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A Study on Jaco-Type Graphs

Research Article

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Abstract. For a sequence $\{a_n\}$ in general where $a_n \in \mathbb{N}$ and $a_{n+1} \geq a_n$, $n = 1,2,3,\dots$, a new population of directed graphs is defined such that for a given sequence $\{a_n\}$ the infinite directed *root*-graph has $d^+(v_n) = a_n$. The infinite directed *root*-graph is denoted $J_\infty(\{a_n\})$. The family of finite Jaco-type graphs is the set of directed graphs $J_n(\{a_n\})$, $n \in \mathbb{N}$ by lobbing off all vertices and arcs in $J_\infty(\{a_n\})$ for vertices v_i, $i > n$. We present introductory results for two families of these new directed graphs.

Keywords. Jaco graph; Linear Jaco graph; Jaco-type graph

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1. Introduction

For general notation and concepts in graphs and digraphs see [1, 2, 3, 4, 9]. Unless mentioned otherwise all graphs (digraphs) will be simple and connected.

The notion of Jaco graphs was introduced in [5] and further research on these types of graphs were carried out in [6, 7, 8].

Definition 1.1 ([6]). *The infinite Jaco graph, denoted by $J_\infty(x)$, $x \in \mathbb{N}$, is defined as a directed graph whose vertex set is $V(J_\infty(x)) = \{v_i : i \in \mathbb{N}\}$ and the arc set is $A(J_\infty(x)) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ such that $(v_i, v_j) \in A(J_\infty(x))$ if and only if $2i - d^-(v_i) \geq j$.*

Definition 1.2 ([6]). *A finite Jaco graph, denoted by $J_n(x)$, is defined to be a finite subgraph of order n of the infinite Jaco graph $J_\infty(x)$, where $n, x \in \mathbb{N}$.*

Definition 1.3 ([7])**.** A *linear Jaco graph*, denoted by $J_\infty(mx+c)$, $x,n \in \mathbb{N}$, $m,c \in \mathbb{N}_0$, is defined as a directed graph whose vertex set is $V(J_\infty(mx+c)) = \{v_i : i \in \mathbb{N}\}$ and the arc set is $A(J_\infty(mx+c)) \subseteq \{(v_i,v_j) : i,j \in \mathbb{N}, i<j\}$ such that $(v_i,v_j) \in A(J_\infty(mx+c))$ if and only if $(m+1)i+c-d^-(v_i) \geq j$.

Definition 1.4 ([7])**.** *A finite linear Jaco graph, denoted by $J_n(mx+c)$, is a finite subgraph of order n of the infinite linear Jaco graph $J_\infty(mx+c)$, where $n,x \in \mathbb{N}$ and $m,c \in \mathbb{N}_0$.*

Inherent to the definition of linear Jaco graphs is the property of the well-defined value of $d(v_i) = d^-(v_i)+d^+(v_i) = f(i)$, $v_i \in V(J_\infty(f(x)))$. Finding closed formula for both $d^-(v_i)$ and $d^+(v_i)$ remained a mystery until recently. In the initial studies mentioned above, an arguably closed formula is given by Bettina's theorem (see [7]).

2. Jaco-type Graphs

Upon finding a closed formula $d^-(v_n) = \left\lfloor \frac{2(n+1)}{3+\sqrt{5}} \right\rfloor$ for the special case $J_n(x)$ (see [8]), redefining linear Jaco graphs in a more generalised way has become possible. Moreover, it is also possible to define linear Jaco graphs as the graphical embodiment of certain specific non-negative integer sequences. This observation opens wide scope for determining the graphical embodiments of countless other such integer sequences. This paper consists of an initial study on these types of graphs and for this study, we consider only non-negative, non-decreasing integer sequences.

The above graphical embodiments of integer sequences are called *Jaco-type graphs*, which can be formally defined as follows.

Definition 2.1. For a non-negative, non-decreasing integer sequence $\{a_n\}$, the *infinite Jaco-type graph*, denoted by $J_\infty(\{a_n\})$, is defined to be a graph whose vertex set is $V(J_\infty(\{a_n\})) = \{v_i : i \in \mathbb{N}\}$ and the arc set is $A(J_\infty(\{a_n\})) \subseteq \{(v_i,v_j) : i,j \in \mathbb{N}, i<j\}$ such that $(v_i,v_j) \in A(J_\infty(\{a_n\}))$ if and only if $i+a_i \geq j$.

