Journal of Informatics and Mathematical Sciences
Vol. 8, No. 3, pp. 189–200, 2016
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Research Article

Some Elementary Combinatorial Properties of Aktas' Soft Groups with Contributions in Soft Actions

A. Pouhassani, H. Doostie* and H. Rasouli

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
***Corresponding author:** doostih@gmail.com

Abstract. As a generic mathematical tool, the concept of soft sets introduced in 1999 by Molodtesov [8], and in continuation of this research the soft groups defined and studied for their nice properties by Aktas [1] in 2007. Because of the extensive applications of soft sets and soft groups in all branches of sciences involving mathematics we prefer to concentrate on the algebraic properties of algebraic structures. The action of groups on sets is an effective instrument in algebra. In this note we drive some basic combinatorial properties of soft groups using Aktas's definition of soft group and soft subgroups. By giving the definition of soft actions of groups and semigroups we managed to exhibit their congruence properties in this paper.

Keywords. Soft set; Soft semigroup; Soft group; Soft subgroup; Soft action; Congruence

MSC. 33C80; 05A20; 05E15

Received: July 16, 2016 **Accepted:** August 31, 2016

1. Introduction

The set-valued function has a longer history in Analysis, but, their algebraic structures began to study by Molodtesov [8] in 1999. During the years, certain authors had some contributions in this area of research that developing algebraic attitudes of Molodtesov, one may consult [1, 2, 6, 12], for examples.

Theory of soft groups began by Aktas [1] in 2007 and followed by many authors such as Aslam [4], and Lin and Wang [5]. Sezgin [10] studied normalistic soft groups. Aktas [3] by defining the power of a soft subset of a soft group and the order of a soft group, investigate their properties. Also, they define cyclic soft groups and prove some of their properties which are analogous to the crisp case. Yin [11] has more specialized study on soft groups.

One of the interesting topics in finite group theory is the enumerating of subgroups of a group. We attempt to enumerate the soft subgroups of finite soft groups. Also, we will extend the well-known notion of group action to soft action of groups.

2. Overview of actions and soft groups

Following [1,7], we recall some fundamental definitions:

Definition 2.1. For a monoid S and a non-emty set Ω, S is said to act on Ω if for some function $S \times \Omega \to \Omega$, where $(s,\omega) \to s\omega$ then the following properties hold:

(i) $1\omega = \omega$,

(ii) $s_2(s_1\omega) = (s_2s_1\omega)$, for all $s_1, s_2 \in S$ and $\omega \in \Omega$.

If such function exists, we use the notation $(S \mid \Omega)$ to declare that S acts on Ω.

Definition 2.2. Let G be an initial set and E be a set of parameters on the elements of G. For each $A \subseteq E$ the pair (F,A) is called a *soft set* (over G) if and only if F is a mapping of E into $P(G)$, the set of all subsets of G where $F(e) = \varnothing$ for all $e \in E - A$.

The soft set (F,A) may be written as the set of ordered pairs $\{(F(a),a); a \in A\}$.

Definition 2.3. A soft set (F,A) over G is said to be *empty* whenever $A = \varnothing$. Symbolically, we write $(\varnothing,\varnothing)$ for the empty soft set over G. The pair (F,A) is called a *Universal* soft set if $A = E$ and $F(a) = G$, for all $a \in A$. The universal soft set over G will be denoted by (G,E). For a subset B of A the B-universal soft set on G is the soft set (F,B) on G such that $F(a) = G$, for all $a \in B$. A-universal soft set on G may be called absolute soft set and denoted by (G,A) or G_A.

Definition 2.4. Let (F,A) and (H,B) be two soft sets over G. We say that (F,A) is a subset of (H,B), denoted by $(F,A) \subseteq (H,B)$, if either $(F,A) = (\varnothing,\varnothing)$ or $A \subseteq B$ and $F(a) \subseteq H(a)$, for every $a \in A$. Two soft sets (F,A) and (H,B) are said to be equal, denoted by $(F,A) = (H,B)$, if and only if $(F,A) \subseteq (H,B)$ and $(H,B) \subseteq (F,A)$.

Definition 2.5. For two soft sets (F,A) and (H,B), the *intersection* $(F,A) \cap (H,B)$ is defined by $(F \cap H, C)$ where $C = \{a \in A \cap B \mid\ F(a) \cap H(a) \neq \varnothing\}$ and $(F \cap H)(a) = F(a) \cap H(a)$, for all $a \in C$. Also, the *union* $(F,A) \cup (H,B)$ is defined by $(F \cup H, D)$ where, $D = A \cup B$ and $(F \cup H)(a) = F(a) \cup H(a)$ for all $a \in D$.

