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(*Invited paper*)

A Sequence of Inequalities among Difference of Symmetric Divergence Measures

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Abstract. In this paper we have considered two one parametric generalizations. These two generalizations have in particular the well known measures such as: *J-divergence*, *Jensen-Shannon divergence* and *arithmetic-geometric mean divergence*. These three measures are with logarithmic expressions. Also, we have particular cases the measures such as: *Hellinger discrimination*, *symmetric* χ^2*-divergence*, and *triangular discrimination*. These three measures are also well-known in the literature of statistics, and are without logarithmic expressions. Still, we have one more non logarithmic measure as particular case calling it *d-divergence*. These seven measures bear an interesting inequality. Based on this inequality, we have considered different *difference of divergence measures* and established a sequence of inequalities among themselves.

1. Introduction

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \dots, p_n) \;\middle|\; p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geq 2,$$

be the set of all complete finite discrete probability distributions. For all $P, Q \in \Gamma_n$, the following measures are well known in the literature on information theory and statistics:

- **Hellinger Discrimination**

$$h(P \,\|\, Q) = 1 - B(P \,\|\, Q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2,$$

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where

$$B(P\,\|\,Q)=\sqrt{p_iq_i},$$

is the well-known Bhattacharyya *coefficient*.

- **Triangular Discrimination**

$$\Delta(P\,\|\,Q)=2[1-W(P\,\|\,Q)]=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i},$$

where

$$W(P\,\|\,Q)=\sum_{i=1}^{n}\frac{2p_iq_i}{p_i+q_i},$$

is the well-known *harmonic mean divergence*.

- **Symmetric Chi-square Divergence**

$$\Psi(P\,\|\,Q)=\chi^2(P\,\|\,Q)+\chi^2(Q\,\|\,P)=\sum_{i=1}^{n}\frac{(p_i-q_i)^2(p_i+q_i)}{p_iq_i},$$

where

$$\chi^2(P\,\|\,Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}=\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1,$$

is the well-known χ^2-*divergence*.

- **J-Divergence**

$$J(P\,\|\,Q)=\sum_{i=1}^{n}(p_i-q_i)\ln(\frac{p_i}{q_i}).$$

- **Jensen-Shannon Divergence**

$$I(P\,\|\,Q)=\frac{1}{2}\left[\sum_{i=1}^{n}p_i\ln\left(\frac{2p_i}{p_i+q_i}\right)+\sum_{i=1}^{n}q_i\ln\left(\frac{2q_i}{p_i+q_i}\right)\right].$$

- **Arithmetic-Geometric Mean Divergence**

$$T(P\,\|\,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\ln\left(\frac{p_i+q_i}{2\sqrt{p_iq_i}}\right).$$

Originally, the *J-divergence* is due to Jeffreys [4]. The measure *Jensen-Shannon divergence* is due to Sibson [5]. Later, Burbea and Rao [2] studied it extensively. The *arithmetic and geometric mean divergence* is due to Taneja [7]. Detailed study of these measures can be seen in Taneja [7, 9, 10]. We call the above six measures *symmetric divergence measures*, since they are symmetric with respect to the probability distributions P and Q. The author [9, 10] obtained an inequality among these six symmetric divergence measures given by

$$\frac{1}{4}\Delta(P\,\|\,Q)\le I(P\,\|\,Q)\le h(P\,\|\,Q)\le\frac{1}{8}J(P\,\|\,Q)\le T(P\,\|\,Q)\le\frac{1}{16}\Psi(P\,\|\,Q).\quad(1)$$

By defining the nonnegative differences among the divergence measures appearing in (1), the author [10] improved the above result (1) obtaining the following sequence of inequalities:

$$\begin{aligned} D_{I\Delta}(P\,||\,Q) &\le \frac{2}{3}D_{h\Delta}(P\,||\,Q) \le \frac{1}{2}D_{J\Delta}(P\,||\,Q) \le \frac{1}{3}D_{T\Delta}(P\,||\,Q) \le D_{TJ}(P\,||\,Q) \\ &\le \frac{2}{3}D_{Th}(P\,||\,Q) \le 2D_{Jh}(P\,||\,Q) \le \frac{1}{6}D_{\Psi\Delta}(P\,||\,Q) \le \frac{1}{5}D_{\Psi I}(P\,||\,Q) \\ &\le \frac{2}{9}D_{\Psi h}(P\,||\,Q) \le \frac{1}{4}D_{\Psi J}(P\,||\,Q) \le \frac{1}{3}D_{\Psi T}(P\,||\,Q), \end{aligned} \tag{2}$$

where, for example, $D_{\Psi T}(P\,||\,Q) = \frac{1}{16}\Psi(P\,||\,Q) - T(P\,||\,Q)$, and similarly others. Still, we have

$$\frac{2}{3}D_{h\Delta}(P\,||\,Q) \le 2D_{hI}(P\,||\,Q) \le D_{TJ}(P\,||\,Q). \tag{3}$$

The proof of the inequalities (1)-(3) is based on the following two lemmas:

Lemma 1.1. *If the function $f : [0,\infty) \to \mathrm{R}$ is convex and normalized, i.e., $f(1) = 0$, then the* f-divergence, *$C_f(P\,||\,Q)$ given by*

$$C_f(P\,||\,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{4}$$

is nonnegative and convex in the pair of probability distribution $(P,Q) \in \Gamma_n \times \Gamma_n$.

Lemma 1.2. *Let $f_1, f_2 : I \subset \mathrm{R}_+ \to \mathrm{R}$ two generating mappings are normalized, i.e., $f_1(1) = f_2(1) = 0$ and satisfy the assumptions:*

(i) *f_1 and f_2 are twice differentiable on (a,b);*

(ii) *there exists the real constants m, M such that $m < M$ and*

$$m \le \frac{f_1''(x)}{f_2''(x)} \le M, \quad f_2''(x) > 0, \quad \forall\ x \in (a,b),$$

then we have the inequalities:

$$mC_{f_2}(P\,||\,Q) \le C_{f_1}(P\,||\,Q) \le MC_{f_2}(P\,||\,Q). \tag{5}$$

The measure (5) is the well-known *Csisz r's f-divergence*. The Lemma 1.1 is due to Csisz r [3] and the Lemma 1.2 is due to author [10]. Some applications of Lemma 1.1 can be seen in Taneja and Kumar [8].

2. Generalized Symmetric Divergence Measures

Let us consider the measure

$$\zeta_s(P\,||\,Q) = \begin{cases} J_s(P\,||\,Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n}(p_i^s q_i^{1-s} + p_i^{1-s}q_i^s) - 2\right], & s \neq 0,1 \\ J(P\,||\,Q) = \sum_{i=1}^{n}(p_i - q_i)\ln\left(\frac{p_i}{q_i}\right), & s = 0,1 \end{cases} \tag{6}$$

for all $P, Q \in \Gamma_n$.

The measure (6) is *generalized J-divergence* or *J-divergence of type s* and is extensively studied in Taneja [7, 10]. The expression (6) admits the following particular cases:

(i) $\zeta_{-1}(P\,||\,Q) = \zeta_2(P\,||\,Q) = \frac{1}{2}\Psi(P\,||\,Q)$,

(ii) $\zeta_0(P\,||\,Q) = \zeta_1(P\,||\,Q) = J(P\,||\,Q)$,

(iii) $\zeta_{1/2}(P\,||\,Q) = 8\,h(P\,||\,Q)$,

where $\Psi(P\,||\,Q)$, $J(P\,||\,Q)$ and $h(P\,||\,Q)$ are as given in section 1.

