Some Functional Equations arising from the Concept of Unexpectedness of an Event in Information Theory

Prem Nath

Abstract. The general solutions of some functional equations, without imposing any regularity condition on any one of unknown functions appearing in them, have been obtained.

1. Introduction

In this section, we mention (i) some basic concepts from probability theory (see [6], p-83) (ii) some basic definitions based upon functional equations (iii) the concept of unexpectedness or self-information of an event and finally functional equations arising from the concept of unexpectedness.

(i) Some concepts from probability theory

Let $(\Omega, \mathscr{B}, \mu)$ be a probability space, that is, Ω is a nonempty set which usually represents the sample space of a random experiment; $\mathscr{B}$ is a nonempty collection of subsets of Ω such that (i) $\Omega \in \mathscr{B}$ (ii) if $E \in \mathscr{B}$, then the set $\widetilde{E} \in \mathscr{B}$ where $\widetilde{E} = \Omega - E = \{w \in \Omega : w \notin E\}$ is the complement of E in Ω (iii) if $E_i \in \mathscr{B}$, $i = 1, 2, \ldots$ then $\bigcup_{i=1}^{\infty} E_i \in \mathscr{B}$; $\mu : \mathscr{B} \to I = [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $\mathbb{R}$ denoting the set of all real numbers, is a nonnegative countably additive set function with $\mu(\Omega) = 1$. The nonempty collection $\mathscr{B}$, of subsets of Ω, possessing the three properties (i), (ii) and (iii) mentioned above is called a σ-algebra or a σ-field of subsets of Ω. It can be shown that the σ-algebra $\mathscr{B}$ is closed under finite unions and finite intersections. In particular, if $A \in \mathscr{B}, B \in \mathscr{B}$, then $A \cup B \in \mathscr{B}$ and also $A \cap B \in \mathscr{B}$. The set function $\mu : \mathscr{B} \to [0,1]$ possessing the above mentioned properties is usually called a probability measure. Any element $w \in \Omega$ is called an elementary event. The elements of the σ-algebra $\mathscr{B}$ are called events and the real

number $\mu(E)$, $E \in \mathscr{B}$, is called the probability of the event $E \in \mathscr{B}$. Finally, any two events $A \in \mathscr{B}$ and $B \in \mathscr{B}$ are said to be independent if $\mu(A \cap B) = \mu(A)\mu(B)$.

(ii) Some definitions

In this paper, we shall be concerned only with those events E for which $0 < \mu(E) \leq 1$. For this reason, we need to consider the interval $I_0 = \{x \in \mathbb{R} : 0 < x \leq 1\} =]0, 1]$. Motivated by functional equations, we give below some definitions:

Definition 1. A function $\ell : I_0 \to \mathbb{R}$ is said to be logarithmic on the interval I_0 if it satisfies the functional equation (see [1], p-26)

(1.1) $\ell(pq) = \ell(p) + \ell(q)$

for all $p \in I_0$, $q \in I_0$.

  Notice that if we put $q = 1$ (or $p = 1$) in (1.1), we get $\ell(1) = 0$.
  The function $\ell : I_0 \to \mathbb{R}$ which vanishes identically on I_0, that is $\ell(p) = 0$ for all $p \in I_0$, is obviously logarithmic on I_0 in the sense of Definition 1.
  The function $p \mapsto -\log_2 p$, $p \in I_0$, is logarithmic on I_0 in the sense of Definition 1. It does not vanish identically on $]0, 1[$.
  We shall denote by the symbol ℓ^*, any function logarithmic on I_0 in the sense of Definition 1 but possessing the additional property that there exists an element $p_0 \in]0, 1[$ such that $\ell(p_0) \neq 0$ where $]0, 1[= \{x \in \mathbb{R} : 0 < x < 1\}$.

Definition 2. A function $M : I_0 \to \mathbb{R}$ is said to be multiplicative on I_0 if $M(1) = 1$ and $M(pq) = M(p)M(q)$ holds for all $p \in]0, 1[$ and $q \in]0, 1[$.

  Notice that according to this Definition 2, the function $p \mapsto p$, $p \in I_0$, is multiplicative on I_0 whereas the function $p \mapsto 0$, $p \in I_0$, is certainly not.

(iii) The concepts of unexpectedness and self-information

Following A. R nyi ([6], p-26), let us associate with every event $A \in \mathscr{B}$, with $0 < \mu(E) \leq 1$, a real number $V(A)$ which represents the unexpectedness of the event A. Based upon intuition, we mention some properties mentioned in [6] which $V(A)$ should possess:

  (I) $V(A)$ depends only on the probability $\mu(A)$ of occurrence of A; that is,

(1.2) $V(A) = f(\mu(A))$

where $f : I_0 \to \mathbb{R}$ is a monotonically decreasing function of $x \in I_0$. This simply means that the more improbable an event A, the greater the unexpectedness of its occurrence.

