Journal of Informatics and Mathematical Sciences
Vol. 11, Nos. 3-4, pp. 407–419, 2019
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
DOI: 10.26713/jims.v11i3-4.419

http://www.rgnpublications.com

Research Article

The Existence and Approximation Fixed Point Theorems for Monotone Nonspreading Mappings in Ordered Banach Spaces

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Abstract. In this paper, we proved some existence theorems of fixed points for monotone nonspreading mappings T in a Banach space E with the partial order $\leq$. In order to finding a fixed point of such a mapping T, moreover we proved the convergence theorem of Mann iterative schemes under the condition $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$, which contain $\beta_n = \frac{1}{n+1}$ as a special case.

Keywords. Ordered Banach space; Fixed point; Monotone nonspreading mapping; Mann iterative scheme

MSC. 47H07; 47H10

Received: June 6, 2016 **Accepted:** October 20, 2018

1. Introduction

Let E be a Banach space and E^* be the dual space of E. For all $x \in E$ and $f \in E^*$, let the value of f at x be denoted by $\langle x, f\rangle$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}$$

for all $x \in E$. A single-valued normalized duality mapping is denoted by j, which means a mapping $j : E \to E^*$ such that, for each $u \in E, j(u) \in E^*$ satisfying the following:

$$\langle j(u), u\rangle = \|j(u)\|\|u\|, \|j(u)\| = \|u\|.$$

In 2008, Kohsaka and Takahashi [3] also introduced the class of mappings called the class of nonspreading mappings to study the resolvent of a maximal monotone operator in Banach spaces. Let E be a smooth, strictly convex and reflexive Banach space and K be a nonempty closed convex subset of E.

Definition 1.1. A mapping $T : K \to K$ is said to be nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where

$$\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2.$$

The set of fixef points of a mapping $T : K \to K$ is defined by

$$F(T) := \{x \in C : Tx = x\}.$$

In 1954, Mann [5] introduced the following iteration to finding a fixed point, which is referred to as the Mann iteration,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n$$

for each $n \ge 1$, where $\beta_n \in [0, 1]$ is a sequence with some condition. However, there are not many convergence theorems of such a iteration in a order Banach space $(E, \le)$. Motivated by the above results, we consider the weak convergence of the Mann iterative scheme for a monotone nonspreading mapping T under the condition

$$\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$$

which contain $\beta_n = \frac{1}{n+1}$ as a special case. By motivation of Mann iteration for a monotone nonexpansive mapping of Dehaish and Khamsi [2].

2. Preliminaries

Let P be a closed convex cone of a real Banach space E. A *partial order* "$\le$" with respect to P in E is defined as follows:

$$x \le y \ (x \le y) \text{ if and on if } y - x \in P \ (y - x \in \mathring{P})$$

for all $x, y \in E$, where $\mathring{P}$ is the interior of P.

Throughout this paper, let E be a Banach space with the norm "$\|\cdot\|$" and the partial order "$\le$". Let $F(T) = \{x \in H : Tx = x\}$ denote the set of all fixed points of a mapping T. Also, we assume that the order intervals are closed and convex. An *order interval* $[x, y]$ for all $x, y \in E$ is given by

$$[x, y] = \{z \in E : x \leq z \leq y\}. \tag{1}$$

Then the convexity of the order interval $[x, y]$ implies that

$$x \leq tx + (1-t)y \leq y \quad \text{for all } x, y \in E \text{ with } x \leq y. \tag{2}$$

Definition 2.1. Let K be a nonempty closed and convex subset of a Banach space E. A mapping $T : K \to K$ is said to be:

(1) *monotone* if $Tx \leq Ty$ for all $x, y \in K$ with $x \leq y$;

(2) *monotone nonspreading* if T is monotone and

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, J(y - Ty)\rangle$$

for all $x, y \in K$ with $x \leq y$ and J is normalized duality mapping.

A Banach space E is said to be:

(1) *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$;

(2) *uniformly convex* if, for all $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$.

The following inequality was showed by Xu [9] in a uniformly convex Banach space E, which is known as *Xu's inequality*.