Let us consider now the another measure

$$\xi_s(P\,||\,Q) = \begin{cases} IT_s(P\,||\,Q) = [s(s-1)t]^{-1}\left[\sum_{i=1}^n \left(\frac{p_i^s+q_i^s}{2}\right)\left(\frac{p_i+q_i}{2}\right)^{1-s} - 1\right], & s \neq 0,1 \\ I(P\,||\,Q) = \frac{1}{2}\left[\sum_{i=1}^n p_i \ln\left(\frac{2p_i}{p_i+q_i}\right) + \sum_{i=1}^n q_i \ln\left(\frac{2q_i}{p_i+q_i}\right)\right], & s = 1 \\ T(P\,||\,Q) = \sum_{i=1}^n \left(\frac{p_i+q_i}{2}\right)\ln\left(\frac{p_i+q_i}{2\sqrt{p_iq_i}}\right), & s = 0 \end{cases} \tag{7}$$

for all $P, Q \in \Gamma_n$.

The measure (7) is new in the literature and is studied for the first time by Taneja [9]. It is called *generalized arithmetic and geometric mean divergence measure.* The measure (7) admits the following particular cases:

(i) $\xi_{-1}(P\,||\,Q) = \frac{1}{4}\Delta(P\,||\,Q)$.

(ii) $\xi_1(P\,||\,Q) = I(P\,||\,Q)$.

(iii) $\xi_{1/2}(P\,||\,Q) = 4\,d(P\,||\,Q)$.

(iv) $\xi_0(P\,||\,Q) = T(P\,||\,Q)$.

(v) $\xi_2(P\,||\,Q) = \frac{1}{16}\Psi(P\,||\,Q)$.

where $\Delta(P\,||\,Q)$, $I(P\,||\,Q)$, $T(P\,||\,Q)$ and $\Psi(P\,||\,Q)$ are as given in section 1. The measure $d(P\,||\,Q)$ appearing in particular case (iii) is given by

$$d(P\,||\,Q) = 1 - \sum_{i=1}^n \left(\frac{\sqrt{p_i}+\sqrt{q_i}}{2}\right)\left(\sqrt{\frac{p_i+q_i}{2}}\right). \tag{8}$$

For simplicity, we call the measure (8) as *d-divergence*. Thus we observe that when we take $s = -1, 0, \frac{1}{2}, 1$ and 2 in (6) and (7), we have *seven* particular cases. The measure $\Psi(P\,||\,Q)$ appears as a particular case in both the measures (6) and (7). An inequality among these seven measures is given by

$$\begin{aligned} &\frac{1}{4}\Delta(P\,||\,Q) \le I(P\,||\,Q) \le h(P\,||\,Q) \le 4\,d(P\,||\,Q) \\ &\qquad \le \frac{1}{8}J(P\,||\,Q) \le T(P\,||\,Q) \le \frac{1}{16}\Psi(P\,||\,Q). \end{aligned} \tag{9}$$

Results appearing in (2) and (3) are based on the inequalities given in (1). In this paper our aim is improve the results given in (2)-(3) and obtain a new sequence of inequalities based on the expression (9).

3. **Difference of Divergence Measures and their Convexity**

The inequality (9) admits many nonnegative difference than the one given (2) and (3), but we shall consider only those having the expression $d(P\,\|\,Q)$. These *nonnegative* differences are given by

$$D_{\Psi d}(P\,\|\,Q)=\frac{1}{16}\Psi(P\,\|\,Q)-4d(P\,\|\,Q),$$

$$D_{Td}(P\,\|\,Q)=T(P\,\|\,Q)-4d(P\,\|\,Q),$$

$$D_{Jd}(P\,\|\,Q)=\frac{1}{8}J(P\,\|\,Q)-4d(P\,\|\,Q),$$

$$D_{dh}(P\,\|\,Q)=4d(P\,\|\,Q)-h(P\,\|\,Q),$$

$$D_{dI}(P\,\|\,Q)=4d(P\,\|\,Q)-I(P\,\|\,Q),$$

and

$$D_{d\Delta}(P\,\|\,Q)=4d(P\,\|\,Q)-\frac{1}{4}\Delta(P\,\|\,Q).$$

Here below we shall prove the convexity of the above *six* measures. The proof is based on the Lemma 1.1. Initially, we shall give the convexity of the measures (6) and (7).

Property 3.1. (i) *The measure $\zeta_s(P\,\|\,Q)$ is* nonnegative *and* convex *in the pair of probability distributions $(P,Q)\in\Gamma_n\times\Gamma_n$ for all $s\in(-\infty,\infty)$.*

(ii) *The measure $\xi_s(P\,\|\,Q)$ is* nonnegative *and* convex *in the pair of probability distributions $(P,Q)\in\Gamma_n\times\Gamma_n$ for all $s\in(-\infty,\infty)$.*

Proof. (i) For all $x>0$ and $s\in(-\infty,\infty)$, let us consider

$$\phi_s(x)=\begin{cases}[s(s-1)]^{-1}[x^s+x^{1-s}-(1+x)], & s\neq 0,1,\\ (x-1)\ln x, & s=0,1,\end{cases}$$

in (4), then we have $C_f(P\,\|\,Q)=\zeta_s(P\,\|\,Q)$, where $\zeta_s(P\,\|\,Q)$ is given by (6).

Moreover,

$$\phi_s'(x)=\begin{cases}[s(s-1)]^{-1}[s(x^{s-1}+x^{-s})+x^{-s}-1], & s\neq 0,1\\ 1-x^{-1}+\ln x, & s=0,1\end{cases},$$

and

$$\phi_s''(x)=x^{s-2}+x^{-s-1}. \tag{10}$$

Thus we have $\phi_s''(x)>0$ for all $x>0$, and hence, $\phi_s(x)$ is convex for all $x>0$. Also, we have $\phi_s(1)=0$. In view of this we can say that the *J-divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$ for any $s \in (-\infty, \infty)$.

For all $x > 0$ and $s \in (-\infty, \infty)$, let us consider

$$\psi_s(x) = \begin{cases} [s(s-1)]^{-1} \left[\left(\frac{x^{1-s}+1}{2}\right) \left(\frac{x+1}{2}\right)^s - \left(\frac{x+1}{2}\right) \right], & s \neq 0,1, \\ \frac{x}{2} \ln x - \left(\frac{x+1}{2}\right) \ln \left(\frac{x+1}{2}\right), & s = 0, \\ \left(\frac{x+1}{2}\right) \ln \left(\frac{x+1}{2\sqrt{x}}\right), & s = 1, \end{cases}$$

in (4), then we have $C_f(P \| Q) = \xi_s(P \| Q)$, where $\xi_s(P \| Q)$ is as given by (7).

Moreover,

$$\psi_s'(x) = \begin{cases} (s-1)^{-1} \left[\frac{1}{s} \left[\left(\frac{x+1}{2x}\right)^s - 1 \right] - \frac{x^{-s}-1}{4} \left(\frac{x+1}{2}\right)^{s-1} \right], & s \neq 0,1, \\ -\frac{1}{2} \ln \left(\frac{x+1}{2x}\right), & s = 0, \\ 1 - x^{-1} - \ln x - 2 \ln \left(\frac{2}{x+1}\right), & s = 1, \end{cases}$$

and

$$\psi_s''(x) = \left(\frac{x^{-s-1}+1}{8}\right) \left(\frac{x+1}{2}\right)^{s-2}. \tag{11}$$

Thus we have $\psi_s''(x) > 0$ for all $x > 0$, and hence, $\psi_s(x)$ is convex for all $x > 0$. Also, we have $\psi_s(1) = 0$. In view of this we can say that *AG and JS-divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$ for any $s \in (-\infty, \infty)$. □

Property 3.2. (i) *The measure $\zeta_s(P \| Q)$ is monotonically increasing in s for all $s \geq \frac{1}{2}$ and decreasing in $s \leq \frac{1}{2}$.*
(ii) *The measure $\xi_s(P \| Q)$ is monotonically increasing in s for all $s \geq -1$.*

The proof of the Properties 3.1 and 3.2 can be seen in Taneja [9]. Here, we have repeated the proof of (6), since we need the expressions given in (10) and (11).

Lemma 3.1. *The above six difference of divergence measures are* convex *in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$.*

Proof. We shall prove the above lemma in each case separately.