  (II) If the events $A \in \mathscr{B}$ and $B \in \mathscr{B}$ are independent, then the unexpectedness associated with the occurrence of the event $A \cap B$ is equal to the sum of their

individual unexpectedness, that is

(1.3) $V(A \cap B) = V(A) + V(B)$.

(III) The unexpectedness of an event having probability $\dfrac{1}{2}$ is one, that is,

$$f\left(\frac{1}{2}\right) = 1.$$

It can be shown that if the three properties (I), (II) and (III) hold, then

(1.4) $V(A) = -\log_2 \mu(A)$.

Indeed, the unexpectedness of an event $A \in \mathscr{B}$ is defined by (1.4). The real number $-\log_2 \mu(A)$ is also called the amount of self-information (see [7], p-78) associated with the event A which occurs with probability $\mu(A)$, $0 < \mu(A) \leq 1$.

For the sake of notational convenience, we write $\mu(A) = p \in I_0$ and

(1.5) $f_1(p) = -\log_2 p$

where $f_1 : I_0 \to \mathbb{R}$. From the point of view of functional equations, it seems desirable to consider the sequence of functions $f_n : I_0 \to \mathbb{R}$, $n = 1, 2, \dots$ defined as

(1.6) $f_n(p) = (-\log_2 p)^n$

for all $p \in I_0$. Clearly, the function f_1 satisfies the logarithmic functional equation

$$f_1(pq) = f_1(p) + f_1(q)$$

for all $p \in I_0$, $q \in I_0$. Moreover, the functions f_1 and f_2, taken together, satisfy the functional equation

(1.7) $f_2(pq) = f_2(p) + f_2(q) + 2f_1(p)f_1(q)$

for all $p \in I_0$, $q \in I_0$. This functional equation motivates us to consider the functional equation

(FE1) $f(pq) = f(p) + f(q) + c\, g(p)g(q)$

where $f : I_0 \to \mathbb{R}$, $g : I_0 \to \mathbb{R}$ are unknown functions, $p \in I_0$, $q \in I_0$ and $c \neq 0$ is a given real number. Besides (FE1), we also consider in this paper four more functional equations, namely,

(FE2) $f(pq) = f(p) + f(q) + g(p)h(q)$

(FE3) $F(pq) = F(p) + F(q) + G(p)H(q) + \alpha\, G(p) + \beta\, H(q)$

(FE4) $F(pq) = K(p) + N(q) + G(p)H(q)$

(FE5) $f(pq) = K(p) + N(q) + S(p)T(q) + aS(p) + bT(q)$

where f, g, F, G, H, K, N, S and T are real-valued functions each with domain I_0; α, β, a and b are given real numbers.

Sections 2 to 6 are devoted respectively to the study of functional equations (FE1), (FE2), (FE3), (FE4) and (FE5).

2. On the functional equation (FE1)

In this section, we investigate all possible solutions of (FE1) without imposing any regularity condition on the functions f and g. We prove:

Theorem 1. *Let $c \neq 0$ be a given real number and $f : I_0 \to \mathbb{R}$, $g : I_0 \to \mathbb{R}$ be functions which satisfy the functional equation* (FE1) *for all $p \in I_0$, $q \in I_0$. Then, any general solution (f, g) of* (FE1) *is of the form*

(2.1) $f(p) = \ell(p)$ $g(p) \equiv 0$

or

(2.2) $f(p) = \ell(p) - cd^2$ $g(p) = d$

or

(2.3) $f(p) = \ell(p) + \dfrac{1}{2}c[\ell^*(p)]^2$ $g(p) = \ell^*(p)$

or

(2.4) $f(p) = \ell(p) + c\lambda_0^2[M(p) - 1]$ $g(p) = \lambda_0[M(p) - 1]$

where λ_0 and d are arbitrary nonzero real constants; $\ell : I_0 \to \mathbb{R}$ is a function logarithmic on I_0 in the sense of Definition 1; $\ell^* : I_0 \to \mathbb{R}$ is a function logarithmic on I_0 in the sense of Definition 1 but it possesses the additional property that it does not vanish identically on the interval $]0, 1[$; and $M : I_0 \to \mathbb{R}$ is a function multiplicative on I_0 in the sense of Definition 2 but it possesses the additional property that the function $q \mapsto M(q) - 1$ does not vanish identically on the interval $]0, 1[$.

Before giving the proof of this theorem, we prove the following:

Lemma 2. *Let $c \neq 0$ be a given real number and $f : I_0 \to \mathbb{R}$, $g : I_0 \to \mathbb{R}$ be functions which satisfy the functional equation* (FE1) *for all $p \in I_0$, $q \in I_0$. Then the function $g : I_0 \to \mathbb{R}$ satisfies the equation*

(2.5) $[g(pq) - g(q)]g(r) = [g(qr) - g(q)]g(p)$

for all $p, q, r \in I_0$.