Throughout this paper, G is a group and S is a monoid. When $A = \Omega$ we simply use F instead of F_Ω.

Definition 2.6. Let G be a group and (F,A) be a soft set over G. Then, (F,A) is said to be a *soft group* over G if and only if $F(a)$ is a subgroup of G, for all $a \in A$. A *soft cyclic* group on G is a soft group (F,A) on G where $F(a)$ is a cyclic subgroup of G. If $F(a) = H$ for all $a \in A$, we say that (F,A) is an H*-Identity* soft group on G. When $H = \{e\}$ where e is the identity element of G and when $H = G, (F,A)$ is said to be absolout soft group [9]. A soft subset (H,B) of (F,A) is a soft subgroup of (F,A) if and only if $H(b) \leq F(b)$ for all $b \in B$. Moreover, the *Order* of (F,A) defines as the number of elements of (F,A), i.e., $|A|$.

Definition 2.7. A soft group (F,A) on a group G is called a *Normal soft group* over G, if and only if $F(a) \trianglelefteq G$ for all $a \in A$.

Definition 2.8. Let (F,A) be a soft group on a group G and (H,B) is a soft subset of (F,A). Soft subgroup of G *generated by* (H,B) is the soft intersection of all soft subgroups of (F,A) containing (H,B) and denoted by $\langle (H,B) \rangle$. If (H,B) is singleton, say $(H,B) = \{(F(a),a)\}$ then $\langle \{(F(a),a)\} \rangle = \langle (F(a),a) \rangle$ will refer as *soft cyclic subgroup* of (F,A).

In Section 5 we will give certain concrete examples concerning the above notations.

3. Enumerating Soft Subgroups

In this section we purpose to study the subgroups of soft groups. Our preliminary result is:

Proposition 3.1. (i) *If A has $|A|$ elements and the group G has m subgroups, then there exists $m^{|A|}$ soft groups (F,A) on G.*

(ii) *Let (F,A) be a soft group on a group G. Then, $(F,A) \leq (F,A)$. Moreover, if $(H,B),(K,C)$ are soft subgroups of G such that $(K,C) \leq (H,B)$, then $(K,C) \leq (F,A)$.*

(iii) *If (F,A) and (H,B) are two soft groups on a group G, then $(F,A) \cap (H,B)$ is a soft group on G.*

(iv) *If (F,A) be a soft group on a group G and for $a \in A, F(a)$ be a subgroup of G, then $\langle (F(a),a) \rangle = \{(F(a),a)\}$.*

(v) *Let (F,A) be a soft group on a group G and $(H,B) \subseteq (F,A)$. Then $\langle (H,B) \rangle$ is the smallest soft subgroup of (F,A) consists of $(\langle F(b) \rangle, b)$ for all $b \in B$.*

Proof. (i) Let $A = \{1,2,\ldots,|A|\}$ and $\Omega = \{H_1,H_2,\ldots,H_m\}$ is the set of all subgroups of G. Then the number of soft groups on G is equal to the number of functions $F : A \to \Omega$ which is $m^{|A|}$.

Parts (ii), (iii), (iv) and (v) are easy. □

Proposition 3.2. *Let $(F,A) = \{(1,H_1),(2,H_2),\ldots,(k,H_k)\}$ be a soft group on a group G, where each H_i has m_i subgroups $(i = 1,2,\ldots,k)$, then (F,A) possesses*

$$\sum_{i=1}^{k} m_i + \sum_{i \neq j} m_i m_j + \sum_{i \neq j, i \neq t, j \neq t} m_i m_j m_t + \ldots + (m_1 m_2 \ldots m_k)$$

soft subgroups.