Lemma 2.2 (Xu [9, Theorem 2]). *For any real numbers $q > 1$ and $r > 0$, a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^q \leq t\|x\|^q + (1-t)\|y\|^q - \omega(q, t)g(\|x - y\|) \tag{3}$$

for all $x, y \in B_r(0) = \{x \in E; \|x\| \leq r\}$ and $t \in [0, 1]$, where $\omega(q, t) = t^q(1-t) + t(1-t)^q$. In particular, take $q = 2$ and $t = \frac{1}{2}$,

$$\left\|\frac{x+y}{2}\right\|^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}g(\|x - y\|). \tag{4}$$

The following conclusion is well known:

Lemma 2.3 (Takahashi [8, Theorem 1.3.11]). *Let K be a nonempty closed convex subset of a reflexive Banach space E. Assume that $\varphi : K \to R$ is a proper convex lower semi-continuous and coercive function. Then the function φ attains its minimum on K, that is, there exists $x \in K$ such that*

$$\varphi(x) = \inf_{y \in K} \varphi(y).$$

Theorem 2.4. *Let $\{x_n\}$ be a bounded above monotone nondecreasing sequence. Then $\{x_n\}$ converges to the supemum of $\{x_n : n \in N\}$.*

Lemma 2.5. *Let K be a nonempty closed convex subset of a uniformly convex Banach space $(E, \leq)$ and $T : K \to K$ be a monotone nonspreading mapping. If $x \in K$ such that $x_{n+1} = Tx_n$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded. Then $\limsup_{n\to\infty} \|x_n - Tx_n\| \to 0$.*

Proof. From Theorem 2.4 $\{x_n\}$ is bounded and monotone increasing then there exists $M > 0$ such that $\|x_n\| \leq M$ so,

$$\limsup_{n\to\infty} \|x_n - M\| \leq 0$$

by analogy we obtain

$$\limsup_{n\to\infty} \|x_{n+1} - M\| \leq 0$$

so,

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - x_{n+1}\| &\leq \limsup_{n\to\infty} [\|x_n - M\| + \|x_{n+1} - M\|] \\ &\leq \limsup_{n\to\infty} \|x_n - M\| + \limsup_{n\to\infty} \|x_{n+1} - M\| \\ &= 0,\end{aligned}$$

therefore, we can conclude that

$$\limsup_{n\to\infty} \|x_n - x_{n+1}\| \to 0. \tag{5}$$

□

Theorem 2.6. *Let $\{x_n\}$ be a bounded above monotone nonincreasing sequence, then $\{x_n\}$ converges to the infimum of $\{x_n : n \in N\}$.*

Lemma 2.7. *Let K be a nonempty closed convex subset of an uniformly convex Banach space $(E, \leq)$ and $T : K \to K$ be a monotone nonspreading mapping. If $x \in K$ such that $x_{n+1} = Tx_n$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded. Then $\limsup_{n\to\infty} \|x_n - Tx_n\| \to 0$.*

Proof. From Theorem 2.6 $\{x_n\}$ is bounded and monotone decreasing then there exists $M > 0$ such that $\|x_n\| \leq M$ so,

$$\liminf_{n\to\infty} \|x_n - M\| \leq 0$$

by analogy we get

$$\liminf_{n\to\infty} \|x_{n+1} - M\| \leq 0$$

so,

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - x_{n+1}\| &\geq \liminf_{n\to\infty} [\|x_n - M\| + \|x_{n+1} - M\|] \\ &\geq \liminf_{n\to\infty} \|x_n - M\| + \liminf_{n\to\infty} \|x_{n+1} - M\| \\ &= 0.\end{aligned}$$

Therefore, we can conclude that

$$\liminf_{n\to\infty} \|x_n - x_{n+1}\| \to 0$$

on the other hand, we can conclude that

$$\liminf_{n\to\infty} \|x_n - x_{n+1}\| = \limsup_{n\to\infty} \|x_n - x_{n+1}\| \to 0. \tag{6}$$

□

Definition 2.8. Let E be a smooth Banach space and define the functional $\phi : E \times E \to R$ by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + |y|^2$$

for $x, y \in E$ from the definition of ϕ, we have

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2,$$

$$\|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \le \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

and, we can conclude that

$$2\langle x, Jy\rangle \le 2\|x\|\|y\|. \tag{7}$$

3. Main Results

3.1 Existence of Fixed Points

In this section, we prove the existence theorem of fixed points of a monotone nonspreading mapping in an uniformly convex Banach space $(E, \le)$.

