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Research Article

The Behavior of Maximum Outcomes in Symmetric R&D Networks

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Abstract. We consider the network game for firms conduct research and development (R&D) to reduce marginal costs of their products. We complete the work by Goyal and Moraga-Gonzalez [6] who introduced a mathematical model to describe R&D partnerships between firms. Our additional results focus on the behavior of the equilibrium outcomes as functions of cooperation activities. We determine optimal activities to have optimal outcomes and study change rates with each new cooperation.

Keywords. Symmetric networks; Maximization; Optimal levels; Equilibria

MSC. 91A13; 91A20; 91A43

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1. Introduction

There has been substantial progress in research in economics networks reported in both empirical and theoretical literature. There are massive examples, including job-contacts, sellers and buyers, diffusion of the knowledge between agents and other relationships that link agents in a market. Among these relationships is R&D cooperation that has been developed in numerous papers. One of the new approaches introduced in this field is the network concept. The idea is that the R&D organization can be described as a network where players (firms) are represented by nodes and the R&D partnerships (cooperation) are represented by links [2, 4, 6, 7, 11].

We use a network game given by Goyal and Moraga-Gonzalez [6]. The game consists of three stages: partnerships selection, R&D investment and market competition. The network formation is in the first stage as bilateral agreements between firms. The whole system in the end forms a network called an R&D cooperation network. The effective effort of each firm depends on the individual effort and efforts of other firms in a market. Specifically, if firms cooperate in R&D, they are linked in the network; otherwise there is a free R&D spillover to ensure knowledge flow between non-cooperating firms. If numbers of links of firms are equal, the network is called symmetric (regular); otherwise it is referred to as an asymmetric network.

Goyal and Moraga-Gonzalez investigated the effect of the cooperative links on the R&D effort of firms and on their incentives to build R&D relationships. They also investigated the situations in which the conflict between the individual and social desires from forming R&D partnerships occurs. The authors did their study for two market structures: independent and homogeneous products. The findings suggest that the equilibrium outcomes is managed by the market structure. For independent products case, the R&D effort and profit and social welfare increase as firms form R&D partnerships. For homogeneous products case, the opposite occurs in terms of the effort, where it is maximized if all firms stay isolated in the network. In terms of the profit and social welfare, each of them is maximized at an intermediate cooperation activity.

The discussion in this paper is limited to symmetric networks. This is because for asymmetric interactions, the equilibria cannot be generalized where they vary according to the network structure. Also, with increasing number of firms, the number and complexity of the network will increase and this makes the presentation and analysis of the outcomes difficult. Moreover, the discussion focuses on the homogeneous products since the equilibrium outcomes in the independent products case are maximized with the cooperation activities.

The outcomes of this paper can be summarized as follows. Firstly, for each market size, the production quantity is maximized at the half of the cooperation activity. Secondly, the production quantity is symmetric around the optimal level. This indicates that the low and intense R&D cooperative organizations do not encourage the production quantity. Also, there is a state of balance in the production quantity with respect to the cooperation activity. This state is not realized in the profit or social welfare case. Finally, the regular increase in the cooperation activity does not guarantee a constant change in the equilibrium outcomes. This refers that the regular developing in the R&D network does not provide steady changes that can be expected from one R&D organization.

This paper is organized as follows. In section 2, we review issues of the social network and introduce the network model by Goyal and Moraga-Gonzalez model. Then, we review some terminologies in economics. In section 3, we provide our outcomes. In section 4, we conclude our study.

2. Background

2.1 Players and Networks

A *network* is a set of objects (called nodes or vertices) that are connected together by the edges or links [8]. In mathematics, networks are often referred as graphs. For the purpose of this article, we focus on undirected networks; meaning that each link between any two vertices runs in both directions. We also focus on simple networks that have neither parallel edges (edges that have the same end vertices) nor loops (edges where their start and end vertices are the same). We define N as a set of all vertices labeled by letters $i, j, k, \ldots$ where $|N| = n$ and $E = \{ij, jk, \ldots\}$ is a set of all edges in the network where $|E| = m$ is the number of links. Then $G(N,E)$ denotes an undirected network with nodes N and links E, and for simplicity the network is denoted by G.