Proof. Let $p \in I_0$, $q \in I_0$ and $r \in I_0$. Then, by associative law of multiplication of real numbers, we have $p(qr) = (pq)r$. Hence,

$$f(p(qr)) = f((pq)r).$$

Now

$$f(p(qr)) = f(p) + f(qr) + c\,g(p)g(qr)$$
$$= f(p) + f(q) + f(r) + c\,g(q)g(r) + c\,g(p)g(qr).$$

Similarly

$$f((pq)r) = f(p) + f(q) + f(r) + c\,g(p)g(q) + c\,g(pq)g(r).$$

Equating the values of $f(p(qr))$ and $f((pq)r)$ and using the fact $c \neq 0$, equation (2.5) follows.

***Proof of Theorem* 1.** If g vanishes identically on I_0, then solution (2.1) is obvious. Now suppose g does not vanish identically on I_0. Then there exists an element $r_0 \in I_0$ such that $g(r_0) \neq 0$.

Let us consider the case when $r_0 = 1$. Then $g(1) \neq 0$. Let $d = g(1)$. Then $d \neq 0$. Putting $r = 1$ in (2.5) and using the fact that $g(1) \neq 0$, it follows that $g(pq) = g(q)$ for all $p \in I_0$, $q \in I_0$. Hence $g(p) = d \neq 0$ for all $p \in I_0$. Using this form of g in (FE1), we get the equation $f(pq) = f(p) + f(q) + cd^2$ which can be written in the form

$$f(pq) + cd^2 = (f(p) + cd^2) + (f(q) + cd^2)$$

valid for all $p \in I_0$, $q \in I_0$. Define $\ell : I_0 \to \mathbb{R}$ as $\ell(p) = f(p) + cd^2$ for all $p \in I_0$. Then $\ell(pq) = \ell(p) + \ell(q)$ for all $p \in I_0$, $q \in I_0$. So, $\ell : I_0 \to \mathbb{R}$ is a function logarithmic on I_0 in the sense of Definition 1. Also, $f(p) = \ell(p) - cd^2$ for all $p \in I_0$. Thus, solution (2.2) is obtained.

Now consider the case when $r_0 \in]0, 1[$. Then $g(r_0) \neq 0$ but now we must have $g(1) = 0$. Substituting $r = r_0 \in]0, 1[$ in (2.5) and using the fact that $g(r_0) \neq 0$, we obtain

$$(2.6) \qquad g(pq) = g(q) + M(q)g(p)$$

where $p \in I_0$, $q \in I_0$ and $M : I_0 \to \mathbb{R}$ is defined as

$$(2.7) \qquad M(q) = [g(r_0)]^{-1}[g(qr_0) - g(q)]$$

for all $q \in I_0$. Since $g(1) = 0$, it follows from (2.7) that $M(1) = 1$. The left hand side of (2.6) is symmetric in p and q. So, should be its right hand side. This gives $g(q) + M(q)g(p) = g(p) + M(p)g(q)$ which can be written in the form

$$(2.8) \qquad [M(q) - 1]g(p) = [M(p) - 1]g(q)$$

valid for all $p \in I_0$, $q \in I_0$.

If $q \mapsto M(q) - 1$, $q \in I_0$, vanishes identically on I_0, then $M(q) = 1$ for all $q \in I_0$ and equation (2.6) reduces to the equation $g(pq) = g(p) + g(q)$ valid for all $p \in I_0$ and $q \in I_0$. So, $g : I_0 \to \mathbb{R}$ is a function logarithmic on I_0 in the sense of Definition 1. But $g(r_0) \neq 0$, $r_0 \in]0, 1[$. So, g is of the form $g(p) = \ell^*(p)$ for all

$p \in I_0$ where $\ell^* : I_0 \to \mathbb{R}$ is a function logarithmic on I_0 in the sense of Definition 1 and does not vanish identically on $]0, 1[$ as $0 \neq g(r_0) = \ell^*(r_0)$, $r_0 \in]0, 1[$. Using this form of g in (FE1), we obtain

$$f(pq) = f(p) + f(q) + c\,\ell^*(p)\ell^*(q)$$

which can be written in the form

$$f(pq) - \frac{1}{2}c[\ell^*(pq)]^2 = \left\{ f(p) - \frac{1}{2}c[\ell^*(p)]^2 \right\} + \left\{ f(q) - \frac{1}{2}c[\ell^*(q)]^2 \right\}$$

valid for all $p \in I_0$, $q \in I_0$. Now define $\ell : I_0 \to \mathbb{R}$ as

$$\ell(p) = f(p) - \frac{1}{2}c[\ell^*(p)]^2$$

for all $p \in I_0$. Then $\ell : I_0 \to \mathbb{R}$ is a function logarithmic on I_0 in the sense of Definition 1. Also, $f(p) = \ell(p) + \frac{1}{2}c[\ell^*(p)]^2$ for all $p \in I_0$. Thus, solution (2.3) is obtained.