Theorem 3.1. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E, \le)$ and $T : K \to K$ be a monotone nonspreading mapping. Assume that there exists $x \in K$ such that $x \le Tx$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded and $Tx_n \le y$ for some $y \in K$ and all $n \ge 1$. Then $F(T) \ne \emptyset$ and $x \le y^*$ for some $y^* \in F(T)$.*

Proof. Let $x_1 = x$, and $x_{n+1} = Tx_n = T^n x$. So, we have $x_1 = x \le Tx = x_2$, and so, we get

$$x_2 = Tx_1 = Tx \le Tx_2 = T^2 x = x_3.$$

By analogy, we must have

$$x = x_1 \le x_2 \le x_3 \le \cdots \le x_n \le x_{n+1} \le \cdots .$$

Let $K_n = \{z \in K; x_n \le z\}$ for all $n \ge 1$. Clearly, for each $n \ge 1$, K_n is closed convex ($K_n \in K$) and K_n is nonempty too ($y \in K_n$). Let $K^* = \bigcap_{n=1}^{\infty} K_n$. Then K^* is a nonempty closed convex subset of K. Since $\{x_n\}$ is bounded, we can define a function $\varphi : K^* \to [0, +\infty)$ as follows:

$$\varphi(z) = \limsup_{n\to\infty} \|x_n - z\|^2$$

for all $z \in K^*$. From Lemma 2.3, it follows that there exists $y^* \in K$ such that

$$\varphi(y^*) = \inf_{z\in K^*} \varphi(z). \tag{8}$$

Now, we show $y^* = Ty^*$. In fact, by the definition of K^*, we obtain

$$x_1 \le x_2 \le x_3 \le \cdots \le x_n \le x_{n+1} \le \cdots \le y^*.$$

Then, we have $x_{n+1} = Tx_n \le Ty^*$ by the monotonicity of T and hence, for each $n \ge 1$, $x_n \le Ty^*$. So we have $Ty^* \in K^*$. From the convexity of K^*, it follows that $\frac{y^*+Ty^*}{2} \in K^*$ and so, by equation (8), we have

$$\varphi(y^*) \le \varphi\Big(\frac{y^* + Ty^*}{2}\Big) \quad \text{and} \quad \varphi(y^*) \le \varphi(Ty^*). \tag{9}$$

By the way, from equations (5) and (7), we obtain

$$\varphi(Ty^*) = \limsup_{n\to\infty} \|x_{n+1} - Ty^*\|^2$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} \|Tx_n - Ty^*\|^2 \\
&\le \limsup_{n\to\infty} [\|x_n - y^*\|^2 + 2\langle x_n - Tx_n, J(y^* - Ty^*)\rangle] \\
&\le \limsup_{n\to\infty} [\|x_n - y^*\|^2 + 2\|x_n - Tx_n\| \|y^* - Ty^*\|] \\
&\le \limsup_{n\to\infty} \|x_n - y^*\|^2 + \limsup_{n\to\infty} 2\|x_n - Tx_n\| \|y^* - Ty^*\| \\
&\le \limsup_{n\to\infty} \|x_n - y^*\|^2 \qquad (10) \\
&= \varphi(y^*).
\end{aligned}
$$

Combining equations (9) and (10), we have

$$\varphi(Ty^*) = \varphi(y^*). \qquad (11)$$

It follows from Lemma 2.7 ($q = 2$ and $t = \frac{1}{2}$) and equation (11) that

$$
\begin{aligned}
\varphi\Big(\frac{y^* + Ty^*}{2}\Big) &= \limsup_{n\to\infty} \Big\| x_n - \frac{y^* + Ty^*}{2} \Big\|^2 \\
&= \limsup_{n\to\infty} \Big\| \frac{x_n - y^*}{2} + \frac{x_n - Ty^*}{2} \Big\|^2 \\
&\le \limsup_{n\to\infty} \Big(\frac{1}{2}\|x_n - y^*\|^2 + \frac{1}{2}\|x_n - Ty^*\|^2 - \frac{1}{4} g(\|y^* - Ty^*\|) \Big) \\
&\le \frac{1}{2}\varphi(y^*) + \frac{1}{2}\varphi(Ty^*) - \frac{1}{4} g(\|y^* - Ty^*\|) \\
&= \varphi(y^*) - \frac{1}{4} g(\|y^* - Ty^*\|).
\end{aligned}
$$