(i) **For $D_{\Psi d}(P \| Q)$:** We can write

$$D_{\Psi d}(P \| Q) = \frac{1}{16} \Psi(P \| Q) - 4d(P \| Q) = \sum_{i=1}^{n} q_i f_{\Psi d}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{\Psi d}(x) = \frac{1}{16} f_\Psi(x) - 4 f_d(x), \quad x > 0.$$

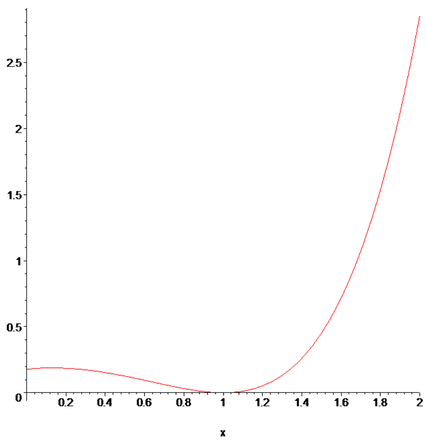
We have

$$
\begin{aligned}
f''_{\Psi d}(x) &= \frac{1}{16} f''_{\Psi}(x) - 4 f''_{d}(x) \\
&= \frac{x^3+1}{8x^3} - \frac{x^{3/2}+1}{2\sqrt{2}\, x^{3/2}(x+1)^{3/2}} \\
&= \frac{(x^3+1)(x+1)\sqrt{2x+2} - 4x^{3/2}(x^{3/2}+1)}{8x^3\sqrt{2x+2}} \\
&= \frac{1}{x^3(x+1)\sqrt{2x+2}} \times m_1(x), \qquad (12)
\end{aligned}
$$

where $m_1(x),\ x>0$ is given by

$$
m_1(x) = \left(\frac{x^3+1}{2}\right)\left(\frac{x+1}{2}\right)^{3/2} - x^{3/2}\left(\frac{x^{3/2}+1}{2}\right).
$$

The measures $f''_{\Psi}(x)$ and $f''_{d}(x)$ appearing in (12) are obtained from (11) by taking $s=2$ and $s=1$ respectively. The graph of the function $m_1(x),\ x>0$ is given by



From the above graph we observe that the function $m_1(x) \geq 0,\ \forall\ x>0$. This allows us to conclude that $f''_{\Psi d}(x) \geq 0,\ x>0$, and hence, $f_{\Psi d}(x)$ is convex for all $x>0$. Also, $f_{\Psi d}(1)=0$. Thus by the application of the Lemma 1.1, we conclude that measure $D_{\Psi d}(P\,\|\,Q)$ is nonnegative and convex for all $(P,Q)\in\Gamma_n\times\Gamma_n$.

(ii) **For $D_{Td}(P\,\|\,Q)$:** We can write

$$
D_{Td}(P\,\|\,Q) = T(P\,\|\,Q) - 4d(P\,\|\,Q) = \sum_{i=1}^{n} q_i f_{Td}\left(\frac{p_i}{q_i}\right),
$$

where

$$
f_{Td}(x) = f_T(x) - 4f_d(x), \quad x>0.
$$

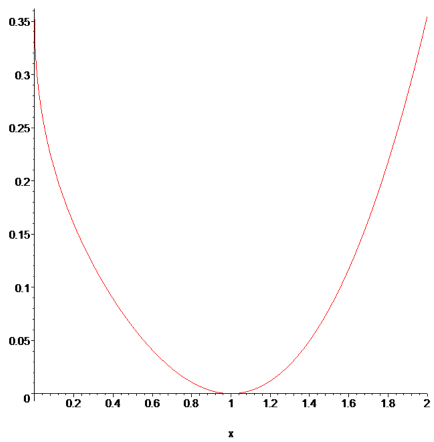
We have

$$f''_{Td}(x) = f''_T(x) - 4f''_d(x) = \frac{1+x^2}{4x^2(x+1)} - \frac{x^{3/2}+1}{2\sqrt{2}\,x^{3/2}(x+1)^{3/2}} = \frac{1}{x^2(x+1)\sqrt{2x+2}} \times m_2(x), \tag{13}$$

where $m_2(x)$, $x > 0$ is given by

$$m_2(x) = \left(\frac{x^2+1}{2}\right)\left(\frac{x+1}{2}\right)^{1/2} - \sqrt{x}\left(\frac{x^{3/2}+1}{2}\right).$$

The measures $f''_T(x)$ and $f''_d(x)$ appearing in (13) are obtained from (11) and by taking $s = 1$ and $s = \frac{1}{2}$ (dividing by 4), respectively. The graph of the function $m_2(x)$, $x > 0$ is given by



From the above graph we observe that the function $m_2(x) \geq 0$, $\forall\ x > 0$. This allows us to conclude that $f''_{Td}(x) \geq 0$, $x > 0$, and hence, $f_{Td}(x)$ is convex for all $x > 0$. Also $f_{Td}(1) = 0$. Thus by the application of the Lemma 1.1, we conclude that measure $D_{Td}(P\,||\,Q)$ is nonnegative and convex for all $(P, Q) \in \Gamma_n \times \Gamma_n$.

(iii) **For $D_{Jd}(P\,||\,Q)$:** We can write

$$D_{Jd}(P\,||\,Q) = \frac{1}{8}J(P\,||\,Q) - 4d(P\,||\,Q) = \sum_{i=1}^{n} q_i f_{Jd}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{Jd}(x) = \frac{1}{8}f_J(x) - 4f_d(x), \quad x > 0.$$

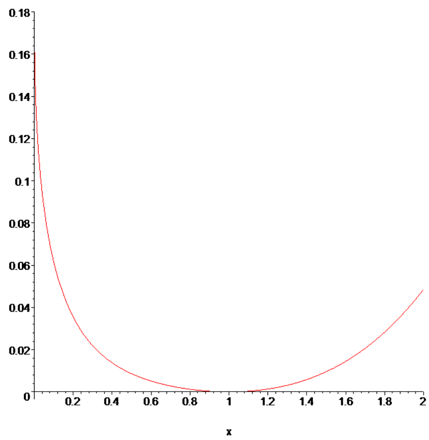
We have

$$\begin{aligned} f''_{Jd}(x) &= \frac{1}{8} f''_J(x) - 4 f''_d(x) \\ &= \frac{x+1}{8x^2} - \frac{x^{3/2}+1}{2\sqrt{2}\, x^{3/2}(x+1)^{3/2}} \\ &= \frac{1}{x^2(x+1)\sqrt{2x+2}} \times m_3(x), \end{aligned} \tag{14}$$

where $m_3(x)$, $x > 0$ is given by

$$m_3(x) = \left(\frac{x+1}{2}\right)^{5/2} - \sqrt{x}\left(\frac{x^{3/2}+1}{2}\right).$$

The measures $f''_J(x)$ and $f''_d(x)$ appearing in (14) are obtained from (10) and (11) by taking $s = 0$ (or $s = 1$) and $s = \frac{1}{2}$ (dividing by 4), respectively. The graph of the function $m_3(x)$, $x > 0$ is given by



From the above graph we observe that the function $m_3(x) \geq 0$, $\forall\ x > 0$. This allows us to conclude that $f''_{Jd}(x) \geq 0$, $x > 0$, and hence, $f_{Jd}(x)$ is convex for all $x > 0$. Also $f_{Jd}(1) = 0$. Thus by the application of the Lemma 1.1, we conclude that measure $D_{Jd}(P\,||\,Q)$ is nonnegative and convex for all $(P, Q) \in \Gamma_n \times \Gamma_n$.

