

For the sake of completeness, we include the following results of Mishra *et al.* [7].

Theorem 2.1. *Suppose that F_i, G_i, $i \in \{1, \dots, p\}$ and H_j, $j \in \{1, \dots, m\}$ are differentiable at $S^* \in \Lambda$, and that for each $i \in \{1, \dots, p\}$ there exists an $S^\dagger \in \Lambda^n$ such that*

$$H_j(S^*) + \Sigma_{k=1}^n \langle D_k H_j(S^*), \eta(S_k^\dagger, S_k^*)\rangle < 0 \quad \textit{for } j \in \{1, \dots, m\},$$

and for each $\ell \in \{1, \dots, p\}\backslash\{i\}$,

$$\Sigma_{k=1}^n \langle D_k F_l(S^*) - \lambda_l^* D_k G_l(S^*), \eta(S_k^\dagger, S_k^*)\rangle < 0.$$

If S^* is an efficient solution of the subset programming problem (P) *and $\lambda_i^* = \frac{F_i(S^*)}{G_i(S^*)}$ for $i \in \{1, \dots, p\}$, then there exists an $u^* \in U = \{u \in R^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in R^m_+$ such that*

$$\Sigma_{k=1}^n \langle \Sigma_{i=1}^p u_i^* [D_k F_i(S^*) - \lambda_i^* D - k G_i(S^*)] + \Sigma_{j=1}^m v_j^* D_k H_j(S^*), \eta(S_k, S_k^*)\rangle \geq 0$$

$v_j^* H_j(S^*) = 0 \quad for\ j \in \{1,\dots,m\}.$

Theorem 2.2. *Suppose that F_i, G_i, $i \in \{1,\dots,p\}$ and H_j, $j \in \{1,\dots,m\}$ are differentiable at $S^* \in \Lambda$, and that for each $i \in \{1,\dots,p\}$ there exists an $S^\dagger \in \Lambda^n$ such that*

$$H_j(S^*) + \Sigma_{k=1}^n \langle D_k H_j(S^*), \eta(S_k^\dagger, S_k^*)\rangle < 0 \ \ for\ j \in \{1,\dots,m\},$$

and for each $l \in \{1,\dots,p\}\backslash\{i\}$,

$$\Sigma_{k=1}^n \langle D_k G_l(S^*) - F_l(S^*) D_k G_l(S^*), \eta(S_k^\dagger, S_k^*)\rangle < 0.$$

If S^* is an efficient solution of the subset programming problem (P), then there exists an $u^* \in U = \{u \in R^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in R_+^m$ such that

$$\Sigma_{k=1}^n \langle \Sigma_{i=1}^p u_i^* [G_i(S^*) D_k F_i(S^*) - F_i(S^*) D - k G_i(S^*)] + \Sigma_{j=1}^m v^* D_k H_j(S^*), \eta(S_k, S_k^*)\rangle \geq 0$$
$$v_j^* H_j(S^*) = 0 \quad for\ j \in \{1,\dots,m\}.$$

3. **Parametric Optimality Conditions**

This section deals with some parametric sufficient optimality conditions for problem (P) under the generalized frameworks for generalized invexity, including the (ρ,η,A)-quasi-invexity and (ρ,η,A)-pseudo-invexity, where $A : \Lambda^n \to R$ is a function on Λ^n. We start with real-valued functions $A_i(\cdot\,;\lambda,u)$ and $B_j(\cdot,v)$ defined by

$$A_i(\cdot\,;\lambda,u) = u_i[F_i(S) - \lambda_i G_i(S)] \ \text{ for } i = 1,\dots,p, \text{ and for fixed } \lambda, u \text{ and } v$$

and

$$B_j(\cdot,v) = v_j H_j(S), \quad j = 1,\dots,m.$$

Theorem 3.1. *Let $S^* \in \Xi$, let F_i, G_i, $i \in \{1,\dots,p\}$, and H_j, $j \in \{1,\dots,m\}$, be differentiable at $S^* \in \Lambda$, and let there exist $u^* \in U$ and $v^* \in R_+^m$ such that*

(3.1) $$\langle \Sigma_{i=1}^p u_i^* [F_i'(S^*) - \lambda_i^* G_i'(S^*)] + \Sigma_{j=1}^m v_j^* H_j'(S^*), \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 \geq 0\ \forall\ S \in \Lambda^n,$$

(3.2) $$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \quad for\ i \in \{1,\dots,p\},$$

(3.3) $$v_j^* H_j(S^*) = 0 \quad for\ j \in \{1,\dots,m\}.$$

Let $A : \Lambda^n \to R$ be a function on Λ^n. Suppose, in addition, that any one of the following assumptions holds:

(i) *$A_i(\cdot\,;\lambda^*,u^*)$ $(\forall\ i = 1,\dots,p)$ are (ρ,η,A)-pseudo-invex at S^* and $B_j(\cdot\,;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ are (ρ,η,A)-quasi-invex at S^*.*

(ii) *$A_i(\cdot\,;\lambda^*,u^*)$ $(\forall\ i \in \{1,\dots,p\}$ are (ρ,η,A)-prestrictly-pseudo-invex at S^* and $B_j(\cdot\,;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ are (ρ,η,A)-prestrictly-quasi-invex at S^*.*

(iii) *$A_i(\cdot\,;\lambda^*,u^*)$ $(\forall\ i \in \{1,\dots,p\}$ are (ρ,η,A)-prestrictly-quasi-invex at S^* and $B_j(\cdot\,;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ are (ρ,η,A)-prestrictly-pseudo-invex at S^*.*

Then S^* is an efficient solution to (P).