Any network G can be represented by an $n \times n$ adjacency symmetric matrix A with elements 0 or 1, depending on whether or not nodes are linked. More formally, each element a_{ij} of the adjacency matrix A, we have $a_{ij} = 1$ if $ij \in E$; otherwise $a_{ij} = 0$. For an undirected network the adjacency matrix, A is symmetric. Nodes linked to node $i \in N$ is defined as a set of *neighbors* of that node: $N_i = \{j \in N : ij \in E\}$. The length of the neighbors' set of node i is a *degree* of that node [10]. Thus, the degree of each node $i \in N$ is denoted by $\deg(i) = |N_i|$ where $0 \le \deg(i) \le n-1$. The *density* of network G gives a ratio of actual links in the network out of possible links $D(G) = 2m/n(n-1)$ where n and m are numbers of firms and links, respectively.

A *symmetric network* is a graph in which each player has the same number of links. If G is a symmetric network such that each play has k links, G is called a symmetric network of degree k. An *empty network* E_n (a graph with no links between players) is an example of the symmetric networks. Also, a *complete network* K_n (a graph such that each two nodes are linked) is a symmetric network of degree $n-1$. A cycle network C_n (a graph contains a single cycle through all nodes) is symmetric of degree 2.

2.2 The Model

The emphasis in this paper is on the linear-quadratic function of consumers given by [1] and [5]:

$$U = a\sum_{i=1}^{n} q_i - \frac{1}{2}\left(\alpha \sum_{i=1}^{n} q_i^2 + 2\lambda \sum_{j\neq i} q_i q_j\right) + I \ . \tag{2.1}$$

Here the demand parameters $a > 0$ denotes the willingness of consumers to pay and $\alpha > 0$ is the diminishing marginal rate of consumption, while q_i is the quantity consumed of product i and I measures the consumer's consumption of all other products. Without loss of generality, it is assumed that $\alpha = 1$ to simplify the analysis. The parameter $-1 \le \lambda \le 1$ captures the marginal rate of substitution between different products. If $\lambda < 0$ ($\lambda > 0$), the products are complements (substitutes). Also, if $\lambda = 0$ ($\lambda = 1$), the products are independent (homogeneous).

If the consumer buys q_i of good i where m is a consumer's income and p_i is the price of good i, the money spent is $p_i q_i$ and the balance is $I = m - p_i q_i$. By substituting into (2.1), we determine the optimal consumption of good i by calculating $\frac{\partial U}{\partial q_i} = a - q_i - \lambda \sum_{j\neq i} q_j - p_i = 0$. This implies the inverse demand function for each good i

$$D_i^{-1} = p_i = a - q_i - \lambda \sum_{j\neq i} q_j, \quad i = 1, \ldots, n \ . \tag{2.2}$$

The profit π_i for firm i is

$$\pi_i = (p_i - c_i)q_i = \left(a - q_i - \lambda \sum_{j\neq i}^{n} q_j - c_i\right) q_i \ , \tag{2.3}$$

where p_i is the price of good i produced by firm i and c_i is the production cost.

The total welfare is expressed as

$$TW = \frac{(1-\lambda)}{2}\sum_{i=1}^{n} q_i^2 + \frac{\lambda}{2}\left(\sum_{i=1}^{n} q_i\right)^2 + \sum_{i=1}^{n}\pi_i \ . \tag{2.4}$$

2.3 R&D Network Model

The focus of this paper is on Goyal and Moraga-Gonzalez model. In their model, if firms cooperate in R&D, they are linked in an undirected network and spillover is set at one where the cost of link formation is assumed to be negligible. If firms do not cooperate, they are not linked and there is a spillover ($\beta \in [0,1)$) between non-linked firms.

Goyal and Moraga-Gonzalez studied two general cases of networks: symmetric and asymmetric networks [6]. For symmetric networks, the spillover term between non-linked firms is set at zero. This enabled them to identify the effect of the cooperation activity on the equilibrium outcomes. For asymmetric networks, the spillover term between non-linked firms is involved. For the asymmetric networks, the authors considered three firms in a market and this is due to difficulty computing the equilibrium for a large number of firms, in addition to the multiplicity of the results as a result of the connectivity changes.

(1) Symmetric R&D network

The discussion in this paper is limited to symmetric networks. This is because for asymmetric interactions, the equilibria cannot be generalized where they vary according to the network structure. Also, with increasing the market size n, the number of involved networks will increase and this makes the presentation and analysis of the outcomes difficult. In a symmetric network of degree k, all firms have an identical number of links k.