Proof. Let $U_{i,1}, U_{i,2}, \ldots, U_{i,m_i}$ be all of the subgroups of H_i then consider below rows:

$$\boxed{\begin{array}{c}(1,U_{1,1}),(1,U_{1,2}),\ldots,(1,U_{1,m_1}),\\(2,U_{2,1}),(2,U_{2,2}),\ldots,(2,U_{2,m_2}),\\ \vdots \\(k,U_{k,1}),(k,U_{k,2}),\ldots,(k,U_{k,m_k}).\end{array}}$$

A singleton $\{\alpha\}$ where α is an entry of the above table, is a soft subgroup of (F,A). Also, a soft set consists of two elements of different rows is also a soft subgroup of (F,A). The number of such soft subsets is $\sum_{i\neq j} m_i m_j$. Analogously, the number of soft subgroups of (F,A) contains three elements is $\sum_{i\neq j, i\neq t, j\neq t} m_i m_j m_t$. By inductive method, we got $m_1 m_2 \ldots m_k$ soft subgroups involve k elements. These are all soft subgroups of (F,A) as desired. □

Definition 3.3. Let (F,A) and (H,B) be two soft groups over G,K respectively. Also, let $f : G \to K$ and $g : A \to B$ be two functions such that $f(F(a)) = H(g(a))$ for all $a \in A$. The pair $\theta = (f,g)$ is a *soft homomorphism*. If $f_{|F(a)}$ is homomorphism, for all $a \in A$. Moreover, if $f_{|F(a)}$ is homomorphism and g is one to one (onto) then θ is soft *monomorphism (epimorphism)*. Furthermore, θ is soft *isomorphism* if f is homomorphism and g is a bijection.

Let G be a group and E is an initial set. The notation $\mathcal{S}(G)$ will be used for the set of all soft groups on G with all soft homomorphisms between them.

Assume that $(F,A),(H,B)$ be two soft groups on G and K respectively. For the soft homomorphism $\theta = (f,g) : (F,A) \to (H,B)$, the *kernel* of θ denoted by $\ker\theta$ is defined as

$$\ker\theta = \{((F(a),a),(F(b),b)) \mid \theta(F(a),a) = \theta(F(b),b)\}.$$

We use the notation $\mathrm{Hom}(F_A,H_B)$ to denote the set of all homomorphisms $\theta : (F,A) \to (H,B)$.

Proposition 3.4. *Combination of soft homomorphism — if exists — is again soft homomorphism. Moreover, if (F,A) be a soft groups on a group G. Then* $\mathrm{Hom}(F_A,F_A) = \mathrm{Hom}(F,A)$ *is a non-commutative regular semigroup if and only if for every $\theta = (f,g)$, at least one of the functions f,g is regular. Moreover, S is finite and $|S| \le |(F,A)|^{|F,A|}$.*

Proof. Suppose that $\mathrm{Hom}(F,A)$ is regular and $\theta = (f,g) \in \mathrm{Hom}(F,A)$. Then there exists an element $\phi = (\alpha,\beta) \in \mathrm{Hom}(F,A)$ such that $\theta\phi\theta = \theta$. So, $(f\alpha f, g\beta g) = (f,g)$, which yields:

$$\begin{cases} f\alpha f_{|F(a)} = f_{|F(a)}, & \forall\, a \in A \\ g\beta g = g \end{cases}$$

f and g are regular because of the commutativity of the diagrams:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & X & \xrightarrow{\alpha} & X & \xrightarrow{f} & X \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle F} & & \uparrow{\scriptstyle F} & & \uparrow{\scriptstyle F} \\ A & \xrightarrow{f} & A & \xrightarrow{\beta} & A & \xrightarrow{g} & A \end{array}$$

Conversely, suppose that $g\beta g = g$, that is g is regular. So, $F(g\beta g(a)) = F(g(a))$, for all $a \in A$. □

Proposition 3.5. *Let $(F,A),(H,B)$ are two soft groups on groups G,K respectively, and $\theta = (f,g) : (F,A) \to (H,B)$ be a soft homomorphism. Then, there is a one to one correspondence between $\frac{(F,A)}{\ker\theta}$ and $\operatorname{Im}\theta$.*

Proof. Define $\phi : \frac{(F,A)}{\ker\theta} \to (H,B)$ such that $\phi([(F(a),a)]_{\ker\theta}) = (H(g(a)),g(a))$ for every $a \in A$. Now, if for $a,b \in A$; $(Hg(a),g(a)) = (Hg(b),b)$ then $\theta(F(a),a) = \theta(F(b),b)$ which means is $[(F(a),a)]_{\ker\theta} = [(F(b),b)]_{\ker\theta}$. □

Let (F,A) be a soft group on a group G. As usual the coset may be defined. For every $x \in G$, the *left coset* (similarly for it right coset) of (F,A) in G denoted by $x(F,A)$ is defined as $x(F,A) := (xF,A)$ such that $xF : A \to P(G)$, $(xF)(a) := x(F(a))$ for all $a \in A$.