(iv) **For $D_{dh}(P\,||\,Q)$:** We can write

$$D_{dh}(P\,||\,Q) = 4d(P\,||\,Q) - h(P\,||\,Q) = \sum_{i=1}^{n} q_i f_{dh}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{dh}(x) = 4 f_d(x) - f_h(x), \quad x > 0.$$

We have

$$
\begin{aligned}
f''_{dh}(x) &= 4f''_d(x) - f''_h(x) \\
&= \frac{x^{3/2}+1}{2\sqrt{2}\,x^{3/2}(x+1)^{3/2}} - \frac{1}{4x^{3/2}} \\
&= \frac{2(x^{3/2}+1) - (x+1)\sqrt{2x+2}}{4x^{3/2}(x+1)\sqrt{2x+2}} \\
&= \frac{1}{x^{3/2}(x+1)\sqrt{2x+2}}\left[\frac{x^{3/2}+1}{2} - \left(\frac{x+1}{2}\right)^{3/2}\right], \quad \forall\ x>0, \qquad (15)
\end{aligned}
$$

where $f''_d(x)$ and $f''_h(x)$ are obtained from (11) and (10) by taking $s=\frac{1}{2}$(dividing by 4) and $s=\frac{1}{2}$ (dividing by 8) respectively. The non-negativity of the expression (15) follows from the fact that the function $\left(\frac{x^s+1}{2}\right)^{1/s}$, $s\neq 0$ is monotonically increasing function of s [1]. Thus, we have $f''_{dh}(x)\geq 0$, $\forall\ x>0$, and hence, $f_{dh}(x)$ is convex for all $x>0$. Also $f_{dh}(1)=0$. Thus by the application of the Lemma 1.1, we conclude that measure $D_{dh}(P\,\|\,Q)$ is nonnegative and convex for all $(P,Q)\in\Gamma_n\times\Gamma_n$.

(v) **For $D_{dI}(P\,\|\,Q)$:** We can write

$$
D_{dI}(P\,\|\,Q) = 4d(P\,\|\,Q) - I(P\,\|\,Q) = \sum_{i=1}^{n} q_i f_{dI}\left(\frac{p_i}{q_i}\right),
$$

where

$$
f_{dI}(x) = f_d(x) - 4f_I(x), \quad x>0.
$$

We have

$$
\begin{aligned}
f''_{dI}(x) &= f''_d(x) - 4f''_I(x) \\
&= \frac{x^{3/2}+1}{2\sqrt{2}\,x^{3/2}(x+1)^{3/2}} - \frac{1}{2x(x+1)} \\
&= \frac{x^{3/2}+1-\sqrt{x}\sqrt{2x+2}}{2x^{3/2}(x+1)\sqrt{2x+2}} \\
&= \frac{1}{x^{3/2}(x+1)\sqrt{2x+2}}\left[\left(\frac{x^{3/2}+1}{2}\right) - \sqrt{x}\sqrt{\frac{x+1}{2}}\right], \qquad (16)
\end{aligned}
$$

where $f''_d(x)$ and $f''_I(x)$ are obtained from (11) by taking $s=\frac{1}{2}$ (dividing by 4) and $s=0$ respectively. Now we shall prove the non-negativity of the expression (16). We know that [9], pp. 209:

$$
\sqrt{x} \leq \left(\frac{\sqrt{x}+1}{2}\right)^2 \quad \text{and} \quad \left(\frac{\sqrt{x}+1}{2}\right)\sqrt{\frac{x+1}{2}} \leq \left(\frac{x+1}{2}\right).
$$

This give

$$\sqrt{x}\sqrt{\frac{x+1}{2}} \leq \left(\frac{\sqrt{x}+1}{2}\right)^2 \sqrt{\frac{x+1}{2}} \leq \left(\frac{x+1}{2}\right)\left(\frac{\sqrt{x}+1}{2}\right). \tag{17}$$

By simple calculations, we can check that

$$\left(\frac{x+1}{2}\right)\left(\frac{\sqrt{x}+1}{2}\right) \leq \frac{x^{3/2}+1}{2}. \tag{18}$$

The expressions (17) and (18) together give the non-negativity of the expression (16), i.e., $f''_{dI}(x) \geq 0$, $\forall x > 0$, and hence, $f_{dI}(x)$ is convex for all $x > 0$. Also, $f_{dI}(1) = 0$. Thus by the application of the Lemma 1.1, we conclude that measure $D_{dI}(P \| Q)$ is nonnegative and convex for all $(P, Q) \in \Gamma_n \times \Gamma_n$.

(vi) **For $D_{d\Delta}(P \| Q)$:** We can write

$$D_{d\Delta}(P \| Q) = 4d(P \| Q) - \Delta(P \| Q) = \sum_{i=1}^{n} q_i f_{d\Delta}\left(\frac{p_i}{q_i}\right),$$

where

$$f_{d\Delta}(x) = 4f_d(x) - f_\Delta(x), \quad x > 0.$$

We have

$$\begin{aligned} f''_{d\Delta}(x) &= 4f''_d(x) - f''_\Delta(x) \\ &= \frac{x^{3/2}+1}{2\sqrt{2}\, x^{3/2}(x+1)^{3/2}} - \frac{2}{(x+1)^3} \\ &= \frac{(x^{3/2}+1)(x+1)^2 - 4x^{3/2}\sqrt{2x+2}}{2x^{3/2}(x+1)^3\sqrt{2x+2}} \\ &= \frac{8}{2x^{3/2}(x+1)^3\sqrt{2x+2}}\sqrt{\frac{x+1}{2}}\left[\left(\frac{x^{3/2}+1}{2}\right)\left(\frac{x+1}{2}\right)^{3/2} - x^{3/2}\right] \end{aligned} \tag{19}$$

where $f''_d(x)$ and $f''_\Delta(x)$ are obtained from (11) and (10) by taking $s = \frac{1}{2}$ (dividing by 4) and $s = -1$ (multiplying by 4) respectively.

Now we shall prove the non-negativity of the expression (19). We can easily check that

$$\left(\frac{\sqrt{x}+1}{2}\right)^3 \leq \frac{x^{3/2}+1}{2}. \tag{20}$$

On the other side we know that [9], pp.209:

$$x^{3/2} \leq \left(\frac{\sqrt{x}+1}{2}\right)^3 \left(\frac{x+1}{2}\right)^{3/2}. \tag{21}$$

Expressions (20) and (21) together give

$$x^{3/2} \leq \left(\frac{\sqrt{x}+1}{2}\right)^3 \left(\frac{x+1}{2}\right)^{3/2} \leq \left(\frac{x^{3/2}+1}{2}\right)\left(\frac{x+1}{2}\right)^{3/2}. \tag{22}$$

Expression (22) proves the non-negativity of the expression (19), i.e., $f''_{d\Delta}(x) \geq 0$, $\forall\ x > 0$, and hence, $f_{d\Delta}(x)$ is convex for all $x > 0$. Also, $f_{d\Delta}(1) = 0$. Thus by the application of the Lemma 1.1, we conclude that measure $D_{d\Delta}(P \,\|\, Q)$ is nonnegative and convex for all $(P, Q) \in \Gamma_n \times \Gamma_n$. □

4. A Sequence of Inequalities among Difference of Divergence Measures

In this section our aim is to establish *a sequence of inequalities* among *difference of divergence measures*. The main result of the paper is summarized in the theorem below.

Theorem 4.1. *The following sequence of inequalities hold:*

$$\begin{aligned} D_{h\Delta}(P \,\|\, Q) &\leq \frac{4}{5} D_{d\Delta}(P \,\|\, Q) \leq 4 D_{dh}(P \,\|\, Q) \leq \frac{12}{7} D_{dI}(P \,\|\, Q) \\ &\leq 3 D_{hI}(P \,\|\, Q) \leq D_{Th}(P \,\|\, Q) \leq \frac{4}{3} D_{Td}(P \,\|\, Q) \leq \frac{1}{4} D_{\Psi\Delta}(P \,\|\, Q) \\ &\leq \frac{1}{3} D_{\Psi h}(P \,\|\, Q) \leq \frac{4}{11} D_{\Psi d}(P \,\|\, Q) \leq \frac{1}{2} D_{\Psi T}(P \,\|\, Q). \end{aligned} \tag{23}$$

The proof of the above theorem is based on the following propositions. In order to prove the propositions, we shall be using frequently, expressions (10) and (11) with particular values.