Now consider the case when the function $q \mapsto M(q) - 1$ does not vanish identically on I_0. In this case, since $M(1) = 1$, there exists an element $q_0 \in]0, 1[$ such that $[M(q_0) - 1] \neq 0$. Setting $q = q_0$ in (2.8) and using $[M(q_0) - 1] \neq 0$, it follows that $g(p) = \lambda_0[M(p) - 1]$ for all $p \in I_0$ with $\lambda_0 = g(q_0)[M(q_0) - 1]^{-1}$. We prove that $\lambda_0 \neq 0$. To the contrary, suppose $\lambda_0 = 0$ then $g(p) = 0$ for all $p \in I_0$ contradicting $g(r_0) \neq 0$, $r_0 \in]0, 1[$. So, we have

(2.9) $g(p) = \lambda_0[M(p) - 1], \quad \lambda_0 \neq 0$

for all $p \in I_0$. Making use of this form of g in (2.6), we have

(2.10) $M(pq) = M(p)M(q)$

for all $p \in I_0$, $q \in I_0$. Hence, in particular, $M(pq) = M(p)M(q)$ for all $p \in]0, 1[$, $q \in]0, 1[$. So $M : I \to \mathbb{R}$, defined by (2.7), is a function multiplicative on I_0 in the sense of Definition 2. Now, from (FE1), (2.9) and (2.10), it follows that

$$f(pq) = f(p) + f(q) + c\lambda_0^2 M(pq) - c\lambda_0^2 M(p) - c\lambda_0^2 M(q) + c\lambda_0^2$$

which can be written in the form

(2.11) $f(pq) - c\lambda_0^2 M(pq) + c\lambda_0^2$

$$= \ [f(p) - c\lambda_0^2 M(p) + c\lambda_0^2] + [f(q) - c\lambda_0^2 M(q) + c\lambda_0^2]$$

valid for all $p \in I_0$, $q \in I_0$. Now define $\ell : I_0 \to \mathbb{R}$ as

(2.12) $\ell(p) = f(p) - c\lambda_0^2 M(p) + c\lambda_0^2$

for all $p \in I$. Then, (2.11) gives $\ell(pq) = \ell(p) + \ell(q)$ for all $p \in I_0$, $q \in I_0$. So, ℓ is a function logarithmic on I_0 in the sense of Definition 1. Now, from (2.12),

$f(p) = \ell(p) + c\lambda_0^2[M(p) - 1]$ where $\lambda_0 \neq 0$ is an arbitrary real constant. Thus, solution (2.4) is also obtained. $\qquad\square$

3. **On the functional equation (FE2)**

Making use of Theorem 1, we prove:

Theorem 3. *Suppose $f : I_0 \to \mathbb{R}$, $g : I_0 \to \mathbb{R}$ and $h : I_0 \to \mathbb{R}$ are functions which satisfy the functional equation* (FE2) *for all $p \in I_0$, $q \in I_0$. Then, any general solution (f, g, h) of* (FE2) *is of the form*

(3.1) $\quad f(p) = \ell(p) \qquad\qquad\qquad g$ *arbitrary* $\qquad\qquad h(p) \equiv 0$

or

(3.2) $\quad f(p) = \ell(p) \qquad\qquad\qquad g(p) \equiv 0 \qquad\qquad h$ *arbitrary*

or

(3.3) $\quad f(p) = \ell(p) - bd^2 \qquad\qquad g(p) = d \qquad\qquad h(p) = bd$

or

(3.4) $\quad f(p) = \ell(p) + \dfrac{1}{2} b[\ell^*(p)]^2 \qquad g(p) = \ell^*(p) \qquad\quad h(p) = b\,\ell^*(p)$

or

(3.5) $f(p) = \ell(p) + b\lambda_0^2[M(p) - 1] \quad g(p) = \lambda_0[M(p) - 1] \quad h(p) = b\lambda_0[M(p) - 1]$

where b, d, λ_0 are arbitrary nonzero real constants and the functions $\ell : I_0 \to \mathbb{R}$, $\ell^* : I_0 \to \mathbb{R}$ and $M : I_0 \to \mathbb{R}$ are as described in the statement of Theorem 1.

Proof. From (FE2), it is obvious that

(3.6) $\quad g(p)h(q) = g(q)h(p)$

for all $p \in I_0$, $q \in I_0$. If $g(p) \equiv 0$ on I_0, then solution (3.2) is obvious. If g does not vanish identically on I_0, then there exists an element $p_0 \in I$ such that $g(p_0) \neq 0$. Setting $p = p_0$ in (3.6) and using the fact that $g(p_0) \neq 0$, it follows that $h(q) = b\,g(q)$ where $b = [g(p_0)]^{-1}h(p_0)$ is an arbitrary real constant. If $b = 0$, then $h(q) \equiv 0$ on I_0. In this case, solution (3.1) follows. If $b \neq 0$, then h does not vanish identically on I_0 and also