Proposition 3.6. *Let (F,A) be a soft group on a group G and $x,y \in G$. Then just one of the followings holds:*

(i) $x(F,A) \cap y(F,A) = (\emptyset,\emptyset)$,

(ii) $x(F,A) = y(F,A)$.

Proof. Note that if $x = y$ then (ii) holds and there is nothing to prove. Thus, assume that $x \neq y$. First, suppose that (i) dose not hold, i.e.; $x(F,A) \cap y(F,A) \neq (\emptyset,\emptyset)$. By the Definitions 2.3, 2.5, we get $A \neq \emptyset$ and

$$(xF,A) \cap (yF,A) \neq (\emptyset,\emptyset) \iff (xF \cap yF,B) \neq (\emptyset,\emptyset)$$

where,

$$B = \{a \in A \cap A; xF(a) \cap yF(a) \neq \emptyset\} = \{a \in A; xF(a) \cap yF(a) \neq \emptyset\}.$$

Since, $F(a)$ is a subgroup of G (for all $a \in A$) then $xF(a)$ and $yF(a)$ are two left cosets of $F(a)$ in G which are not disjoint. So, $xF(a) = yF(a)$ therefore, $B = \{a \in A; xF(a) \neq \emptyset\} = A$.

Moreover, for all $b \in B = A$ we get that:

$$\begin{aligned}(xF \cap yF)(b) &= xF(b) \cap yF(b)\\ &= xF(b) \cap yF(b)\\ &\neq \emptyset.\end{aligned}$$

Thus, the intersection of two left cosets of $F(b)$ in G is nonempty. This implies that $xF(b) = yF(b)$. Since b is arbitrary, then $xF = yF$ and $(xF,A) = (yF,A)$. Now, the result follows at once. □

Proposition 3.7. *Let (F,A) be a soft group on a group G and $x \in X$. Then:*

(i) $\bigcup_{x\in G} x(F,A) = (G,A)$,*i.e., the collection of all left (right) cosets of (F,A) in G is a partition of A-universal soft set (G,A).*

(ii) $x(F,A) = (F,A)$ *if and only if* $x \in \bigcap_{a\in A} F(a)$.

Proof. (i) It is clear.

(ii) Perceive that

$$\begin{aligned} x(F,A)=(F,A) &\iff (xF,A)=(F,A) \\ &\iff (xF)(a)=F(a) \quad \text{(for every } a\in A) \\ &\iff x(F(a))=F(a) \\ &\iff x\in F(a). \end{aligned}$$

The converse is clear. □

4. Soft Actions

Our definition of soft action is:

Definition 4.1. Let $\Omega\in S-Act$ and $F:\Omega\to P(\Omega)$ be a soft set. The action $(S\mid\Omega)$ is called an *F-soft action if $F(s\omega)=s(F(\omega))$, for every $\omega\in\Omega$ and $s\in S$. Here, $s(F(\omega))=\{sx\mid x\in F(\omega)\}$.*

If $\Omega\in S-Act$ then $P(\Omega)\in S-Act$ too. Thus, the soft set $F:\Omega\to P(\Omega)$ is an $S-Act$ homomorphism.

Let us examine some simple examples. Let F be a constant soft set, then, every action of G on Ω is an F-soft action. Moreover, assume that $F(\omega)=\{\omega\}$ for each $\omega\in\Omega$ (which may refers to the identity soft set). Then every group action $(G\mid\Omega)$ is an F-soft action. A nontrivial example is given as follows. Let N be a normal subgroup of a group G. Define $F_N:N\to P(N)$ by $F_N(n):=G_n\cap N$ where G_n is the stabilizer of n in the action $(G\mid N)$ defined by $gn:=g^{-1}ng$, $n\in N$, $g\in G$. Clearly, $(G\mid N)$ is an F_N-soft action. Furthermore, one can easily show that $(G\mid N)$ yields $(G\mid P(N))$.

Let $(S\mid\Omega)$ be an F-soft action and $\Delta\subseteq\Omega$. Then, $F(s\Delta)=sF(\Delta)$ for all $s\in S$.

Indeed, $F(s\Delta)=\{F(s\delta)\mid\delta\in\Delta\}=\{sF(\delta)\mid\delta\in\Delta\}=sF(\Delta)$. Now, if Δ is a stable block under this action (i.e., $s\Delta=\Delta$ for all $s\in S$), then we get $F(\Delta)=sF(\Delta)$.

It is easy to show that if $(S\mid\Omega)$ be an F-soft action and $H\subseteq F$ then $(S\mid\Omega)$ is an H-soft action.