(2) Stages of the model

In Goyal and Moraga-Gonzalez paper, firms strategically form bilateral collaborative links with other firms where the collaboration of firms is modeled as a three-stage game:

The first stage: Each firm chooses its research partners (network formation). The cooperation in the symmetric networks is called activity levels.

The second stage: Given the R&D network, each firm chooses the amounts of investment (effort) in R&D simultaneously and independently in order to reduce the cost of production.

The third stage: Given the R&D investments of each firm, firms compete in the product market by setting quantities (Cournot competition) in order to maximize their profits.

(3) Cost reduction

According to Goyal and Moraga-Gonzalez paper, the effective R&D effort for each firm is defined by the following equation:

$$X_i = x_i + \sum_{j\in N_i} x_j + \beta \sum_{k\notin N_i} x_k, \quad i = 1,\dots,n \ , \tag{2.5}$$

where x_i denotes R&D effort of firm i, N_i is the set of firms participating in a joint venture with firm i and $\beta \in [0,1)$ is an exogenous parameter that captures knowledge spillovers acquired from firms not engaged in a joint venture with firm i. The effective R&D effort reduces firm i's marginal cost ($\overline{c}$) of production

$$c_i = \overline{c} - x_i - \sum_{j\in N_i} x_j - \beta \sum_{k\notin N_i} x_k, \quad i = 1,\dots,n \,. \tag{2.6}$$

The effort is assumed to be costly and the function of the cost is quadratic, so that the cost of R&D is γx_i^2, where $\gamma > 0$ indicates the effectiveness of R&D expenditure [3]. The profit π_i for firm i is the difference between revenue and production cost minus the cost of R&D

$$\pi_i = \left(a - \sum_{i=1}^{n} q_i - \overline{c} + x_i + \sum_{j\in N_i} x_j + \beta \sum_{k\notin N_i} x_k\right) q_i - \gamma x_i^2 \quad i = 1,\dots,n \,, \tag{2.7}$$

where the marginal cost satisfies $a > \overline{c}$.

(4) Equilibria for symmetric networks

We assume that the marginal cost $\overline{c}$ is constant and equal for all firms. We also assume that for each firm $i \in N$, $\deg(i) = k$. The sub-game perfect Nash equilibrium is identified by using backwards induction. Under homogeneous Cournot competition, we show the final list of the equilibria and the detail is given in [6] paper.

$$\textbf{R\&Deffort}: \qquad x^* = \frac{(n-k)(a-\overline{c})}{\gamma(n+1)^2 - (k+1)(n-k)}\,, \tag{2.8}$$

$$\textbf{ProductionCost}: \qquad c^* = \frac{\overline{c}\gamma(n+1)^2 - a(n-k)(k+1)}{\gamma(n+1)^2 - (k+1)(n-k)}\,, \tag{2.9}$$

$$\textbf{Quantity}: \qquad q^* = \frac{\gamma(n+1)(a-\overline{c})}{\gamma(n+1)^2 - (k+1)(n-k)}\,, \tag{2.10}$$

$$\textbf{Profit}: \qquad \pi^* = \frac{\gamma\left[\gamma(n+1)^2 - (n-k)^2\right](a-\overline{c})^2}{\left[\gamma(n+1)^2 - (k+1)(n-k)\right]^2}\,, \tag{2.11}$$

$$\textbf{Totalwelfare}: \qquad TW^* = \frac{n\gamma\left[\gamma(n+1)^2(n+2) - 2(n-k)^2\right](a-\overline{c})^2}{\left[\gamma(n+1)^2 - (k+1)(n-k)\right]^2}\,. \tag{2.12}$$

Under the network game, the concepts of pairwise stability and efficiency are considered in this paper. The pairwise stability of the network depends on the profit of firm [9] as follows:

Definition 1 (Pairwise Stability)**.** For any network G to be stable, the following two conditions need to be satisfied for any two firms $i, j \in G$:

(1) If $ij \in G$, $\pi_i(G) \geq \pi_i(G - ij)$ and $\pi_j(G) \geq \pi_j(G - ij)$,

(2) If $ij \notin G$ and if $\pi_i(G) < \pi_i(G + ij)$, then $\pi_j(G) > \pi_j(G + ij)$,

$G - ij$ is the network resulting from deleting a link ij from the network G and $G + ij$ is the network resulting from adding a link ij to the network G. From this definition, network G is stable if no firm can obtain higher profit from deleting one of its links; and any other link between two firms would benefit only one of them.