Proposition 4.1. *We have*

$$D_{h\Delta}(P \,\|\, Q) \leq \frac{4}{5} D_{d\Delta}(P \,\|\, Q).$$

Proof. Let us consider

$$\begin{aligned} g_{h\Delta\_d\Delta}(x) &= \frac{f''_{h\Delta}(x)}{f''_{d\Delta}(x)} \\ &= \frac{\sqrt{(2x+2)}\left[(x+1)^3 - 8x^{3/2}\right]}{2\left[(x^{3/2}+1)(x+1)^2 - 4x^{3/2}\sqrt{2x+2}\right]} \\ &= \frac{\sqrt{(2x+2)}\left[(\sqrt{x}+1)^2(x+1) + 4x(\sqrt{x}-1)^2\right]}{2\left[(x^{3/2}+1)(x+1)^2 - 4x^{3/2}\sqrt{2x+2}\right]}, \quad x \neq 1 \end{aligned}$$

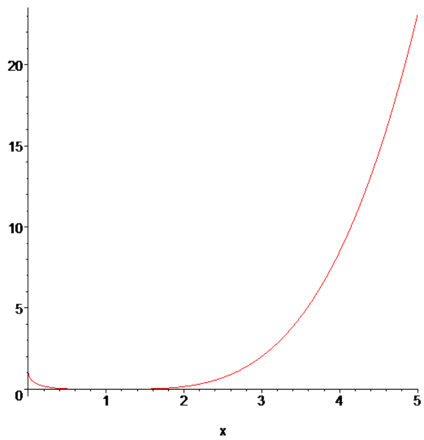
for all $x \in (0, \infty)$, where $f''_{h\Delta}(x) = \varphi''_{-1}(x) - \psi''_{-1}(x)$ and $f''_{d\Delta}(x) = \psi''_{1/2}(x) - \psi''_{-1}(x)$. Calculating the first order derivative of the function $g_{I\Delta\_h\Delta}(x)$ with respect to x, one gets

$$g'_{h\Delta\_d\Delta}(x) = \frac{-3(x+1)(\sqrt{x}-1)\sqrt{2x+2}}{4[(x^{3/2}+1)(x+1)^2 - 4x^{3/2}\sqrt{2x+2}]^2} \times k_1(x), \tag{24}$$

where

$$k_1(x) = x^3 - 8x^{5/2} - 5x^2 - 8x^{3/2} - 5x - 8x^{1/2} + 1 + 4\sqrt{2x(x+1)}(\sqrt{x}+1)(x+1). \tag{25}$$

The graph of the function $k_1(x)$, $x > 0$ is given by



We observe from the above graph that the function $k_1(x) \geq 0$, $\forall\ x > 0$. Thus from the from the expression (24), we conclude that

$$g'_{h\Delta\_d\Delta}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \tag{26}$$

Let us calculate now $g_{h\Delta\_d\Delta}(1)$. We observe that

$$g_{h\Delta\_d\Delta}(x)\Big|_{x=1} = \frac{f''_{h\Delta}(x)}{f''_{d\Delta}(x)}\Big|_{x=1} = .\frac{(f''_{h\Delta}(x))'}{(f''_{d\Delta}(x))'}\Big|_{x=1} = \text{indermination}.$$

Calculating the second order derivatives of numerator and denominator of the function $g_{I\Delta\_h\Delta}(x)$, we have

$$g_{h\Delta\_d\Delta}(1) = \frac{(f''_{h\Delta}(x))''}{(f''_{d\Delta}(x))''}\Big|_{x=1} = \frac{\frac{3}{4}}{\frac{15}{16}} = \frac{4}{5}. \tag{27}$$

By the application of the inequalities (5) with (27) we get the required result. □

Proposition 4.2. *We have*

$$D_{d\Delta}(P\,||\,Q) \leq 5\,D_{dh}(P\,||\,Q).$$

Proof. Let us consider

$$g_{d\Delta\_dh}(x) = \frac{f''_{d\Delta}(x)}{f''_{dh}(x)} = \frac{2\left[(x+1)^2(x^{3/2}+1) - 4x^{3/2}\sqrt{2x+2}\right]}{(x+1)^2\left[2(x^{3/2}+1) - (x+1)\sqrt{2x+2}\right]}, \quad x \neq 1,$$

for all $x \in (0,\infty)$, where $f''_{d\Delta}(x) = \psi''_{1/2}(x) - \psi''_{-1}(x)$ and $f''_{dh}(x) = \psi''_{1/2}(x) - \varphi''_{1/2}(x)$.

Calculating the first order derivative of the function $g_{d\Delta\_dh}(x)$ with respect to x, one gets

$$g'_{d\Delta\_dh}(x) = \frac{-3\left(\sqrt{x}-1\right)\sqrt{2x+2}}{(x+1)^3\left[2(x^{3/2}+1)-(x+1)\sqrt{2x+2}\right]^2} \times k_1(x), \tag{28}$$

where $k_1(x)$, $x > 0$ is as given in (25). Since $k_1(x) \geq 0$, $\forall x > 0$. Thus from the from the expression (28), we conclude that

$$g'_{d\Delta\_dh}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \tag{29}$$

Let us calculate now $g_{d\Delta\_dh}(1)$. We observe that

$$g_{d\Delta\_dh}(x)\Big|_{x=1} = \frac{f''_{d\Delta}(x)}{f''_{dh}(x)}\Bigg|_{x=1} = \frac{(f''_{d\Delta}(x))'}{(f''_{dh}(x))'}\Bigg|_{x=1} = \text{indermination}.$$

Calculating the second order derivatives of numerator and denominator of the function $g_{d\Delta\_dh}(x)$, we have

$$g_{d\Delta\_dh}(1) = \frac{(f''_{d\Delta}(x))''}{(f''_{dh}(x))''}\Bigg|_{x=1} = \frac{\frac{15}{16}}{\frac{3}{16}} = 5. \tag{30}$$

By the application of the inequalities (5) with (30) we get the required result. □

Proposition 4.3. *We have*

$$D_{dh}(P \,\|\, Q) \leq \frac{3}{7} D_{dI}(P \,\|\, Q).$$

Proof. Let us consider

$$g_{dh\_dI}(x) = \frac{f''_{dh}(x)}{f''_{dI}(x)} = \frac{2(x^{3/2}+1)-(x+1)\sqrt{2x+2}}{2\left[(x^{3/2}+1)-\sqrt{2x(x+1)}\right]}, \quad x \neq 1$$

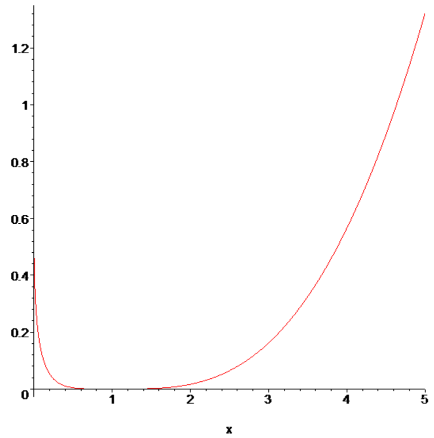
for all $x \in (0,\infty)$, where $f''_{dh}(x) = \psi''_{1/2}(x) - \varphi''_{1/2}(x)$ and $f''_{dI}(x) = \psi''_{1/2}(x) - \varphi''_{1}(x)$. Calculating the first order derivative of the function $g_{dh\_dI}(x)$ with respect to x, one gets

$$g'_{dh\_dI}(x) = -\frac{(\sqrt{x}-1)\sqrt{2x+2}}{4\sqrt{x}(x+1)\left[\left((x^{3/2}+1)-\sqrt{2x(x+1)}\right)\right]^2} \times k_2(x) \tag{31}$$

where

$$k_2(x) = x^2 - x^{3/2} + 6x - \sqrt{x} + 2 - (x+1)(\sqrt{x}+1)\sqrt{2x+2}. \tag{32}$$