Noticing equation (9), we have

$$g(\|y^* - Ty^*\|) \le \varphi(y^*) - \varphi\Big(\frac{y^* + Ty^*}{2}\Big) \le 0$$

and from Lemma 2.2, we have $g(\|y^* - Ty^*\|) = 0$. Thus we have $y^* = Ty^*$ by the property of g. This yields the desired conclusion. □

Theorem 3.2. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E, \le)$ and $T : K \to K$ be a monotone nonspreading mapping. Assume that there exists $x \in K$ such that $Tx \le x$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded and $y \le Tx_n$ for some $y \in K$ and all $n \ge 1$. Then $F(T) \ne \emptyset$ and $y^* \le x$ for some $y^* \in F(T)$.*

Proof. Let $x_1 = x$ and $x_{n+1} = Tx_n = Tx_n$. Then $Tx = x_2 \le x_1 = x$, and so,

$$Tx_2 = T^2 x = x_3 \le x_2 = Tx_1 = Tx.$$

By analogy, we have

$$\cdots \le x_{n+1} \le x_n \le \cdots \le x_3 \le x_2 \le x = x_1.$$

Let $K_n = \{z \in K; z \le x_n\}$ for all $n \ge 1$. Clearly, for each $n \ge 1$, K_n is closed convex ($K_n \in K$) and K_n is nonempty too ($y \in K_n$). Let $K^* = \bigcap_{n=1}^{\infty} K_n$. Then K^* is a nonempty closed convex subset of

K. Since $\{x_n\}$ is bounded, we can define a function $\varphi : K^* \to [0,+\infty)$ as follows:

$$\varphi(z) = \limsup_{n\to\infty} \|x_n - z\|^2$$

for all $z \in K^*$. From Lemma 2.2, it follows that there exists $y^* \in K$ such that

$$\varphi(y^*) = \inf_{z\in K^*} \varphi(z). \tag{12}$$

Now, we show $y^* = Ty^*$. In fact, by the definition of K^*, we obtain

$$y^* \le \cdots \le x_{n+1} \le x_n \le \cdots \le x_3 \le x_2 \le x_1.$$

Then, we have $Ty^* \le x_{n+1} = Tx_n$ by the monotonicity of T and hence, for each $n \ge 1$, $Ty^* \le x_n$. So we have $Ty^* \in K^*$. From the convexity of K^*, it follows that $\frac{y^*+Ty^*}{2} \in K^*$ and so, by equation (12), we have

$$\varphi(y^*) \le \varphi\Big(\frac{y^*+Ty^*}{2}\Big) \quad \text{and} \quad \varphi(y^*) \le \varphi(Ty^*). \tag{13}$$

On the other hand, by using equations (6) and (7) we get

$$\begin{aligned}
\varphi(Ty^*) &= \limsup_{n\to\infty} \|x_{n+1} - Ty^*\|^2 = \limsup_{n\to\infty} \|Tx_n - Ty^*\|^2\\
&\le \limsup_{n\to\infty} [\|x_n - y^*\|^2 + 2\langle x_n - Tx_n, J(y^* - Ty^*)\rangle]\\
&\le \limsup_{n\to\infty} [\|x_n - y^*\|^2 + 2\|x_n - Tx_n\|\|y^* - Ty^*\|]\\
&\le \limsup_{n\to\infty} \|x_n - y^*\|^2 + \limsup_{n\to\infty} 2\|x_n - Tx_n\|\|y^* - Ty^*\|\\
&\le \limsup_{n\to\infty} \|x_n - y^*\|^2\\
&= \varphi(y^*).
\end{aligned} \tag{14}$$