Proof. If (i) holds, and if S is an arbitrary solution to (P), then it follows from (3.1) that

$$(3.4) \quad \langle \Sigma_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 + \langle \Sigma_{j=1}^{m} v_j^* H_j'(S^*), \eta(S,S^*)\rangle \geq 0 \;\forall\; S \in \Lambda^n.$$

Since $v^* \geq 0$, $S \in \Xi$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j'(S) \leq \Sigma_{j=1}^{m} v_j^* H_j'(S^*),$$

and in light of the (ρ,η,A)-quasi-invexity of $B_j(\cdot, v^*)$ at S^*, we arrive at

$$\langle \Sigma_{j=1}^{m} v_j^* H_j'(S^*), \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 \leq 0.$$

Consequently, we have

$$(3.5) \quad \langle \Sigma_{j=1}^{m} v_j^* H_j'(S^*), \eta(S,S^*)\rangle \leq 0.$$

It follows from (3.4) and (3.5) that

$$(3.6) \quad \langle \Sigma_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S,S^*)\rangle + \rho\|A(S) - A(S^*)\|^2 \geq 0.$$

Next, applying the (ρ,η,A)-pseudo-invexity at S^* to (3.6), we have

$$\Sigma_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geq \Sigma_{i=1}^{p} u_i^*[F_i(S^*) - \lambda_i^* G_i(S^*)],$$

and applying (3.2), it reduces to

$$(3.7) \quad \Sigma_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geq 0.$$

Since $u_i^* > 0$ for each $i \in \{1, \ldots, p\}$, we have from (3.7) that

$$(F_1(S) - \lambda_1^* G_1(S), \ldots, F_p(S) - \lambda_1^* G_p(S)) \not\leq (0, \ldots, 0).$$

Thus, we conclude that

$$\phi(S) = \left(\frac{F_1(S)}{G_1(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \not\leq \lambda^*$$

At this stage, as we observe that $\lambda^* = \phi(S^*)$ and $S \in \Xi$ is arbitrary, it implies that S^* is an efficient solution to (P). Similar proofs hold for (ii) and (iii).

For $A = I$, we have [10]. □

Theorem 3.2 ([10, Theorem 3.1]). *Let $S^* \in \Xi$, let F_i, G_i, $i \in \{1, \ldots, p\}$, and H_j, $j \in \{1, \ldots, m\}$, are differentiable at $S^* \in \Lambda$, and let there exist $u^* \in U$ and $v^* \in R_+^m$ such that*

$$(3.8) \quad \langle \Sigma_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)] + \Sigma_{j=1}^{m} v_j^* H_j'(S^*), \eta(S,S^*)\rangle + \rho\|S - S^*\|^2 \geq 0 \;\forall\; S \in \Lambda^n,$$

$$(3.9) \quad F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \; for \; i \in \{1, \ldots, p\},$$

$$(3.10) \quad v_j^* H_j(S^*) = 0 \; for \; j \in \{1, \ldots, m\}.$$

Suppose, in addition, that any one of the following assumptions holds:

(i) $A_i(\cdot;\lambda^*,u^*)$ $(\forall\ i = 1,\dots,p)$ *are* (ρ,η)*-pseudo-invex at* S^* *and* $B_j(\cdot;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are* (ρ,η)*-quasi-invex at* S^**.*

(ii) $A_i(\cdot;\lambda^*,u^*)$ $(\forall\ i \in \{1,\dots,p\}$ *are* (ρ,η)*-prestrictly-pseudo-invex at* S^* *and* $B_j(\cdot;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are* (ρ,η)*-prestrictly-quasi-invex at* S^**.*

(iii) $A_i(\cdot;\lambda^*,u^*)$ $(\forall\ i \in \{1,\dots,p\}$ *are* (ρ,η)*-prestrict-quasi-invex at* S^* *and* $B_j(\cdot;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are* (ρ,η)*-prestrictly-pseudo-invex at* S^**.*

Then S^* *is an efficient solution to (P).*

Note that for $\rho = 0$ in Theorem 3.1, it reduces to the result of Mishra *et al.* [7] on the quasi-invexity and pseudo-invexity.

Theorem 3.3 ([7, Theorem 3.1]). *Let* $S^* \in \Xi$*, let* F_i, G_i*,* $i \in \{1,\dots,p\}$*, and* H_j*,* $j \in \{1,\dots,m\}$*, are differentiable at* $S^* \in \Lambda$*, and let there exist* $u^* \in U$ *and* $v^* \in R^m_+$ *such that*

(3.11) $\langle \Sigma_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)] + \Sigma_{j=1}^{m} v_j^* H_j'(S^*), \eta(S,S^*)\rangle \geq 0 \,\forall\, S \in \Lambda^n,$

(3.12) $F_i(S^*) - \lambda_i^* G_i(S^*) = 0\, for\ i \in \{1,\dots,p\},$

(3.13) $v_j^* H_j(S^*) = 0\, for\ j \in \{1,\dots,m\}.$

Suppose, in addition, that any one of the following assumptions holds:

(i) $A_i(\cdot;\lambda^*,u^*)$ $(\forall\ i = 1,\dots,p)$ *are pseudo-invex at* S^* *and* $B_j(\cdot;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are quasi-invex at* S^**.*

(ii) $A_i(\cdot;\lambda^*,u^*)$ $(\forall\ i \in \{1,\dots,p\}$ *are prestrictly-pseudo-invex at* S^* *and* $B_j(\cdot;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are prestrictly-quasi-invex at* S^**.*

(iii) $A_i(\cdot;\lambda^*,u^*)$ $(\forall\ i \in \{1,\dots,p\}$ *are prestrict-quasi-invex at* S^* *and* $B_j(\cdot;\lambda^*,u^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are prestrictly-pseudo-invex at* S^**.*

Then S^* *is an efficient solution to* (P).

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