(3.7) $\quad h(q) = b\,g(q), \quad b \neq 0$

for all $q \in I_0$. From (FE2) and (3.7), the equation

$$f(pq) = f(p) + f(q) + b\,g(p)g(q), \quad b \neq 0$$

follows. Making use of Theorem 1 and (3.7), the solutions (3.3) to (3.5) follow. $\qquad\square$

Corollary 4. *Suppose $f : I_0 \to \mathbb{R}$, $g : I_0 \to \mathbb{R}$ and $h : I_0 \to \mathbb{R}$ are functions which satisfy the functional equation* (FE2) *for all $p \in I_0$, $q \in I_0$. If*

(3.8) $f(1) = 0,$ $g(1) = 0,$ $h(1) = 0$

then any general solution (f, g, h) of (FE2) *is of the form*

(3.9) $f(p) = \ell(p)$ *g arbitrary with $g(1) = 0$* $h(p) \equiv 0$

or

(3.10) $f(p) = \ell(p)$ $g(p) \equiv 0$ *h arbitrary with $h(1) = 0$*

or (3.4) *or* (3.5).

The proof of Corollary 4 is simple and hence omitted.

Note 1. From (3.7), we have $g(q) = \dfrac{1}{b}h(q)$ for all $q \in I_0$ with $b \neq 0$. Using this form of g in (FE2), we obtain the functional equation

(3.11) $f(pq) = f(p) + f(q) + \dfrac{1}{b}h(p)h(q)$

in which $p \in I_0$, $q \in I_0$, $b \neq 0$, $h : I_0 \to \mathbb{R}$ does not vanish identically on I_0 and $f : I_0 \to \mathbb{R}$. Making use of Theorem 1, all solutions (f, h), of (3.11), in which h does not vanish identically on I_0, can be determined. There are only three such solutions.

From these three solutions, making use of the fact that $g(q) = \dfrac{1}{b}h(q)$, the following three solutions can be obtained:

(3.12) $f(p) = \ell(p) - \dfrac{d^2}{b},$ $g(p) = \dfrac{d}{b},$ $h(p) = d$

(3.13) $f(p) = \ell(p) + \dfrac{1}{2b}[\ell^*(p)]^2,$ $g(p) = \dfrac{1}{b}\ell^*(p),$ $h(p) = \ell^*(p)$

(3.14) $f(p) = \ell(p) + \dfrac{1}{b}\lambda_0^2[M(p) - 1],$ $g(p) = \dfrac{1}{b}\lambda_0[M(p) - 1],$ $h(p) = \lambda_0[M(p) - 1].$

Indeed, one can obtain (i)(3.12) by replacing d by $\dfrac{d}{b}$ in (3.3), (ii)(3.13) by replacing $\ell^*(p)$ by $\dfrac{1}{b}\ell^*(p)$ in (3.4) and (iii)(3.14) by replacing λ_0 by $\dfrac{\lambda_0}{b}$ in (3.5). Thus, (3.12) to (3.14) are not new solutions.

4. On the functional equation (FE3)

In this section, making use of Theorem 3, we prove the following:

Theorem 5. *Let α and β be given real numbers. Suppose $F : I_0 \to \mathbb{R}$, $G : I_0 \to \mathbb{R}$ and $H : I_0 \to \mathbb{R}$ are functions which satisfy the functional equation* (FE3) *for all $p \in I_0$, $q \in I_0$. Then, any general solution (F, G, H) of* (FE3) *is of the form*

(4.1) $F(p) = \ell(p) + \alpha\beta$ *G arbitrary* $H(p) = -\alpha$

or

(4.2) $F(p) = \ell(p) + \alpha\beta$ $G(p) = -\beta$ *H arbitrary*

or

(4.3) $F(p) = \ell(p) - bd^2 + \alpha\beta$ $G(p) = d - \beta$ $H(p) = bd - \alpha$

or

(4.4) $F(p) = \ell(p) + \dfrac{1}{2} b [\ell^*(p)]^2 + \alpha\beta$ $G(p) = \ell^*(p) - \beta$ $H(p) = b\ell^*(p) - \alpha$

or

(4.5) $F(p) = \ell(p) + b\lambda_0^2 [M(p) - 1] + \alpha\beta$ $G(p) = \lambda_0 [M(p) - 1] - \beta$

$$H(p) = b\lambda_0 [M(p) - 1] - \alpha$$

where b, d, λ_0 are arbitrary nonzero real constants and $\ell : I_0 \to \mathbb{R}$, $\ell^* : I_0 \to \mathbb{R}$ and $M : I_0 \to \mathbb{R}$ are as mentioned in the statement of Theorem 1.