The efficiency of the network is determined by comparing the total welfare of all possible networks generated from a certain number of firms.

Definition 2 (Network Efficiency)**.** Network G is said to be efficient if no other network $\acute{G}$ can be generated by adding or deleting links, such that $TW(\acute{G}) > TW(G)$.

3. The Outcomes

In this section, we show how the network structure shapes the equilibrium outcomes. Specifically, we study two related issues the behavior of the equilibria and the change rate of the outcomes with respect to the cooperation activity.

3.1 Behavior of the Equilibria under the Network Framework

Assume a market consists of a finite number of firms where each firm chooses R&D partners and the amount of its output to produce. The optimal value of the equilibrium outcomes is affected by several factors: market structure, marginal cost and number of firms. Since the focus of this paper is on homogeneous products for an identical marginal cost, the concern will be about the effect of the number of firms on the outcomes.

Proposition 1 states that the optimal cooperation activity to maximize the production quantity is equal to the half of firms in the network. This result will vary with the number type i.e., odd or even number. If the size of the market is odd, there is only one optimal cooperation activity, but if the size is even, there are two optimal activities.

Proposition 1. *With n firms in a market, assume the R&D partnerships form symmetric networks. If n is an odd (even) number, the optimal quantity of the production is at the cooperation activity (activities) $k^{q^*} = (n-1)/2$ ($k_1^{q^*} = (n-2)/2$ and $k_1^{q^*} = n/2$).*

The proof is given in the Appendix.

The following result states that growing the cooperation activity carries positive and negative impact on the production quantity. In particular, if the cooperation activity is lower than the optimal activity k^{q^*}, the optimal production quantity increases; however, if the activity is higher than the optimal activity, the quantity decreases.

Corollary 1. *With n firms in a market, assume the R&D partnerships form symmetric networks. The production quantity is symmetric about the optimal activity k^{q^*}.*

The proof is given in the Appendix.

Example 1. For seven and eight firms in a market, suppose the R&D partnerships form symmetric networks. Figure 2 shows the equilibrium quantity for symmetric networks with those two different sizes.

(1) As shown in the figure, if $n = 7$ ($n = 8$), the optimal cooperation activity (activities) $k^{q^*} = 3$ ($k^{q^*} = 3$ and $k^{q^*} = 4$).

(2) It can be observed that the equilibrium quantity is symmetric about the optimal level k^{q^*}.

The symmetry of the equilibrium quantity about the optimal cooperation activity is not observed if the R&D spillover is considered. This indicates that this external parameter has an effect on the behavior of the optimal quantity.

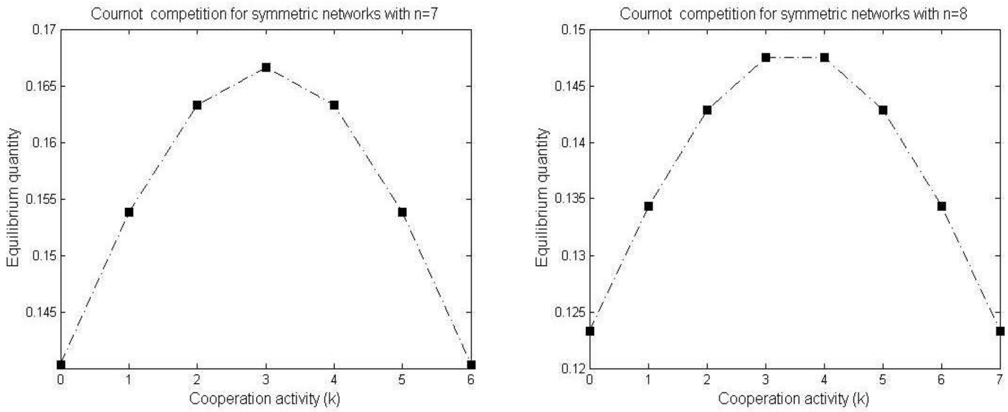

Figure 1. The cooperation activity at which the equilibrium quantity is maximized with $n = 7$ and $n = 8$. The parameters used to plot the figure are $a = 2$, $\overline{c} = 1$ and $\gamma = 1$.