Definition 2.2. For a non-negative, non-decreasing integer sequence $\{a_n\}$, a *finite Jaco-type graph* of order n, denoted by $J_n(\{a_n\})$, is defined to be a finite subgraph of order n of the infinite Jaco-type graph $J_\infty(\{a_n\})$.

All other definitions and properties of linear Jaco graphs are described in [5, 6, 7] and either hold exactly or similarly good for Jaco-type graphs. Define the sequence $\{a_n\}$, by $a_n = n - \left\lfloor \frac{2(n+1)}{3+\sqrt{5}} \right\rfloor$, $n = 1,2,3,\ldots$. Hence, we have the sequence $1,1,2,3,3,4,4,5,6,6,7,8,8,9,9,\ldots, n - \left\lfloor \frac{2(n+1)}{3+\sqrt{5}} \right\rfloor, \ldots$. Clearly, the linear Jaco graphs and the Jaco-type graphs for the sequence $\{a_n\}$, $a_n = n - \left\lfloor \frac{2(n+1)}{3+\sqrt{5}} \right\rfloor$, $n = 1,2,3,\ldots$ are identical. Therefore, an alternative definition for a finite linear Jaco graph $J_n(x)$ can be made as follows.

Definition 2.3. For a non-negative, non-decreasing integer sequence $\{a_n\}$, the *infinite Jaco-type graph*, denoted by $J_\infty(\{a_n\})$, is defined as the directed graph whose vertex set is $V(J_\infty(\{a_n\})) = \{v_i : i \in \mathbb{N}\}$ and the arc set is $A(J_\infty(\{a_n\})) \subseteq \{(v_i,v_j) : i,j \in \mathbb{N}, i<j\}$ such that $(v_i,v_j) \in A(J_\infty(\{a_n\}))$ if and only if $2i - \left\lfloor \frac{2(i+1)}{3+\sqrt{5}} \right\rfloor \geq j$.

Definition 2.4. A finite Jaco graph $J_n(x)$ may also be defined by $J_n(x) \subseteq J_\infty(\{a_n\})$ such that $a_n = n - \left\lfloor \frac{2(n+1)}{3+\sqrt{5}} \right\rfloor$, $n = 1,2,3,\ldots$.

Note that Definition 2.3 is function specific; that is, $f(x) = x$, $x \in \mathbb{N}$. Generalisations for $f(x) = mx + c$, $x \in \mathbb{N}$ and $m, c \in \mathbb{N}_0$ remain open. In the following section, we discuss the application of Definition 2.3 and Definition 2.4 with respect to two well-known, non-negative, non-decreasing integer sequences. The main purpose of the discussion is to illustrate the analysis in respect of certain important invariants of the corresponding Jaco-type graphs.

2.1 Jaco-type Graph for Sequence $\{n\}$, $n = 1,2,3,\ldots$

For ease of notation, all sequences will be labeled $s_i, i \in \mathbb{N}$. The definition of the infinite Jaco-type graph corresponding to the captioned sequence can be derived from Definition 2.1.

We know that $J_\infty(s_1)$ is the graph with vertex set $V(J_\infty(s_1)) = \{v_i : i \in \mathbb{N}\}$ and the arc set $A(J_\infty(s_1)) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ such that $(v_i, v_j) \in A(J_\infty(s_1))$ if and only if $2i \geq j$. Note that a finite Jaco-type graph $J_n(s_1)$ in this family is obtained from $J_\infty(s_1)$ by lobbing off all vertices v_k (with incident arcs) $\forall\ k > n$.

Table 1 shows the results for the application of a derivative of the Fisher algorithm for $n \leq 20$ (see [5, 6]).