Proof. Let us write (FE3) in the form

$$F(pq) = F(p) + F(q) + [G(p) + \beta][H(q) + \alpha] - \alpha\beta.$$

Then

(4.6) $F(pq) - \alpha\beta = [F(p) - \alpha\beta] + [F(q) - \alpha\beta] + [G(p) + \beta][H(q) + \alpha].$

Define $f : I_0 \to \mathbb{R}$, $g : I_0 \to \mathbb{R}$ and $h : I_0 \to \mathbb{R}$ as

(4.7) $f(p) = F(p) - \alpha\beta,$ $g(p) = G(p) + \beta,$ $h(p) = H(p) + \alpha.$

Then (4.6) reduces to the functional equation (FE2) valid for all $p \in I_0$, $q \in I_0$. Also, from (4.7),

(4.8) $F(p) = f(p) + \alpha\beta$ $G(p) = g(p) - \beta$ $H(p) = h(p) - \alpha$

for all $p \in I_0$. The required solutions (4.1) to (4.5) of (FE3) can now be found by making use of (4.8) and Theorem 3. $\square$

Note 2. Let $\mathbb{R}^* = \mathbb{R} - \{0\}$ denote the set of all nonzero real numbers. Suppose $f : \mathbb{R}^* \to \mathbb{R}$ is a function which satisfies the Pompieu functional equation ([3], [4], [5])

$$f(x + y + xy) = f(x) + f(y) + f(xy)$$

for all $x \in \mathbb{R}^*$, $y \in \mathbb{R}^*$. Kannappan and Sahoo [2] considered the functional equation

(4.9) $f(x + y + xy) = p(x) + q(y) + g(x)h(y)$

for all $x \in \mathbb{R}^*$, $y \in \mathbb{R}^*$ in which all the five unknown functions f, p, q, g, h are real-valued; each with domain $\mathbb{R}^*$. During the process of investigating the solutions of this equation, without imposing any regularity condition on any of these five functions, they came across the functional equation of the form

(4.10) $F(uv) = F(u) + F(v) + \alpha\, G(u) + \beta\, H(v) + G(u)H(v)$

in which $u \in \mathbb{R}^*$, $v \in \mathbb{R}^*$, $\alpha := -h(0)$, $\beta := -g(0)$ and F, G, H are real-valued functions each with domain $\mathbb{R}^*$. The functional equation (4.10) differs from (FE3) in the sense that in the former case, the functions F, G, H are real-valued each with domain $\mathbb{R}^*$ whereas in the latter case, these functions are real-valued but each with domain I_0. Our method of obtaining the general solutions of (FE3) is quite different from that employed in [2] to obtain the general solutions of (4.10).

5. **On the functional equation (FE4)**

In this section, using Corollary 4, we prove:

Theorem 6. *Suppose the functions $F : I_0 \to \mathbb{R}$, $G : I_0 \to \mathbb{R}$, $H : I_0 \to \mathbb{R}$, $K : I_0 \to \mathbb{R}$ and $N : I_0 \to \mathbb{R}$ satisfy the functional equation* (FE4) *for all $p \in I_0$, $q \in I_0$. Then, any general solution (F, K, N, G, H) of* (FE4) *is of the form*

(5.1)
$$\begin{cases} F(p) = \ell(p) + \beta_1 \\[4pt] K(p) = \ell(p) - \beta_5 G(p) + \beta_4\beta_5 + \beta_2 \quad N(p) = \ell(p) + \beta_3 \\[4pt] G \text{ arbitrary with } G(1) = \beta_4 \qquad\qquad H(p) = \beta_5 \end{cases}$$

or

(5.2)
$$\begin{cases} F(p) = \ell(p) + \beta_1 \\[4pt] K(p) = \ell(p) + \beta_2 \quad N(p) = \ell(p) - \beta_4 H(p) + \beta_4\beta_5 + \beta_3 \\[4pt] G(p) = \beta_4 \qquad\qquad H \text{ arbitrary with } H(1) = \beta_5 \end{cases}$$

or

$$(5.3) \quad \begin{cases} F(p) = \ell(p) + \dfrac{1}{2} b[\ell^*(p)]^2 + \beta_1 \\[2mm] K(p) = \ell(p) - \beta_5 \ell^*(p) + \dfrac{1}{2} b[\ell^*(p)]^2 + \beta_2 \\[2mm] N(p) = \ell(p) - \beta_4 b\, \ell^*(p) + \dfrac{1}{2} b[\ell^*(p)]^2 + \beta_3 \\[2mm] G(p) = \ell^*(p) + \beta_4 \qquad\qquad\qquad H(p) = b\, \ell^*(p) + \beta_5 \end{cases}$$

or

$$(5.4) \quad \begin{cases} F(p) = \ell(p) + b\lambda_0^2[M(p) - 1] + \beta_1 \\[2mm] K(p) = \ell(p) + (b\lambda_0^2 - \beta_5\lambda_0)[M(p) - 1] + \beta_2 \\[2mm] N(p) = \ell(p) + (b\lambda_0^2 - b\lambda_0\beta_4)[M(p) - 1] + \beta_3 \\[2mm] G(p) = \lambda_0[M(p) - 1] + \beta_4 \qquad\quad H(p) = b\lambda_0[M(p) - 1] + \beta_5 \end{cases}$$

where b and λ_0 are arbitrary nonzero real constants; $\beta_i (i = 1, 2, 3, 4, 5)$ are arbitrary real constants such that

$$(5.5) \qquad \beta_1 = \beta_2 + \beta_3 + \beta_4\beta_5$$

and the functions $\ell : I_0 \to \mathbb{R}$, $\ell^* : I_0 \to \mathbb{R}$ and $M : I_0 \to \mathbb{R}$ are as described in the statement of Theorem 1.