Combining equations (13) and (14), we have

$$\varphi(Ty^*) = \varphi(y^*). \tag{15}$$

It follows from Lemma 2.7 ($q = 2$ and $t = \frac{1}{2}$) and (15) that

$$\begin{aligned}
\varphi\Big(\frac{y^*+Ty^*}{2}\Big) &= \limsup_{n\to\infty} \Big\|x_n - \frac{y^*+Ty^*}{2}\Big\|^2\\
&= \limsup_{n\to\infty} \Big\|\frac{x_n - y^*}{2} + \frac{x_n - Ty^*}{2}\Big\|^2\\
&\le \limsup_{n\to\infty} \Big(\frac{1}{2}\|x_n - y^*\|^2 + \frac{1}{2}\|x_n - Ty^*\|^2 - \frac{1}{4}g(\|y^* - Ty^*\|)\Big)\\
&\le \frac{1}{2}\varphi(y^*) + \frac{1}{2}\varphi(Ty^*) - \frac{1}{4}g(\|y^* - Ty^*\|)\\
&= \varphi(y^*) - \frac{1}{4}g(\|y^* - Ty^*\|).
\end{aligned}$$

Noticing equation(13), we have

$$g(\|y^* - Ty^*\|) \le \varphi(y^*) - \varphi\Big(\frac{y^*+Ty^*}{2}\Big) \le 0$$

and from Lemma 2.2, we have $g(\|y^* - Ty^*\|) = 0$. Thus we have $y^* = Ty^*$ by the property of g. This yields the desired conclusion. This completes the proof. □

3.2 The Convergence of the Mann Iteration

In this section, for a monotone nonspreading mapping T, we consider the Mann iteration sequence defined by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n \tag{16}$$

for each $n \geq 1$, where $\{\beta_n\}$ in $(0,1)$ satisfies the following condition:

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty.$$

Clearly, the above condition contains $\beta_n = \frac{1}{n+1}$ as a special case.

Lemma 3.3 (Dehaish and Khamsi [2, Lemma 3.1])**.** *Let K be a nonempty and closed convex subset of a Banach space $(E,\leq)$ and $T: K \to K$ be a monotone mapping. Assume that the sequence $\{x_n\}$ is defined by equation* (16) *and $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). If $F(T) \neq \emptyset$ and $p \leq x_1$ (or $x_1 \leq p$) for some $p \in F(T)$, then*

(1) *$\{x_n\}$ is bounded and $x_n \leq x_{n+1} \leq Tx_n$ (or $Tx_n \leq x_{n+1} \leq x_n$);*

(2) *$x_n \leq x$ (or $x \leq x_n$) for all $n \geq 1$ provided $\{x_n\}$ weakly converges to a point $x \in K$.*

Lemma 3.4. *Let K be a nonempty and closed convex subset of a Banach space $(E,\leq)$ and $T: K \to K$ be a monotone nonspreading mapping. Assume that the sequence $\{x_n\}$ is defined by equation* (16) *and $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). If $F(T) \neq \emptyset$ and $p \leq x_1$ (or $x_1 \leq p$) for some $p \in F(T)$, then*

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists.}$$

Proof. Assume that $p \leq x_n$ for any $n \geq 1$. Since T is monotone, then we obtain $p = Tp \leq Tx_1$. Since the order interval $[p, \rightarrow)$ is convex, assume $p \leq x_2$ since T is monotone, then we have $Tp \leq Tx_2$. By induction we will show that $p \leq x_n$ for any $n \geq 1$, as claimed. Since T is monotone nonspreading, from equation (7), we have $p = Tp$ and we get

$$\begin{aligned}
\|Tx_n - p\|^2 &= \|Tx_n - Tp\|^2 \\
&\leq \|x_n - p\|^2 + 2\langle x_n - Tx_n, J(p - Tp)\rangle \\
&\leq \|x_n - p\|^2 + 2\|x_n\|\|p - Tp\| \\
&\leq \|x_n - p\|^2
\end{aligned}$$

it follows that,

$$\|Tx_n - p\| \leq \|x_n - p\|.$$

Since equation (16), which implies

$$\begin{aligned}
\|x_{n+1} - p\| &\leq t_n\|Tx_n - p\| + \|(1-t_n)\|x_n - p\| \\
&\leq t_n\|x_n - p\| + \|(1-t_n)\|x_n - p\| \\
&\leq \|x_n - p\|
\end{aligned}$$

for any $n \geq 1$. This means that $\|x_n - p\|$ is a monotone sequence, which implies that $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Theorem 3.5. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\leq)$ and $T:K\to K$ be a monotone nonspreading mapping. Assume that the sequence $\{x_n\}$ is defined by equation (16) and $x_1\leq Tx_1$ (or $Tx_1\leq x_1$). If $F(T)\neq\emptyset$ and $p\leq x_1$ (or $x_1\leq p$) for some $p\in F(T)$, then*