As a quick result of the concept of soft action, we can get the following lemmas:

Lemma 4.2. *Let $(S\mid\Omega)$ be an F-soft action as well as an H-soft action where $F,H:\Omega\to P(\Omega)$ are two soft sets. Then for every $\omega\in\Omega$ and $s\in S$:*

(1) *$(S\mid\Omega)$ is an $(F\bigcup H)$-soft action,*

(2) *If $st=sv$ implies that $t=v$ for all $s\in S$ and $t,v\in\Omega$, then $s(F\bigcap H)(\omega)=(F\bigcap H)(s\omega)$.*

Proof. The assertion (1) is straightforward. For (2) it is easy to show that $s((F\bigcap H)(\omega))\subseteq(F\bigcap H)(s\omega)$. Conversely, for each $x\in(F\cap H)(s\omega)=F(s\omega)\cap H(s\omega)$ we have, $x\in sF(\omega)$ and $x\in sH(\omega)$ because of F-softness and H-softness of $(S\mid\Omega)$. Thus there exist $t\in F(\omega)$ and $v\in H(\omega)$ such that $x=st,x=sv$; which gives that $t=v$ by the assumption. Then $t\in F(\omega)\bigcap H(\omega)$ and so $x\in s(F\bigcap H)(\omega)$. □

Lemma 4.3. *Let $\Omega_1,\Omega_2 \in S-Act$ and $(S \mid \Omega_1)$, $(S \mid \Omega_2)$ be F-soft actions. Then*

(1) *$(S \mid \Omega_1\dot{\cup}\Omega_2)$ and $(S \mid \Omega_1 \cap \Omega_2)$ are F-soft actions, where $\Omega_1\dot{\cup}\Omega_2$ denote the disjoint union of Ω_1,Ω_2.*

(2) *$(S \mid \Omega_1 \times \Omega_2)$ which is defined by $s(\omega_1,\omega_2) := (s\omega_1, s\omega_2)$ for all $s \in S, (\omega_1,\omega_2) \in \Omega_1 \times \Omega_2$ is a T-soft action where $T : \Omega_1 \times \Omega_2 \to P(\Omega_1 \times \Omega_2)$ is given by:*

$$T(\omega_1,\omega_2) = F(\omega_1) \times F(\omega_2).$$

Let $\Omega_1,\Omega_2 \in S-Act$. Assume that F_A and H_B are two soft sets on Ω_1 and Ω_2, respectively. By a *soft map* from F_A to H_B we mean a pair of functions $\theta = (f,g)$ where $f : \Omega_1 \to \Omega_2$ and $g : A \to B$ such that $fF(a) = Hg(a)$, for all $a \in A$, i.e., we have the commutative diagram

$$\begin{tikzcd} P(\Omega_1) \arrow[r, "f"] & P(\Omega_2) \\ A \arrow[u, "F"] \arrow[r, "g"] & B \arrow[u, "H"'] \end{tikzcd}$$

When $A = \Omega_1$, $B = \Omega_2$ and $f = g$, we simply use the notation f instead of (f,f). In this case, f is said to be a *soft homomorphism* if it is an $S-Act$ homomorphism, i.e., $f(s\omega) = sf(\omega)$. If f is a surjective (injective) soft homomorphism then we say that f is a soft *epimorphism (monomorphism)*. A soft isomorphism is a soft epimorphism as well as a soft monomorphism.

Proposition 4.4. *Let $\Omega_1 \in S_1-Act$, $\Omega_2 \in S_2-Act$, $\epsilon : S_1 \to S_2$ a monoid epimorphism, and F_1,F_2 be two soft sets on Ω_1 and Ω_2, respectively. Moreover, assume that the action $(S \mid \Omega_1)$ is an F_1-soft action and $f : \Omega_1 \to \Omega_2$ is a soft epimorphism such that $f(s\omega) = \epsilon(s)f(\omega)$ for all $s \in S_1, \omega \in \Omega_1$. Then the action $(S \mid \Omega_2)$ is an F_2-soft action.*

Proof. Consider the diagram

$$\begin{tikzcd} P(\Omega_1) \arrow[r, "f"] & P(\Omega_2) \\ \Omega_1 \arrow[u, "F_1"] \arrow[r, "f"] & \Omega_2 \arrow[u, "F_2"'] \end{tikzcd}$$