The graph of the function $k_2(x)$, $x > 0$ is given by



We observe from the above graph that the function $k_2(x) \geq 0$, $\forall\ x > 0$. Thus from the from the expression (37), we conclude that

$$g'_{dh\_dI}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \tag{33}$$

Let us calculate now $g_{dh\_dI}(1)$. We observe that

$$g_{dh\_dI}(x)\big|_{x=1} = \frac{f''_{dh}(x)}{f''_{dI}(x)}\bigg|_{x=1} = \frac{(f''_{dh}(x))'}{(f''_{dI}(x))'}\bigg|_{x=1} = \text{indermination.}$$

Calculating the second order derivatives of numerator and denominator of the function $g_{dh\_dI}(x)$, we have

$$g_{dh\_dI}(1) = \frac{(f''_{dh}(x))''}{(f''_{dI}(x))''}\bigg|_{x=1} = \frac{\frac{3}{64}}{\frac{7}{64}} = \frac{3}{7}. \tag{34}$$

By the application of the inequalities (5) with (34) we get the required result. $\square$

Proposition 4.4. *We have*

$$D_{dI}(P\,\|\,Q) \leq \frac{7}{4} D_{hI}(P\,\|\,Q).$$

Proof. Let us consider

$$g_{dI\_hI}(x) = \frac{f''_{dI}(x)}{f''_{hI}(x)} = \frac{2\left(x^{3/2} + 1 - \sqrt{2x(x+1)}\right)}{\sqrt{2x+2}\,\left(\sqrt{x}-1\right)^2}, \quad x \neq 1,$$

for all $x \in (0,\infty)$, where $f''_{dI}(x) = \psi''_{1/2}(x) - \varphi''_1(x)$ and $f''_{hI}(x) = \varphi''_{1/2}(x) - \varphi''_1(x)$. Calculating the first order derivative of the function $g_{dI\_hd}(x)$ with respect to x, one gets

$$g'_{dI\_hI}(x) = -\frac{\left(\sqrt{x}-1\right)\sqrt{x}}{x(x+1)\sqrt{2x+2}\,\left(2\sqrt{x}-x-1\right)^2} \times k_2(x), \quad x \neq 1. \tag{35}$$

where $k_2(x)$, $x>0$ is as given by (38). Since $k_2(x)\geq 0$, $\forall x>0$. Thus from the from the expression (35), we conclude that

$$g'_{dI\_hI}(x)\begin{cases}>0, & x<1\\ <0, & x>1\end{cases} \tag{36}$$

Let us calculate now $g_{dI\_hI}(1)$. We observe that

$$g_{dI\_hI}(x)\Big|_{x=1}=\left.\frac{f''_{dI}(x)}{f''_{dI}(x)}\right|_{x=1}=\left.\frac{(f''_{dI}(x))'}{(f''_{hI}(x))'}\right|_{x=1}=\text{indermination}.$$

Calculating the second order derivatives of numerator and denominator of the function $g_{dI\_hI}(x)$, we have

$$g_{dI\_hI}(1)=\left.\frac{(f''_{dI}(x))''}{(f''_{hI}(x))''}\right|_{x=1}=\frac{\frac{7}{64}}{\frac{1}{16}}=\frac{7}{4}. \tag{37}$$

By the application of the inequalities (5) with (37) we get the required result. □

Proposition 4.5. *We have*

$$D_{Th}(P\,||\,Q)\leq\frac{4}{3}D_{Td}(P\,||\,Q).$$

Proof. Let us consider

$$g_{Th\_Td}(x)=\frac{f''_{Th}(x)}{f''_{Td}(x)}=\frac{\left(\sqrt{x}-1\right)^2\left(x+\sqrt{x}+1\right)\sqrt{2x+2}}{(x^2+1)\sqrt{2x+2}-2\sqrt{x}\left(x^{3/2}+1\right)},\quad x\neq 1$$

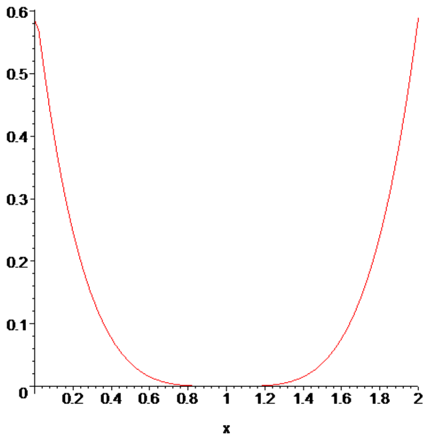
for all $x\in(0,\infty)$, where $f''_{Th}(x)=\psi''_0(x)-\varphi''_{1/2}(x)$ and $f''_{Td}(x)=\psi''_0(x)-\psi''_{1/2}(x)$. Calculating the first order derivative of the function $g_{Th\_Td}(x)$ with respect to x, one gets

$$g'_{Th\_Td}(x)=\frac{-2x^2(x+1)\left(\sqrt{x}-1\right)\sqrt{2x+2}}{2\sqrt{x}\left[2\sqrt{x}\left(x^{3/2}+1\right)-(x^2+1)\sqrt{x+1}\right]^2}\times k_3(x) \tag{38}$$

where

$$\begin{aligned}k_3(x)=&\,2(x^2+1)(x^2+x+1)+2\sqrt{x}(x+1)(x^2+7x+1)\\&-(x+1)(x^2+4x+1)\left(\sqrt{x}+1\right)\sqrt{2x+2}.\end{aligned}$$

The graph of the function $k_3(x), x>0$ is given by



We observe from the above graph that the function $k_3(x) \geq 0$, $\forall x > 0$. Thus from the from the expression (38), we conclude that

$$g'_{Th\_Td}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \tag{39}$$

Let us calculate now $g_{Th\_Td}(1)$. We observe that

$$g_{Th\_Td}(x)\Big|_{x=1} = \frac{f''_{Th}(x)}{f''_{Td}(x)}\bigg|_{x=1} = \frac{(f''_{Th}(x))'}{(f''_{Td}(x))'}\bigg|_{x=1} = \text{indermination.}$$

Calculating the second order derivatives of numerator and denominator of the function $g_{Th\_Td}(x)$, we have

$$g_{Th\_Td}(1) = \frac{(f''_{Th}(x))''}{(f''_{Td}(x))''}\bigg|_{x=1} = \frac{\frac{3}{16}}{\frac{9}{64}} = \frac{4}{3}. \tag{40}$$

By the application of the inequalities (5) with (40) we get the required result. □

Proposition 4.6. *We have*

$$D_{Td}(P\,\|\,Q) \leq \frac{3}{16} D_{\Psi\Delta}(P\,\|\,Q).$$

Proof. Let us consider

$$\begin{aligned} g_{Td\_\Psi\Delta}(x) &= \frac{f''_{Td}(x)}{f''_{\Psi\Delta}(x)} \\ &= \frac{2x(x+1)^2\left[(x^2+1)(2x+2)^{3/2} - 4\sqrt{x}(x+1)(x^{3/2}+1)\right]}{(x-1)^2(2x+2)^{3/2}\left(x^4+5x^3+12x^2+5x+\right)} \\ &= \frac{2x(x+1)^2\left[(x^2+1)\sqrt{2x+2} - 2\sqrt{x}\left(x^{3/2}+1\right)\right]}{(x-1)^2\sqrt{2x+2}\left(x^4+5x^3+12x^2+5x+1\right)}, \quad x \neq 1 \end{aligned}$$