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

Proof. It follows from Lemma 3.4 that

$$p\leq x_1\leq x_n \text{ (or } x_n\leq x_1\leq p)$$

for all $n\geq 1$. Then it follows from the nonspreadingness of T, $p=Tp$ and an application of Lemma 2.7 ($q=2$ and $t=\beta_n$) that

$$\begin{aligned}\|x_{n+1}-p\|^2&=\|\beta_n(x_n-p)+(1-\beta_n)(Tx_n-Tp)\|^2\\&\leq\beta_n\|x_n-p\|^2+(1-\beta_n)\|Tx_n-Tp\|^2-\beta_n(1-\beta_n)g(\|x_n-Tx_n\|)\\&\leq\beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2+2\|x_n-Tx_n\|\|p-Tp\|-\beta_n(1-\beta_n)g(\|x_n-Tx_n\|)\\&\leq\beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2-\beta_n(1-\beta_n)g(\|x_n-Tx_n\|)\\&\leq\|x_n-p\|^2-\beta_n(1-\beta_n)g(\|x_n-Tx_n\|)\end{aligned}$$

and so

$$\beta_n(1-\beta_n)g(\|x_n-Tx_n\|)\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Therefore, we have

$$\sum_{n=1}^{\infty}\beta_n(1-\beta_n)g(\|x_n-Tx_n\|)\leq\|x_1-p\|^2<+\infty. \tag{17}$$

Now, we claim that there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty}g(\|x_{n_k}-Tx_{n_k}\|)=0. \tag{18}$$

Suppose that the conclusion is not true. Then, for all subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty}g(\|x_{n_k}-Tx_{n_k}\|)>0$, we have

$$\liminf_{n\to\infty}g(\|x_n-Tx_n\|)>0.$$

Thus there exists a positive number a and a positive integer N such that $g(\|x_n-Tx_n\|)>a>0$ for all $n>N$. Consequently, we have

$$\beta_n(1-\beta_n)g(\|x_n-Tx_n\|)\geq a\beta_n(1-\beta_n)$$

and hence, by the condition $\sum_{n=1}^{\infty}\beta_n(1-\beta_n)=+\infty$, we obtain

$$\sum_{n=1}^{\infty}\beta_n(1-\beta_n)g(\|x_n-Tx_n\|)=+\infty.$$

This contradicts equation (17). So equation (18) holds and hence, by the property of $g(0)=0$, we have

$$\lim_{k\to\infty}\|x_{n_k}-Tx_{n_k}\|=0.$$

Otherwise, we obtain

$$\|x_{n+1}-Tx_{n+1}\|=\|\beta_n(x_n-Tx_n)+(Tx_n-Tx_{n+1})\|$$

$$\begin{aligned}&\le \beta_n\|x_n - Tx_n\| + \|x_{n+1} - x_n\|\\&= \beta_n\|x_n - Tx_n\| + (1-\beta_n)\|x_n - Tx_n\|\\&= \|x_n - Tx_n\|.\end{aligned}$$

Therefore, the sequence $\{\|x_n - Tx_n\|\}$ is monotonically nonincreasing and hence it follows that $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists. This yields the desired conclusion.

Recall that a Banach space E is said to satisfy *Opial's condition* ([7]) if a sequence $\{x_n\}$ with $\{x_n\}$ weakly converges to a point $x \in E$ implies

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$$

for all $y \in E$ with $y \neq x$. □

Next, we show the weak convergence of the sequence $\{x_n\}$ defined by equation (16). The proof is similar to ones of Dehaish and Khamsi [2], but, for more details, we give the proof.