Proof. Let

$$(5.6) \qquad F(1) = \beta_1, \quad K(1) = \beta_2, \quad N(1) = \beta_3, \quad G(1) = \beta_4, \quad H(1) = \beta_5.$$

Putting $p = q = 1$ in (FE4) and using (5.6), (5.5) follows. Putting $p = 1$ respectively $q = 1$ in (FE4) and using (5.6), we get respectively

$$(5.7) \qquad N(q) = F(q) - \beta_2 - \beta_4 H(q)$$

and

$$(5.8) \qquad K(p) = F(p) - \beta_3 - \beta_5 G(p)$$

for all $q \in I_0$ and $p \in I_0$. From (FE4), (5.5), (5.6), (5.7) and (5.8), it follows that

$$F(pq) = F(p) + F(q) - \beta_1 + (G(p) - \beta_4)(H(q) - \beta_5)$$

which can be written in the form

$$(5.9) \qquad F(pq) - \beta_1 = (F(p) - \beta_1) + (F(q) - \beta_1) + (G(p) - \beta_4)(H(q) - \beta_5)$$

for all $p \in I_0$, $q \in I_0$. Define the functions $f : I_0 \to \mathbb{R}$, $g : I_0 \to \mathbb{R}$ and $h : I_0 \to \mathbb{R}$ as

$$(5.10) \quad f(p) = F(p) - \beta_1, \qquad g(p) = G(p) - \beta_4, \qquad h(p) = H(p) - \beta_5$$

for all $p \in I_0$. Then (5.9) reduces to the functional equation (FE2). Moreover, from (5.6) and (5.10), (3.8) follows. The required solutions (5.1) to (5.4) follow from Corollary 4, equations (5.10), (5.7) and (5.8). The details are omitted for the sake of brevity.   □

Note 3. Let $\mathbb{C}$ be the field of complex numbers and S be a nonempty subset of $\mathbb{C}$ such that $S = (S, *)$ is an abelian semigroup with binary operation $*$. Suppose there is an element $a \in S$ such that the equation $a * z = \zeta$ has at least one solution for every given ζ. Let $F : S \to \mathbb{C}$, $G : S \to \mathbb{C}$, $H : S \to \mathbb{C}$, $K : S \to \mathbb{C}$, $L : S \to \mathbb{C}$ be functions which satisfy the functional equation

$$(5.11) \quad F(z_1 * z_2) = G(z_1) + H(z_2) + K(z_1)L(z_2)$$

for all $z_1 \in S$, $z_2 \in S$. E. Vincze [8] gave a method, known as the method of determinants, and used it to determine all general solutions of (5.11) without imposing any regularity condition on any one of the five functions F, G, H, K and L (see Satz 2 in [8]).

The functional equation (FE4) is similar to (5.11). Using the symbols K, N, G, H in (FE4) respectively in place of G, H, K, L as in (5.11), is a matter of convenience.

In (FE4), all the functions are real-valued and each has domain $I_0 =]0, 1]$. Indeed, I_0 is an abelian semigroup with identity 1, the binary operation $*$ being the usual multiplication of real numbers. We have obtained all solutions of (FE4) without imposing any regularity condition on any one of the five functions F, K, N, G, H and without making use of Vincze's method of determinants.

Finally, (FE4) may be regarded as a most general Pexiderized form of (FE2).

6. On the functional equation (FE5)

In this section, we make use of Theorem 6 to prove:

Theorem 7. *Let a and b be given real numbers. Suppose the functions $f : I_0 \to \mathbb{R}$, $K : I_0 \to \mathbb{R}$, $N : I_0 \to \mathbb{R}$, $S : I_0 \to \mathbb{R}$ and $T : I_0 \to \mathbb{R}$ satisfy the functional equation* (FE5) *for all $p \in I_0$, $q \in I_0$. Then, any general solution (f, K, N, S, T) of* (FE5) *is of the form*

$$(6.1) \quad \begin{cases} f(p) = \ell(p) + \beta_1 - ab \\ K(p) = \ell(p) - \beta_5 S(p) - \beta_5 b + \beta_4 \beta_5 + \beta_2 \qquad N(p) = \ell(p) + \beta_3 \\ S \text{ arbitrary with } S(1) = \beta_4 - b \qquad\qquad\qquad T(p) = \beta_5 - a \end{cases}$$