Table 1

$\phi(v_i) \to i \in \mathbb{N}$	$d^-(v_i) = \lfloor \frac{i}{2} \rfloor$	$d^+(v_i) = i$	$\mathbb{J}(J_i(s_1))$	$\Delta(J_i(s_1))$
1	0	1	$\{v_1\}$	0
2	1	2	$\{v_1, v_2\}$	1
3	1	3	$\{v_2\}$	2
4	2	4	$\{v_2\}$	3
5	2	5	$\{v_2, v_3, v_4\}$	3
6	3	6	$\{v_3, v_4\}$	4
7	3	7	$\{v_4\}$	5
8	4	8	$\{v_4\}$	6
9	4	9	$\{v_4, v_5, v_6\}$	6
10	5	10	$\{v_5, v_6\}$	7
11	5	11	$\{v_6\}$	8
12	6	12	$\{v_6\}$	9
13	6	13	$\{v_6, v_7, v_8\}$	9
14	7	14	$\{v_7, v_8\}$	10
15	7	15	$\{v_8\}$	11
16	8	16	$\{v_8\}$	12
17	8	17	$\{v_8, v_9, v_{10}\}$	12
18	9	18	$\{v_9, v_{10}\}$	13
19	9	19	$\{v_{10}\}$	14
20	10	20	$\{v_{10}\}$	15

Figure 1 depicts a Jaco-type graph $J_8(s_1)$.

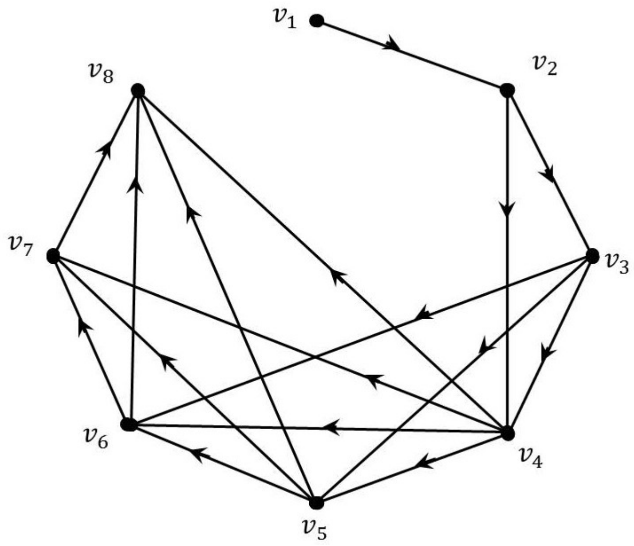

Figure 1. $J_8(s_1)$

Lemma 2.1. *For the Jaco-type graph $J_n(s_1)$ with prime Jaconian vertex v_p, we have*

(i) $d^-(v_n) = \lfloor \frac{n}{2} \rfloor$, $d^+(v_n) = n$.

(ii) $\Delta(J_n(s_1)) = n - \lfloor \frac{n+3}{4} \rfloor$ *or* $\left\lfloor \frac{d^-(v_n)+d^+(v_n)}{2} \right\rfloor$.

(iii) *For $n \geq 2$ let $t = \min\{k \in \mathbb{N} : n = 2k + \ell, \ell \in \{0,1,3\}, k \text{ is odd}\}$.*

Then, we have

$$p = \begin{cases} t, & \text{if } n = 2t, \\ t+1, & \text{if } n \in \{2t+1, 2t+3\}. \end{cases}$$

Proof. Part (i)(a): For any $k \in \mathbb{N}$ and $J_{n \geq 2k}(s_1)$ the arcs, $(v_k, v_{2k}), (v_{k+1}, v_{2k}), (v_{k+2}, v_{2k}), \ldots, (v_{2k-1}, v_{2k})$ exist and the arc (v_{k-1}, v_{2k}) does not exists. Hence $d^-(v_{2k}) = k$. By similar reasoning it follows that $d^-(v_{2k+1}) = k$ whilst $d^-(v_{2k+2}) = k+1$. Hence, through immediate induction we have $d^-(v_n) = \lfloor \frac{n}{2} \rfloor$.

Part (i)(b): Follows from the definition of $J_n(s_1)$.

Part (ii): For $1 \leq n = i \leq 5$ the result follows easily through constructive proof (see Table 1). Assume the result holds for $1 \leq n = i \leq k$. Therefore, $\Delta(J_k(s_1)) = \left\lfloor \frac{d^-(v_k)+d^+(v_k)}{2} \right\rfloor$. Now consider $n = i = k+1$.