Theorem 3.6. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\le)$ and $T : K \to K$ be a monotone nonspreading mapping. Assume that E satisfies Opial's condition and the sequence $\{x_n\}$ is defined by equation (16) with $x_1 \le Tx_1$ (or $Tx_1 \le x_1$). If $F(T) \neq \emptyset$ and $p \le x_1$ (or $x_1 \le p$) for some $p \in F(T)$, then $\{x_n\}$ weakly converges to a fixed point x^* of T.*

Proof. It follows from Lemma 3.4 and Theorem 3.5 that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to a point $x^* \in K$. Following Lemma 3.4, we have $x_1 \le x_{n_k} \le x^*$ (or $x^* \le x_{n_k} \le x_1$) for all $k \ge 1$. In particular, we have

$$\lim_{k\to\infty}\|x_{n_k} - Tx_{n_k}\| = 0.$$

Now, we claim that $x^* = Tx^*$. In fact, assume that this is not true. Then, from the nonspreadingness of T and Opial's condition, it follows that

$$\begin{aligned}\limsup_{k\to\infty}\|x_{n_k} - x^*\| &< \limsup_{k\to\infty}\|x_{n_k} - Tx^*\|\\&\le \limsup_{k\to\infty}(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tx^*\|)\\&\le \limsup_{k\to\infty}(\|Tx_{n_k} - Tx^*\|).\end{aligned}\tag{19}$$

Consider equation (19),

$$\begin{aligned}\limsup_{k\to\infty}\|Tx_{n_k} - Tx^*\|^2 &\le \limsup_{k\to\infty}[\|x_{n_k} - x^*\|^2 + 2\|x_{n_k} - Tx_{n_k}\|\|x^* - Tx^*\|]\\&\le \limsup_{k\to\infty}\|x_{n_k} - x^*\|^2 + \limsup_{k\to\infty}2\|x_{n_k} - Tx_{n_k}\|\|x^* - Tx^*\|\\&\le \limsup_{k\to\infty}\|x_{n_k} - x^*\|^2.\end{aligned}\tag{20}$$

So, we get

$$\limsup_{k\to\infty}\|Tx_{n_k} - Tx^*\| \le \limsup_{k\to\infty}\|x_{n_k} - x^*\|.\tag{21}$$

From equations (19) and (21), we obtain

$$\limsup_{k\to\infty} \|x_{n_k} - x^*\| \le \limsup_{k\to\infty} \|x_{n_k} - x^*\|$$

which is a contradiction. Thus, by Lemma 3.4, it follows that the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Now, we show that $\{x_n\}$ weakly converges to the point x^*. Suppose that this is not true. Then There exists a subsequence $\{x_{n_j}\}$ to converge weakly to a point $z \in K$ and $z \neq x^*$. Similarly, it follows that $z = Tz$ and $\lim_{n\to\infty} \|x_n - z\|$ exists. It follows from Opial's condition that

$$\lim_{n\to\infty} \|x_n - z\| < \lim_{n\to\infty} \|x_n - x^*\| = \limsup_{i\to\infty} \|x_{n_i} - x^*\| < \lim_{n\to\infty} \|x_n - z\|.$$

This is a contradiction and hence $x^* = z$. This completes the proof. □

4. Conclusions

We prove some existence theorems of fixed point for monotone nonspreading mapping in a Banach space E with the partial order $\le$.

Theorem 4.1. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\le)$ and $T : K \to K$ be a monotone nonspreading mapping. Assume that there exists $x \in K$ such that $x \le Tx$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded and $Tx_n \le y$ for some $y \in K$ and all $n \ge 1$. Then $F(T) \neq \emptyset$ and $x \le y^*$ for some $y^* \in F(T)$.*

Theorem 4.2. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\le)$ and $T : K \to K$ be a monotone nonspreading mapping. Assume that there exists $x \in K$ such that $Tx \le x$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded and $y \le Tx_n$ for some $y \in K$ and all $n \ge 1$. Then $F(T) \neq \emptyset$ and $y^* \le x$ for some $y^* \in F(T)$.*

In part of convergence theorem, we prove a weak convergence theorem for monotone nonspreading in order Banach space $(E,\le)$.

Theorem 4.3. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\le)$ and $T : K \to K$ be a monotone nonspreading mapping. Assume that E satisfies Opial's condition and the sequence $\{x_n\}$ is defined by equation* (16) *with $x_1 \le Tx_1$ (or $Tx_1 \le x_1$). If $F(T) \neq \emptyset$ and $p \le x_1$ (or $x_1 \le p$) for some $p \in F(T)$, then $\{x_n\}$ weakly converges to a fixed point x^* of T.*

And we can get some results if we reduce some conditions for prove some existence theorems of fixed point by using a monotone nonexpansive mapping T in a Banach space E with the partial order "$\le$", in Theorem 4.1 and 4.2, we have following corollaries respectively.