Remark 1. With n firms in a market, assume the R&D partnerships form symmetric networks. The production quantity is not symmetric about the optimal activity k^{q^*} if the R&D spillover is applied.

Example 2. Assume four firms participate in R&D and the cooperation forms a symmetric network. Figure 2 shows all possible symmetric networks generated from four firms. Figure 3 displays the equilibrium quantity in those networks for some values of the spillover.

(1) If the spillover $\beta = 0$ (i.e., symmetric networks without spillover), the quantity is symmetric around the activity levels $k_1^{q^*} = 1$ and $k_2^{q^*} = 2$.

(2) If the spillover $\beta \neq 0$, the quantity is not symmetric.

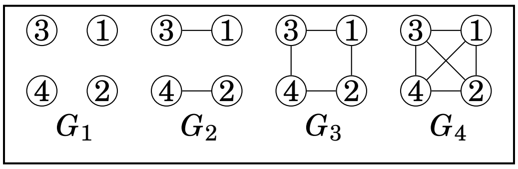

Figure 2. The symmetric networks with four firms.

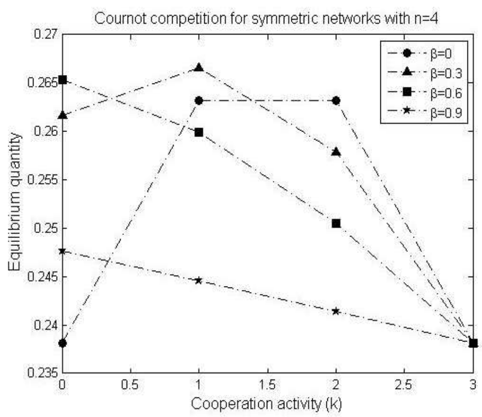

Figure 3. The equilibrium quantity in the networks given in Figure 3 for $\beta = 0$, $\beta = 0.3$, $\beta = 0.6$ and $\beta = 0.9$. The parameters used to plot the figure are $a = 2$, $\overline{c} = 1$ and $\gamma = 1$.

3.2 Comparison of the Behavior of the Equilibria

The behavior of the equilibrium outcomes with respect to the cooperation activity are different. Firstly, the R&D effort decreases as the cooperation activity increases [6]. This indicates that the R&D effort is maximized when the cooperation forms an empty network i.e., $k^{x^*} = 0$. Secondly, according to Goyal and Moraga-Gonzalez, the profit and total welfare are maximized at different intermediate cooperation activities k^{π^*} and k^{TW^*}. They stated that the activities $k^{\pi^*}, k^{TW^*} \in (0, n-1)$ and this indicates that the significant difference between the optimal activities at which the equilibria are maximized. While the effort are maximized at $k = 0$, the other equilibria (the production quantity, profit and total welfare) are maximized at different intermediate cooperation activities.

Example 3. Suppose two symmetric networks one consists of six firms and the other consists of eight firms. Table 1 shows the optimal cooperation activities at which the equilibria are maximized.

Table 1. The optimal cooperation activities for $n = 6$ and $n = 8$.

Optimal activity	k^{x^*}	k^{q^*}	k^{π^*}	k^{TW^*}
Six firms	0	2 & 4	4	3
Eight firms	0	3 & 4	5	4

In the following, we show that the profit and total welfare are not symmetric about their optimal activities. First, the profit and the total welfare functions can be rewritten as follows:

$$\pi^* = \gamma\left[\gamma(n+1)^2 - (n-k)^2\right](a-\overline{c})^2\left(\frac{1}{\phi^2}\right), \tag{3.1a}$$

$$TW^* = n\gamma\left[\gamma(n+1)^2(n+2) - 2(n-k)^2\right](a-\overline{c})^2\left(\frac{1}{\phi^2}\right), \tag{3.1b}$$

where $\phi = \gamma(n+1)^2 - (k+1)(n-k)$. From the first condition $\left(\frac{\partial\phi}{\partial k} = 0\right)$, we have $2k - n + 1 = 0$ and this implies $k = \frac{n-1}{2}$. For maximizing profit, the second derivative should satisfy $2(2\gamma(n+1)^2 - 2(k+1)(n-k) + (2k-n+1)^2) > 0$. This implies that the function $\frac{1}{\phi^2}$ is concave at the activity $k = \frac{n-1}{2}$. As we stated above the equilibrium quantity is maximized at that activity. For the profit and the total welfare functions (equations (3.1a) and (3.1b)), the numerator has a term containing k and because of that, the two functions are not necessary maximized at $k = \frac{n-1}{2}$. However, we found that the cooperation activity $k = \frac{n-1}{2}$ is the lower bound of k^{π^*} and k^{TW^*}.