Case (ii)(a): Let k be even and $\Delta(J_k(s_1) = \ell$. Clearly from Part (i) it follows that $d^-(v_{k+1}) = d^-(v_k)$. Since $d^+(v_{k+1}) = d^+(v_k) + 1$ and $\Delta(J_{k+1}(s_1)) \in \{\Delta(J_k(s_1)), \Delta(J_k(s_1)) + 1\}$ only $\left\lfloor \frac{d^-(v_{k+1})+d^+(v_{k+1})}{2} \right\rfloor$ is possible. Hence, the result holds $\forall\ k+1$ if k is even.

Case (ii)(b): Let k be odd. Similar reasoning as in Case (a) follows except to note that from k to $k+1$ both the corresponding in- and out-degrees increase by $+1$ hence the floor of the fraction increases by $+1$. Therefore $\Delta(J_{k+1}(c_1)) = \Delta(J_k(s_1))$. Hence the result holds $\forall\ k+1$ if k is odd. Hence, through strong induction the result holds for $n \in \mathbb{N}$.

Now that the validity of Table 1 is proven we consider the sequence $\{1-1,\ 2-1,\ 3-1,\ 4-1,\ 5-2,\ 6-2,\ 7-2,\ 8-2,\ 9-3,\ 10-3,\ 11-3,\ 12-3,\dots\}$. Since the sequence $1,1,1,1,2,2,2,2,3,3,3,3,\dots,m,m,m,m,\dots = \{\lfloor \frac{n+3}{4} \rfloor\}$, for $n = 1,2,3,\dots$ the result $\Delta(J_n(s_1)) = n - \lfloor \frac{n+3}{4} \rfloor$ follows.

Part (iii)(a): For any $k \in \mathbb{N}$, k is even, there exists an exact $t \in \mathbb{N} \Rightarrow k = 2t$. Hence, t is a minimum so from the definitions of $J_k(s_1)$ and of the corresponding prime Jaconian vertex v_p of a Jaco-type graph it follows immediately that $p = t$.

Part (iii)(b): For any $k \in \mathbb{N}$, k is odd, either $k = 2t+1$ or $k = 2t+3$ to ensure minimum odd t and $\ell \in \{0,1,3\}$. If $k = 2t+1$ then $d(v_t) = \lfloor \frac{t}{2} \rfloor + t < \lfloor \frac{t+1}{2} \rfloor + (t+1) > k$. Hence, $d(v_{t+1}) = \Delta(J_k(x))$ so $p = t+1$. Through immediate induction the result follows for all odd k of the form $2t+1$. The other cases follow by similar reasoning. □

Corollary 2.2. *For $t \in \mathbb{N}$, t is odd, we have*

(a) *Jaco-type graphs $J_{2t+1}(s_1)$, $J_{2t+2}(s_1)$, have a unique Jaconian vertex, v_{t+1}.*

(b) *Jaco-type graphs $J_{t\geq 3}(s_1)$, t odd, have a prime Jaconian vertex v_p, p is even.*

(c) *A prime Jaconian vertex v_p, p odd, never repeats in the family of Jaco-type graphs, $J_n(s_1)$.*

Proof. Results follow directly from the proof of Lemma 2.1(iii). □

We also observe that for $n \geq 2$, $n \in \mathbb{N}$ the cardinality of the Jaconian set, $\psi(n) = |\mathbb{J}(J_n(s_1))|$ is given by the map:

$$\psi(n) \mapsto \begin{cases} 1, & \text{if } n(\text{mod } 4) = 0, \\ 3, & \text{if } n(\text{mod } 4) = 1, \\ 2, & \text{if } n(\text{mod } 4) = 2, \\ 1, & \text{if } n(\text{mod } 4) = 3. \end{cases}$$

The observation above together with Lemma 2.1(iii) easily provide the Jaconian set of $\mathbb{J}(J_{n\geq 2}(s_1))$.

Example 2.1. Consider $n = 17$. Since $17 = 2\cdot 7 + 3$ (Lemma 2.1(iii)), we have $t = 7 \Rightarrow p = 8$ hence $v_p = v_8$. Furthermore, $17(\text{mod } 4) = 1 \Rightarrow \psi(17) \mapsto 3$. Therefore, $\mathbb{J}(J_{17}(s_1)) = \{v_8, v_9, v_{10}\}$.