Corollary 4.4. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\le)$ and $T : K \to K$ be a monotone nonexpansive mapping. Assume that there exists $x \in K$ such that $x \le Tx$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded and $Tx_n \le y$ for some $y \in K$ and all $n \ge 1$. Then $F(T) \neq \emptyset$ and $x \le y^*$ for some $y^* \in F(T)$.*

Corollary 4.5. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\le)$ and $T : K \to K$ be a monotone nonexpansive mapping. Assume that there exists $x \in K$*

such that $Tx \le x$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded and $y \le Tx_n$ for some $y \in K$ and all $n \ge 1$. Then $F(T) \ne \emptyset$ and $y^* \le x$ for some $y^* \in F(T)$.

And if we consider the convergence of Mann iteration for a monotone nonexpansive mapping T, in Theorem 3.5 and 4.3 by using Dehaish and Khamsi [2, Lemmas 3.1 and 3.2], we have following corollary:

Corollary 4.6. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\le)$ and $T: K \to K$ be a monotone nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by equation (16) and $x_1 \le Tx_1$ (or $Tx_1 \le x_1$). If $F(T) \ne \emptyset$ and $p \le x_1$ (or $x_1 \le p$) for some $p \in F(T)$, then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Corollary 4.7. *Let K be a nonempty and closed convex subset of an uniformly convex Banach space $(E,\le)$ and $T: K \to K$ be a monotone nonexpansive mapping. Assume that E satisfies Opial's condition and the sequence $\{x_n\}$ is defined by equation (16) with $x_1 \le Tx_1$ (or $Tx_1 \le x_1$). If $F(T) \ne \emptyset$ and $p \le x_1$ (or $x_1 \le p$) for some $p \in F(T)$, then $\{x_n\}$ weakly converges to a fixed point x^* of T.*

Acknowledgement

The author would like to thank Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Research Cluster (CLASSIC), Faculty of Science, KMUTT for financial support.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Bachar and M. A. Khamsi, Fixed points of monotone mappings and application to integral equations, *Fixed Point Theory Appl.* **2015** (2015), 110, DOI: 10.1186/s13663-015-0362-x.

[2] B. A. B. Dehaish, M. A. Khamsi, Mann iteration process for monotone nonexpansive mappings, *Fixed Point Theory Appl.* **2015** (2015), 177, DOI: 10.1186/s13663-015-0416-0.

[3] F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings relate to maximal monotone operators in Banach spaces, *Arch. Math.* (Basel) **91** (2008), 166 – 177, DOI: 10.1007/s00013-008-2545-8.

[4] L. S. Liu, Ishikawa and Mann iteration process with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.* **194** (1995), 114 – 125, DOI: 10.1006/jmaa.1995.1289.

[5] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953), 506 – 510, DOI: 10.1090/S0002-9939-1953-0054846-3.

[6] E. Naraghirad, On an open question of Takasashi for nonpsreading mapping in Banach spaces, *Fixed Point Theory Appl.* **2013** (2013), 228, DOI: 10.1186/1687-1812-2013-228.

[7] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings in Banach spaces, *Bull. Amer. Math. Soc.* **73** (1967), 591 – 597, DOI: 10.1090/S0002-9904-1967-11761-0.

[8] W. Takahashi, *Nonlinear Functional Analysis – Fixed Point Theory and its Applications*, Yokohama Publishers Inc., Yokohama (2000).

[9] H. K. Xu, Inequality in Banach spaces with applications, *Nonlinear Anal.* **16** (1991), 1127 – 1138, DOI: 10.1016/0362-546X(91)90200-K.

[10] H. Zhang and Y. Su, Convergence theorems for strict pseudo-contractions in q-uniformly smooth Banach spaces, *Nonlinear Anal.* **71** (2009), 4572 – 4580, DOI: 10.1016/j.na.2009.03.033.

[11] H. Zhou, Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces, *Nonlinear Anal.* **69** (2008), 3160 – 3173, DOI: 10.1016/j.na.2007.09.009.