In addition, the profit and the total welfare are not symmetric around the optimal activity level. This result can be proven by comparing the outcomes under the empty network and the complete network.

Proposition 2. *With n firms in a market, assume the R&D partnerships form symmetric networks. The profit and the total welfare are asymmetric about the cooperation activities k^{π^*} and k^{TW^*}.*

The proof is given in the Appendix.

Example 4. For two different sizes of a market $n = 10$ and $n = 20$, suppose the R&D partnerships form symmetric networks. Figure 4 shows the effort, the profit and the total welfare for all possible symmetric interactions generated from these sizes.

(1) The equilibrium outcomes are not symmetric around the activity levels k^{x^*}, k^{π^*} and k^{TW^*}.

(2) When all firms do not form cooperative links, the effort reaches the highest value; whereas the profit and the total welfare reach the lowest value.

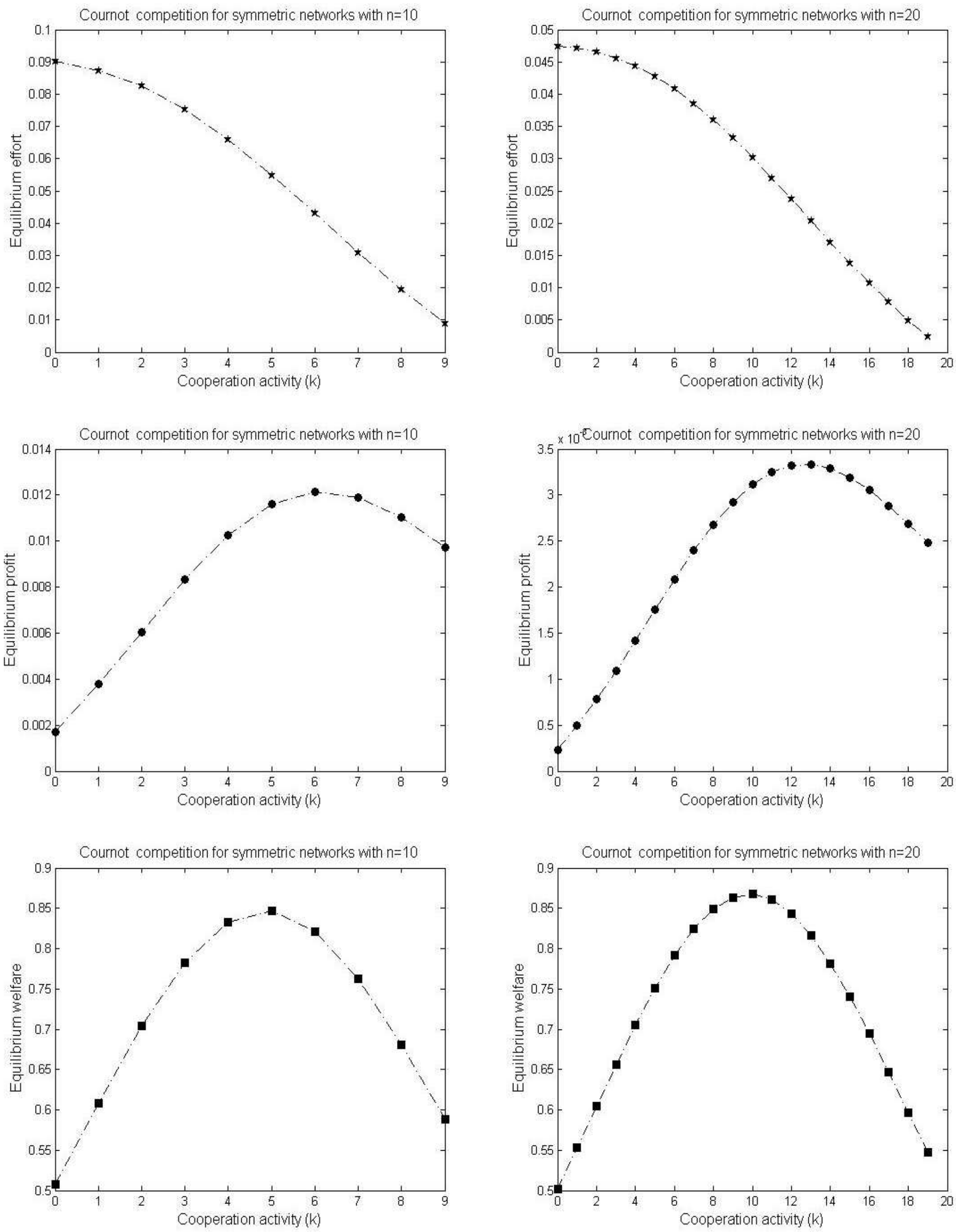

Figure 4. The effort, the profit and the total welfare with $n = 10$ and $n = 20$. The parameters used to plot the figures are $a = 2$, $\overline{c} = 1$ and $\gamma = 1$.

One can conclude that any two symmetric networks generated from any market size provide the same production quantity, but different outcomes for other equilibria. For example, consider the cases when all firms cooperate with each other (complete network) or stay without the cooperation (empty network). The production quantity in those networks is identical, but the complete network is stable. Also, the R&D effort is maximized in the empty network, but this network is not stable or efficient.

3.3 The Change Rate of the Equilibria with Respect to the Cooperation

According to our discussion in the previous sections, the equilibrium outcomes are affected by increasing the density of the network.