or

$$(6.2) \quad \begin{cases} f(p) = \ell(p) + \beta_1 - ab \\ K(p) = \ell(p) + \beta_2 \qquad N(p) = \ell(p) - \beta_4 T(p) - \beta_4 a + \beta_4 \beta_5 + \beta_3 \\ S(p) = \beta_4 - b \qquad T \text{ arbitrary with } T(1) = \beta_5 - a \end{cases}$$

or

$$(6.3) \quad \begin{cases} f(p) = \ell(p) + \dfrac{1}{2} b[\ell^*(p)]^2 + \beta_1 - ab \\[2mm] K(p) = \ell(p) - \beta_5 \ell^*(p) + \dfrac{1}{2} b[\ell^*(p)]^2 + \beta_2 \\[2mm] N(p) = \ell(p) - \beta_4 b\, \ell^*(p) + \dfrac{1}{2} b[\ell^*(p)]^2 + \beta_3 \\[2mm] S(p) = \ell^*(p) + \beta_4 - b \qquad\qquad T(p) = b\, \ell^*(p) + \beta_5 - a \end{cases}$$

or

$$(6.4) \quad \begin{cases} f(p) = \ell(p) + b\lambda_0^2[M(p) - 1] + \beta_1 - ab \\[2mm] K(p) = \ell(p) + (b\lambda_0^2 - \beta_5\lambda_0)[M(p) - 1] + \beta_2 \\[2mm] N(p) = \ell(p) + (b\lambda_0^2 - b\lambda_0\beta_4)[M(p) - 1] + \beta_3 \\[2mm] S(p) = \lambda_0[M(p) - 1] + \beta_4 - b \\[2mm] T(P) = b\lambda_0[M(p) - 1] + \beta_5 - a \end{cases}$$

where the constants b, λ_0, $\beta_i (i = 1, 2, 3, 4, 5)$ *and the functions* $\ell : I_0 \to \mathbb{R}$, $\ell^* : I_0 \to \mathbb{R}$ *and* $M : I_0 \to \mathbb{R}$ *are as described in the statement of Theorem* 6.

Proof. The functional equation (FE5) can be written in the form

$$(6.5) \quad f(pq) + ab = K(p) + N(q) + [S(p) + b][T(q) + a].$$

Define $F : I_0 \to \mathbb{R}$, $G : I_0 \to \mathbb{R}$, $H : I_0 \to \mathbb{R}$ as

$$(6.6) \quad F(p) = f(p) + ab, \qquad G(p) = S(p) + b, \qquad H(p) = T(p) + a$$

for all $p \in I$. Then (6.5) reduces to (FE4) valid for all $p \in I_0$, $q \in I_0$. The required solutions (6.1) to (6.4) can now be obtained by making use of (6.6) and Theorem 6. $\qquad\qquad\qquad\square$

7. Comments

In this paper, we have restricted our discussion to the functional equation (FE1) and some of its generalizations (FE2) to (FE5). Indeed, (FE1) is a generalization of the functional equation (1.7) which arises on considering only the first two members f_1 and f_2 of the sequence $f_n : I_0 \to \mathbb{R}$, $n = 1, 2 \ldots$ defined by (1.6). There are several functional equations which arise when we consider together the members $f_1, \ldots, f_n$ $(n = 3, 4, \ldots)$ of the sequence $f_n : I_0 \to \mathbb{R}$, $n = 1, 2, \ldots$ defined by (1.6). Some of these functional equations and their generalizations will be discussed in our subsequent work.

Acknowledgment

The author is grateful to one of the referees for his valuable comments.

References

[1] PL. Kannappan, *Functional equations and inequalities with applications*, Springer Monographs in Mathemaics, 2009.

[2] PL. Kannappan and P.K. Sahoo, On generalizations of the Pompeiu functional equation, *Internat. J. Math & Math. Sci.* **21** (1) (1998), 117–124.

[3] M. Neagu, *About the Pompeiu equation in distributions*, Inst. Politeh. "Traian Vuia" Timisoara, Lucrar. Semin. Mat. Fiz, 62–66, 1984.

[4] D. Pompeiu, Sur une equation fonctionelle qui s'introduit dans un probleme de moyenne, *C.R. Paris* 190 (1930), 1107–1109.

[5] D. Pompeiu, Les fonctions ind finiment symm triques det les quations diff rentielles, *Bull. Sec. Sci. Acad. Roum.* **24** (1941), 291–296.

[6] A. R nyi, *A Diary on Information Theory*, John Wiley and Sons, Chichester – New York – Brisbane – Toronto – Singapore, 1987.

[7] F.M. Reza, *An introduction to Information Theory*, McGraw-Hill Book Company, Inc., New York – Toronto – London, 1961.

[8] E. Vincze, Eine allgemeinere methode in der theorie der functional gleichungen-I, *Publ. Math. Debrecen* **9** (1962), 149–163.

Prem Nath, *Department of Mathematics, University of Delhi, Delhi 110 007, India*.
E-mail: `pnathmaths@gmail.com`