2.2 Jaco-type Graph for the Fibonacci Sequence

The definition of the infinite Jaco-type graph corresponding to the captioned sequence can be derived from Definition 2.1. We have the graph $J_\infty(s_2)$, defined by $V(J_\infty(s_2)) = \{v_i : i \in \mathbb{N}\}$, $A(J_\infty(s_2)) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in A(J_\infty(s_2))$ if and only if $i + f_i \geq j$.

Table 2 shows the results for the application of a derivative of the Fisher algorithm for $n \leq 30$ (see [5, 6]).

Table 2

$\phi(v_i) \to i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i) = f_i$	$\mathbb{J}(J_i(s_2))$	$\Delta(J_i(s_2))$
1	0	1	$\{v_1\}$	0
2	1	1	$\{v_1, v_2\}$	1
3	1	2	$\{v_2\}$	2
4	2	3	$\{v_2, v_3\}$	2
5	2	5	$\{v_3\}$	3
6	2	8	$\{v_3, v_4, v_5\}$	3
7	3	13	$\{v_4, v_5\}$	4
8	3	21	$\{v_5\}$	5
9	4	34	$\{v_5\}$	6
10	5	55	$\{v_5\}$	7
11	5	89	$\{v_5, v_6, v_7\}$	7
12	6	144	$\{v_6, v_7\}$	8
13	7	233	$\{v_6, v_7\}$	9
14	8	377	$\{v_6, v_7\}$	10
15	8	610	$\{v_7\}$	11
16	9	987	$\{v_7\}$	12
17	10	1597	$\{v_7\}$	13
18	11	2584	$\{v_7\}$	14
19	12	4181	$\{v_7\}$	15
20	13	6765	$\{v_7\}$	16
21	13	10946	$\{v_7, v_8, v_9, v_{10}\}$	16
22	14	17711	$\{v_8, v_9, v_{10}\}$	17
23	15	28657	$\{v_8, v_9, v_{10}\}$	18
24	16	46368	$\{v_8, v_9, v_{10}\}$	19
25	17	76025	$\{v_8, v_9, v_{10}\}$	20
26	18	122396	$\{v_8, v_9, v_{10}\}$	21
27	19	198424	$\{v_8, v_9, v_{10}\}$	22
28	20	320820	$\{v_8, v_9, v_{10}\}$	23
29	21	519244	$\{v_8, v_9, v_{10}\}$	24
30	21	840064	$\{v_9, v_{10}\}$	24

For the first row ($i = 1$) the entries are obviously correct. Because for $\forall\ n \in \mathbb{N}$, $n \geq 2$, there exists a minimum i such that $i + f_i \geq n$ it follows that $d^-(v_n) = n - i$. This fact together with the trivial correctness of the first and third columns entries, propagate recursively through columns 4 and 5 and all rows hence, the table is correct.

Observation 2.3. We observe that for $6 \le i \le 29$ the in-degrees $d^-(v_i)$ follow sequentially in strings:

$$\underbrace{2,3}_{(f_2+1)\text{-entries}}, \underbrace{3,4,5}_{(f_3+1)\text{-entries}}, \underbrace{5,6,7,8}_{(f_4+1)\text{-entries}}, \underbrace{8,9,10,11,12,13}_{(f_5+1)\text{-entries}}, \underbrace{13,14,15,16,17,18,19,20,21}_{(f_6+1)\text{-entries}}$$

.

Denote the consecutive strings $\mathscr{S}_i$, $i \in \mathbb{N}$. So $|\mathscr{S}_i| = f_{(i+1)} + 1$ and the string has 1-st entry value, $f_{(i+2)}$. Further analysis convinces that the pattern holds.

Figure 2 depicts a Jaco-type graph $J_{12}(s_2)$.