From our results in the previous section and from the results of [6], we found that the equilibrium outcomes are affected by the growing density of the network. The question that can be raised here is that does the uniform increase in the cooperation activity k cause a constant change in the equilibrium outcomes? The following proposition answers this question for any market size.

Proposition 3. *With n firms in a market, assume the R&D partnerships form symmetric networks. The change rate of the equilibria with respect to the cooperation activity is not constant.*

The proof is given in the Appendix.

4. Conclusion

In this paper, we used an R&D network model to describe the behavior of the equilibrium outcomes with respect to the cooperation activity. We found that for each market size, the middle cooperation activity is the optimal activity at which the quantity is maximized. We also found that the production quantity is symmetric around the optimal level. Moreover, we found that the regular growth of the cooperation activity does not generate a constant change in the equilibrium outcomes.

Appendix

Proof of Proposition 1. Since the numerator of the quantity function (equation (2.10)) does not depend on the activity level k, we want to show that the denominator is minimized at $0 < k_1^{q^*} = (n-1)/2 < n-1$.

Assume ϕ is the denominator of the quantity function. We want to prove that for each network size n, the function ϕ is convex on its domain $[0, n-1]$. From the first derivative,

$$\frac{\partial \phi}{\partial k} = 0 \quad \Rightarrow \quad 2k - n + 1 = 0 \quad \Rightarrow \quad k = \frac{n-1}{2}\,.$$

The second derivative $\partial^2\phi/\partial k^2 = 2$ is always positive. This means that the function ϕ is convex on $[0, n-1]$ and its lowest value is at $(n-1)/2$. This implies the function $1/\phi$ is concave and its maximum value is at $(n-1)/2$.

From equation (2.10), we can rewrite the quantity function as follows

$$q^* = \gamma(n+1)(a-\overline{c})\left[\frac{1}{\gamma(n+1)^2 - (k+1)(n-k)}\right] = \gamma(n+1)(a-\overline{c})\left(\frac{1}{\phi}\right). \tag{4.1}$$

This indicates that the quantity function is maximized at $k^{q^*} = (n-1)/2$ as the function $1/\phi$.