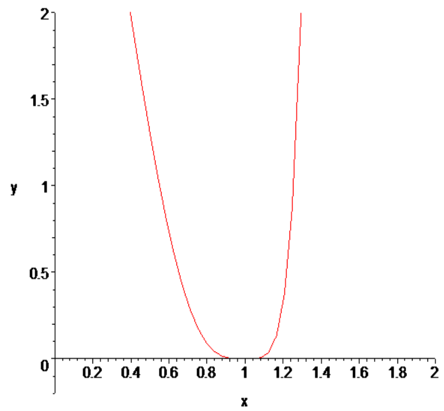
for all $x \in (0, \infty)$, where $f''_{Td}(x) = \psi''_0(x) - \psi''_{1/2}(x)$ and $f''_{\Psi\Delta}(x) = \psi''_2(x) - \psi''_{-1}(x)$. Calculating the first order derivative of the function $g_{Td\_\Psi\Delta}(x)$ with respect to x, one gets

$$g'_{Td\_\Psi\Delta}(x) = -\frac{2}{\begin{pmatrix} x(x-1)^3(2x+2)^{5/2}(x+1)^3 \\ \times(x^4+5x^3+12x^2+5x+1)^2 \end{pmatrix}} \times k_4(x), \tag{41}$$

where $k_4(x)$ is given by

$$\begin{aligned} k_4(x) = &-12\sqrt{x}(\sqrt{x}+1)(x+1)^2\Big[(x+1)(x^6+4x^5+6x^4+18x^3+6x^2+4x+1) \\ &-x^{(1/2)}(x^2+1)(x^4+3x^3+3x+1)\Big] \\ &+(2x+2)^{(5/2)}(1+4x+10x^2+52x^3+58x^4+52x^5+10x^6+4x^7+x^8). \end{aligned}$$

The graph of the function $k_4(x)$, $x > 0$ is given by



We observe from the above graph that the function $k_4(x) \geq 0$, $\forall x > 0$. Thus from the from the expression (41), we conclude that

$$g'_{Td\_\Psi\Delta}(x)\begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \tag{42}$$

Let us calculate now $g_{Td\_\Psi\Delta}(1)$. We observe that

$$g_{Td\_\Psi\Delta}(x)\Big|_{x=1} = \frac{f''_{Td}(x)}{f''_{\Psi\Delta}(x)}\Bigg|_{x=1} = \frac{(f''_{Td}(x))'}{(f''_{\Psi\Delta}(x))'}\Bigg|_{x=1} = \text{indermination}.$$

Calculating the second order derivatives of numerator and denominator of the function $g_{Td\_\Psi\Delta}(x)$, we have

$$g_{Td}(1) = \frac{(f''_{Td}(x))''}{(f''_{\Psi\Delta}(x))''}\Bigg|_{x=1} = \frac{\frac{9}{4}}{12} = \frac{3}{16}. \tag{43}$$

By the application of the inequalities (5) with (43) we get the required result. $\square$

Proposition 4.7. *We have*

$$D_{\Psi h}(P\,\|\,Q) \le \frac{12}{11} D_{\Psi d}(P\,\|\,Q).$$

Proof. Let us consider

$$g_{\Psi h\_\Psi d}(x) = \frac{f''_{\Psi h}(x)}{f''_{\Psi d}(x)} = \frac{(x^{3/2}-1)^2(2x+2)^{3/2}}{(x^3+1)(2x+2)^{3/2}-8x^{3/2}(x^{3/2}+1)}, \quad x \neq 1$$

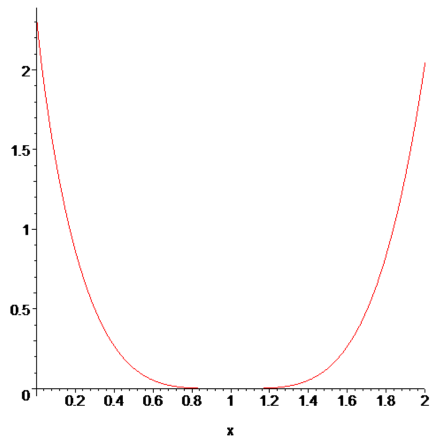
for all $x \in (0,\infty)$, where $f''_{\Psi h}(x) = \psi''_2(x) - \varphi''_{1/2}(x)$ and $f''_{\Psi d}(x) = \psi''_2(x) - \psi''_{1/2}(x)$. Calculating the first order derivative of the function $g_{\Psi h\_\Psi d}(x)$ with respect to x, one gets

$$g'_{\Psi h\_\Psi d}(x) = -\frac{3(\sqrt{x}-1)(x+\sqrt{x}+1)\sqrt{2x+2}}{\sqrt{x}\left[(x^3+1)(2x+2)^{3/2}-8x^3(x^{3/2}+1)\right]^2} \times k_5(x) \tag{44}$$

where

$$k_5(x) = \left[8(x^4+3x^{5/2}+3x^{3/2}+1)-(x^{3/2}+1)(2x+2)^{5/2}\right].$$

The graph of the function $k_5(x)$, $x>0$ is given by



We observe from the above graph that the function $k_5(x) \ge 0$, $\forall\ x > 0$. Thus from the from the expression (44), we conclude that

$$g'_{\Psi h\_\Psi d}(x) \begin{cases} >0, & x<1 \\ <0, & x>1 \end{cases} \tag{45}$$

Let us calculate now $g_{\Psi h\_\Psi d}(1)$. We observe that

$$g_{\Psi h\_\Psi d\Delta}(x)\Big|_{x=1} = \frac{f''_{\Psi h}(x)}{f''_{\Psi d}(x)}\bigg|_{x=1} = \frac{(f''_{\Psi h}(x))'}{(f''_{\Psi d}(x))'}\bigg|_{x=1} = \text{indermination.}$$

Calculating the second order derivatives of numerator and denominator of the function $g_{\Psi h\_\Psi d}(x)$, we have

$$g_{\Psi h\_\Psi d}(1) = \left.\frac{(f''_{\Psi h}(x))''}{(f''_{\Psi d}(x))''}\right|_{x=1} = \frac{9}{\frac{33}{4}} = \frac{12}{11}. \tag{46}$$

By the application of the inequalities (5) with (46) we get the required result. □

Proposition 4.8. *We have*

$$D_{\Psi d}(P \,\|\, Q) \leq \frac{11}{8} D_{\Psi T}(P \,\|\, Q).$$

Proof. Let us consider

$$g_{\Psi d\_\Psi T}(x) = \frac{f''_{\Psi d}(x)}{f''_{\Psi T}(x)} = \frac{(x+1)\left[(x^3+1)(2x+2)^{3/2} - 8x^{3/2}(x^{3/2}+1)\right]}{(2x+2)^{3/2}(x-1)^2(x^2+x+1)}, \quad x \neq 1$$

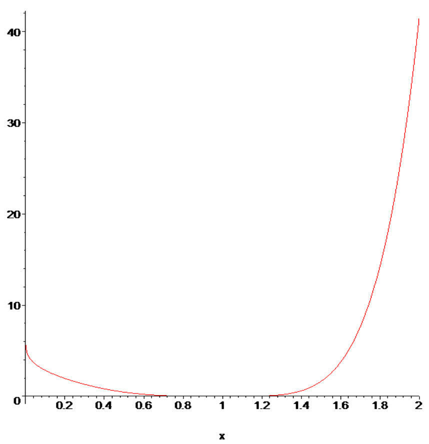
for all $x \in (0, \infty)$, where $f''_{\Psi d}(x) = \psi''_2(x) - \psi''_{1/2}(x)$ and $f''_{\Psi T}(x) = \psi''_2(x) - \psi''_0(x)$. Calculating the first order derivative of the function $g_{\Psi d\_\Psi T}(x)$ with respect to x, one gets

$$g'_{\Psi d\_\Psi T}(x) = -\frac{2x^{5/2}\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)(x+1)}{x^{5/2}(x-1)^4(x^2+x+1)^2(2x+2)^{5/2}} \times k_6(x), \tag{47}$$

where $k_6(x)$, $x > 0$ is given by

$$\begin{aligned} k_6(x) &= (2x+2)^{5/2}(x^4+4x^2+1) - 4x^2(3x^4+4x^3+4x^2+7x+6) \\ &\quad - 4\sqrt{x}(3+4x+4x^2+7x^3+6x^4). \end{aligned}$$

The graph of $k_6(x)$, $x > 0$ is given by



We observe from the above graph that the function $k_6(x) \geq 0$, $\forall\ x > 0$. Thus from the from the expression (47), we conclude that

$$g'_{\Psi d\_\Psi T}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \tag{48}$$