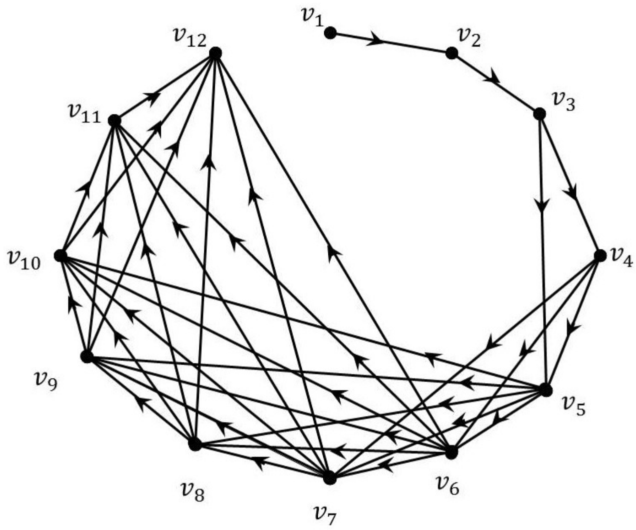

Figure 2. $J_{12}(s_2)$

Observation 2.4. Let $d'_i = d^-(v_i)+1, d''_i = d^-(v_i)+2, d'''_i = d^-(v_i)+3, \ldots$ and so on. We observe that for $6 \le i \le 43$ the maximum degrees $\Delta(v_i)$ follow sequentially in strings:

$$\underbrace{d'_6, d'_7}_{(f_2+1)\text{-entries}}, \underbrace{d''_8, d''_9, d''_{10}, d''_{11}, d''_{12}, d''_{13}, d''_{14}}_{(f_3+f_4+2)\text{-entries}},$$

$$(f_5+f_6+f_7+3)\text{-entries}\begin{cases} \underbrace{d'''_{15}, d'''_{16}, d'''_{17}, d'''_{18}, d'''_{19}, d'''_{20}}_{(f_5+1)\text{-entries}}, \underbrace{d'''_{21}, d'''_{22}, d'''_{23}, d'''_{24}, d'''_{25}, d'''_{26}, d'''_{27}, d'''_{28}, d'''_{29}}_{(f_6+1)\text{-entries}}, \\ \underbrace{d'''_{30}, d'''_{31}, d'''_{32}, d'''_{33}, d'''_{34}, d'''_{35}, d'''_{36}, d'''_{37}, d'''_{38}, d'''_{39}, d'''_{40}, d'''_{41}, d'''_{42}, d'''_{43}}_{(f_7+1)\text{-entries}}. \end{cases}$$

Denote the consecutive strings $\mathscr{S}^*_i$, $i \in \mathbb{N}$. Further analysis suggests the following conjectures.

Conjecture 2.1. *Let* $a_i = \frac{i(i-1)+4}{2}$, $n, i \in \mathbb{N}$. *The* i*-string's string-length is given by* $|\mathscr{S}^*_i| = \sum\limits_{j=0}^{i-1} f_{(a_i+j)} + i$.

Conjecture 2.2. *For the Fibonacci sequence the vertices* v_n, $n \ge 6$ *of the Jaco-type graph* $J_n(s_2)$ *have* $\Delta(J_n(s_2)) = d^-(v_n) + i$, $i \in \mathbb{N}$ *if and only if* $n \in \mathscr{S}^*_i$.

3. Conclusion

This paper discusses the introduction to the concept of Jaco-type graph embodiment of integer sequences. For the Fibonacci sequence we close with two challenging conjectures. Resolving these conjectures will enable further analysis similar to that for the sequence $s_1 = \{n\}$, $n = 1, 2, 3, \ldots$.

Two non-negative, non-decreasing integer sequences were studied. It is evident that a negative entry will only change the orientation of the arc in the embodiment. The restriction of *non-decreasing* can easily be relaxed as well. This opens a wide scope of further studying.

It will be interesting to study properties of some famous sequences in relation to graph theoretic properties of the corresponding Jaco-type graph. A famous sequence that implies a finite maximal Jaco-type graph is the decimal Gray code sequence because the range (number of vertices) is dependent on the *bit width*. A bit width of n allows a maximal Jaco-type graph of order 2^n.

The sequence of primes, triangular numbers and the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$ could provide interesting results. Similar to finding the Golden Ratio in many natural structures and patterns, a big challenge is to find representation of some Jaco-type graphs in nature. Research into the Jaco-type graph of the Fibonacci sequence or perhaps a derivative thereof to approximate galaxy spirals could prove to be a worthy study.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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