If the network size n is odd, the optimal activity level $k^{q^*} = (n-1)/2$ for each firm can be drawn and this means that the quantity is maximized at k^{q^*}. However, if n is even, k^{q^*} will not be an integer number. Therefore, for an even number n, the equilibrium quantity is maximized at two activity levels $k_1^{q^*} = (n-2)/2$ and $k_2^{q^*} = n/2$ where the equilibrium quantity at the two levels are equal. □

Proof of Corollary 1. To prove this proposition, we need to show that the function ϕ is symmetric around the optimal level k^{q^*}. If n is odd, then $k^{q^*} = (n-1)/2$ is the activity level at which the function ϕ is minimized. Take two activity levels lower and higher than k^{q^*} i.e., $k_1 = (n-2)/3$ and $k_2 = (n+1)/2$. By substituting the activity levels k_1 and k_2 into ϕ, we have

$$\phi(k_1) = \gamma(n+1)^2 - \frac{(n-1)(n+3)}{4} = \phi(k_2)\,.$$

We have the same result if we take activity levels from $[0, k^{q^*})$ and their corresponding by the symmetry around k^{q^*} (i.e., from $[k^{q^*}, n-1]$. If n is even, we have the same result where $\phi(k_1^{q^*}) = \phi(k_2^{q^*})$. This implies that the function ϕ is symmetric around the optimal level k^{q^*}. Hence, from equation (2.10), the equilibrium quantity is symmetric around the optimal activity level k^{q^*}. □

Proof of Proposition 2. For the profit function, it is sufficient to show that the values of the function (2.11) under the empty and the complete network (i.e., at $k = 0$ and $k = n-1$) are different. When substituting the activity levels $k = 0$ and $k = n-1$ into the profit function, we have

$$\pi^*_{k=0} = \frac{\gamma[\gamma(n+1)^2 - n^2](a-\bar{c})^2}{\left(\gamma(n+1)^2 - n\right)^2}\,,$$

$$\pi^*_{k=n-1} = \frac{\gamma[\gamma(n+1)^2 - 1](a-\bar{c})^2}{\left(\gamma(n+1)^2 - n\right)^2}\,.$$

For comparison, calculate

$$\pi^*_{k=n-1} - \pi^*_{k=0} = \frac{\gamma(n^2-1)(a-\bar{c})^2}{\left(\gamma(n+1)^2 - n\right)^2} > 0$$

This implies that $\pi^*_{k=n-1} > \pi^*_{k=0}$. Similarly, we prove that the total welfares at $k = 0$ and $k = n-1$ are different. By substituting the activity levels $k = 0$ and $k = n-1$ into (2.12), we have

$$TW^*_{k=0} = \frac{n\gamma[\gamma(n+1)^2(n+2) - 2n^2](a-\bar{c})^2}{2\left(\gamma(n+1)^2 - n\right)^2}\,,$$

$$TW^*_{k=n-1} = \frac{n\gamma[\gamma(n+1)^2(n+2) - 2](a-\bar{c})^2}{2\left(\gamma(n+1)^2 - n\right)^2}\,.$$

By comparing the two equilibria, we have

$$TW^*_{k=n-1} - TW^*_{k=0} = \frac{n\gamma(n^2-1)(a-\bar{c})^2}{\left(\gamma(n+1)^2 - n\right)^2} > 0$$

This implies that $TW^*_{k=n-1} > TW^*_{k=0}$. □

Proof of Proposition 3. To prove the proposition, we need to calculate the derivative of the equilibria with respect to the activity level k.

$$\frac{\partial x^*}{\partial k}=\frac{\left[(1-\gamma)n^2-2(\gamma+k)n+k^2-\gamma\right](a-\overline{c})}{\left(\gamma(n+1)^2-(n-k)(k+1)\right)^2}$$

$$\frac{\partial q^*}{\partial k}=\frac{(n+1)(n-2k+1)(a-\overline{c})}{\left(\gamma(n+1)^2-(n-k)(k+1)\right)^2}$$

$$\frac{\partial \pi^*}{\partial k}=\frac{\left[2\gamma(n-k)\left(\gamma(n+1)^2-(k+1)(n-k)\right)-2\gamma(2k-n+1)\left(\gamma(n+1)^2-(n-k)\right)\right](a-\overline{c})^2}{\left(\gamma(n+1)^2-(n-k)(k+1)\right)^3},$$

$$\frac{\partial TW^*}{\partial k}=\frac{4n\gamma\left[(n-k)+(n-2k+1)\left(\gamma(n+1)^2(n+2)-2(n-k)^2\right)\right](a-\overline{c})^2}{\left(\gamma(n+1)^2-(n-k)(k+1)\right)^3}.$$

Since the derivative of each equilibrium is a function depends on the activity level k, then the change of the equilibrium with respect to the change of k is not fixed. □

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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