Let us calculate now $g_{\Psi d\_\Psi T}(1)$. We observe that

$$g_{\Psi d\_\Psi T}(x)\Big|_{x=1} = \frac{f''_{\Psi d}(x)}{f''_{\Psi T}(x)}\Bigg|_{x=1} = \frac{(f''_{\Psi d}(x))'}{(f''_{\Psi T}(x))'}\Bigg|_{x=1} = \text{indermination}.$$

Calculating the second order derivatives of numerator and denominator of the function $g_{\Psi d\_\Psi T}(x)$, we have

$$g_{\Psi d\_\Psi T}(1) = \frac{(f''_{\Psi d}(x))''}{(f''_{\Psi T}(x))''}\Bigg|_{x=1} = \frac{\frac{33}{4}}{6} = \frac{11}{4}. \tag{49}$$

By the application of the inequalities (5) with (49) we get the required result. □

***Proof of the Theorem* 4.1.** We know that

$$D_{\Psi\Delta}(P \,\|\, Q) \leq \frac{4}{3} D_{\Psi h}(P \,\|\, Q). \tag{50}$$

Propositions 4.1-4.8 together the inequality (50) completes the proof of the theorem.

From the sequence of inequalities given in Theorem 4.1, we noted the absence of the measure $D_{Jd}(P \,\|\, Q)$. Here below is an inequality relating the measures $D_{Jd}(P \,\|\, Q)$ and $D_{Td}(P \,\|\, Q)$. □

Proposition 4.9. *We have*

$$\frac{1}{9} D_{Td}(P \,\|\, Q) \leq D_{Jd}(P \,\|\, Q).$$

Proof. Let us consider

$$\begin{aligned} g_{Td\_Jd}(x) &= \frac{f''_{Td}(x)}{f''_{Jd}(x)} \\ &= \frac{2\left[4\sqrt{x}(x+1)\left(x^{3/2}+1\right) - \left(x^2+1\right)(2x+2)^{3/2}\right]}{(x+1)\left[8\sqrt{x}\left(x^{3/2}+1\right) - (x+1)(2x+2)^{3/2}\right]}, \quad x \neq 1 \end{aligned}$$

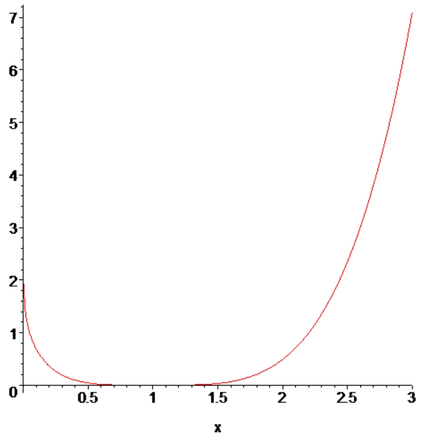
for all $x \in (0, \infty)$, where $f''_{Td}(x)$ and $f''_{Jd}(x)$. Calculating the first order derivative of the function $g_{Td\_Jd}(x)$ with respect to x, one gets

$$g'_{Td\_Jd}(x) = -\frac{4(\sqrt{x}-1)(\sqrt{x}+1)(x+1)\sqrt{2x+2}}{\sqrt{x}(x+1)^2\left[8\sqrt{x}(x^{3/2}+1) - (x+1)(2x+2)^{3/2}\right]^2} \times k_7(x), \tag{51}$$

where $k_7(x)$, $x > 0$ is given by

$$k_7(x) = 2(x^{7/2} + 3x^{5/2} + 4x^2 + 4x^{3/2} + 3x + 1) - \sqrt{x}(2x+2)^{5/2}.$$

The graph of $k_7(x)$, $x > 0$ is given by



We observe from the above graph that the function $k_7(x) \geq 0$, $\forall\ x > 0$. Thus from the from the expression (51), we conclude that

$$g'_{Td\_Jd}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \tag{52}$$

Let us calculate now $g_{Td\_Jd}(1)$. We observe that

$$g_{Td\_Jd}(x)\big|_{x=1} = \left.\frac{f''_{Td}(x)}{f''_{Jd}(x)}\right|_{x=1} = \left.\frac{(f''_{Td}(x))'}{(f''_{Jd}(x))'}\right|_{x=1} = \text{indermination}.$$

Calculating the second order derivatives of numerator and denominator of the function $g_{Td\_Jd}(x)$, we have

$$g_{Td\_Jd}(1) = \left.\frac{(f''_{Jd}(x))''}{(f''_{Td}(x))''}\right|_{x=1} = \frac{\frac{9}{8}}{\frac{1}{8}} = 9. \tag{53}$$

By the application of the inequalities (5) with (53) we get the required result. □

5. Final Remarks

Remark 5.1. In view of (23) and Proposition 4.10, we have

$$D_{h\Delta}(P \| Q) \leq \frac{4}{5} D_{d\Delta}(P \| Q) \leq 4 D_{dh}(P \| Q) \leq \frac{12}{7} D_{dI}(P \| Q)$$
$$\leq 3 D_{hI}(P \| Q) \leq D_{Th}(P \| Q) \leq \frac{4}{3} D_{Td}(P \| Q) \leq 12 D_{Jd}(P \| Q). \tag{54}$$

We can easily find examples where the measure $D_{Jd}(P \| Q)$ don't have relations with the other measures appearing the rest of sequence given in (23), such as $D_{\Psi\Delta}(P \| Q)$, etc.

Remark 5.2. Following the similar lines of above propositions, we can also prove the following inequality for the measure $D_{Jd}(P\,\|\,Q)$

$$\frac{1}{4}D_{Jh}(P\,\|\,Q) \leq D_{Jd}(P\,\|\,Q). \tag{55}$$

Remark 5.3. As a consequence of Propositions 4.1-4.9 and the expression (23), we have the following inequalities:

(i) $\frac{16d+3I}{7} \leq h \leq \frac{64d+3\Delta}{20}$.

(ii) $h \leq \frac{T+3I}{4}$.

(iii) $h \leq \frac{\Psi+12\Delta}{64}$.

(iv) $4d \leq \frac{T+3h}{4}$.

(v) $4d \leq \frac{\Psi+176h}{192}$.

(vi) $4d \leq \frac{3J+8h}{8}$.

(vii) $T \leq \frac{3\Psi+512d}{176}$.

(viii) $\frac{32d+T}{9} \leq \frac{J}{8}$.

(ix) $4T + \frac{3\Delta}{16} \leq \frac{3\Psi}{64} + 16d$.

In view of above above results we have the following inequalities:

$$I \leq \frac{16d+3I}{7} \leq h \leq \frac{64d+\Delta}{20} \leq 4d \leq \frac{32d+T}{9} \leq \frac{1}{8}J,$$
$$h \leq \frac{\Psi+12\Delta}{64} \leq 4d \leq \frac{T+3h}{4} \leq T \leq 4d + \frac{3}{16}\left(\frac{1}{16}\Psi - \frac{1}{4}\Delta\right) \leq \frac{3J+8h}{8},$$
$$4d \leq \frac{\Psi+176h}{192} \leq \frac{1}{16}\Psi,$$

and

$$T \leq \frac{3\Psi+512d}{176} \leq \frac{1}{16}\Psi.$$

Remark 5.4. As a consequence of previous results (2) and (3), we have the following inequalities:

$$\frac{1}{4}\Delta \leq I \leq \frac{9h+\Delta}{12} \leq h \leq \frac{J+8I}{16} \leq \frac{T+2h}{3} \leq \frac{1}{8}J \leq \frac{8T+\Delta}{12} \leq T \leq \frac{\Psi+6J}{642} \leq \frac{1}{16}\Psi$$

and

$$\frac{1}{8}J \leq \frac{\Psi+192h-4\Delta}{192} \leq \frac{\Psi+128h}{144} \leq \frac{1}{16}\Psi.$$

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