We prove that for each $s_2 \in S_2$ and $\omega_2 \in \Omega_2, s_2(F_2(\omega_2)) = (F_2(s_2\omega_2))$. Since ϵ, f are epimorphisms, there exist $s_1 \in S_1$ and $\omega_1 \in \Omega_1$ such that $\epsilon(s_1) = s_2, f(\omega_1) = \omega_2$. Hence,

$$\begin{aligned} F_2(s_2\omega_2) &= F_2(\epsilon(s_1)(f(\omega_1))) \\ &= F_2(f(s_1\omega_1)) \\ &= f(F_1(s_1\omega_1) \\ &= f(s_1(F_1(\omega_1))) \\ &= \epsilon(s_1)(f(F_1(\omega_1))) \\ &= \epsilon(s_1)(F_2 f(\omega_1)) \\ &= s_2(F_2(\omega_2)) \end{aligned}$$

and the proof is complete. □

Proposition 4.5. *For an action $(S \mid \Omega)$, $\mathcal{S}_{\mathcal{F}}(\Omega) \leq G$. Where $S_F\Omega$, the F-softener of Ω is defined to be*

$$\mathcal{S}_{\mathcal{F}}(\Omega) = \{s \in S \mid F(s\omega) = sF(\omega),\ \forall\ \omega \in \Omega\}.$$

Proof. First, for every $g_1, g_2 \in \mathcal{S}_{\mathcal{F}}(\Omega)$ we get

$$\begin{aligned} F(g_2g_1^{-1}\omega) &= F(g_2(g_1^{-1}\omega)) && (G \mid \Omega) \\ &= g_2F(g_1^{-1}\omega) && g_2 \in \mathcal{S}_{\mathcal{F}}(\Omega) \\ &= g_2(g_1^{-1}F(\omega)) && g_1^{-1} \in \mathcal{S}_{\mathcal{F}}(\Omega) \\ &= g_2g_1^{-1}F(\omega) && (G \mid P(\Omega)). \end{aligned}$$

It implies that $g_2g_1^{-1} \in \mathcal{S}_{\mathcal{F}}(\Omega)$ and then the assertion holds. □

Obviously, for an action $(S \mid \Omega)$, $\mathcal{S}_{\mathcal{F}}(\Omega)$ is a submonoid of S and so $\mathcal{S}_{\mathcal{F}}(\Omega)$ acts on Ω in a natural way.

Let $\Omega \in S-Act$. An equivalence relation ρ on Ω is called an *S-congruence* or simply a *congruence* on Ω, if $\omega_1\rho\ \omega_2$ implies $s\omega_1\rho\ s\omega_2$ for all $\omega_1, \omega_2 \in \Omega, s \in S$. Moreover, for a soft set $F : \Omega \to P(\Omega)$, the *kernel* of F is denoted by $\ker F$ and is defined by

$$\ker F = \{(\omega_1, \omega_2) \in \Omega \mid F(\omega_1) = F(\omega_2)\}.$$

For each $\omega \in \Omega$, the equivalent class of ω by $\ker F$ denoted by $[\omega]_{\ker F}$ is

$$[\omega]_{\ker F} = \{\omega' \in \Omega \mid \omega'(\ker F)\omega\} = \{\omega' \in \Omega \mid F(\omega) = F(\omega')\}.$$

Lemma 4.6. (i) *Let $(G \mid \Omega)$ be an F-soft action. For the congruence $\ker F$ on Ω, $g[\omega]_{\ker F} = [g\omega]_{\ker F}$, for all $g \in G$, $\omega \in \Omega$.*

(ii) *Let $(G \mid \Omega)$ be an F-soft action. Then, the hereditary action $(G \mid \Omega/\ker F)$ is an F^*-soft action, where*

$$\begin{aligned} &G \times \Omega/\ker F \to \Omega/\ker F \\ &(g, [\omega]_{\ker F}) \to g[\omega]_{\ker F} := [g\omega]_{\ker F} \end{aligned}$$

and

$$\begin{aligned} &F^* : \Omega/\ker F \to P(\Omega/\ker F) \\ &F^*([\omega]_{\ker F}) := \{[\omega']_{\ker F} \mid \omega' \in F(\omega)\}. \end{aligned}$$

Proof. (i) First, for all $g \in G$ and $\omega \in \Omega$ we have

$$\begin{aligned} x \in g[\omega]_{\ker F} &\Rightarrow \exists\ \ y \in [\omega]_{\ker F}, \quad x = gy \\ &\Rightarrow y\ \ker F\ \omega, \qquad gy\ \ker F\ g\omega \\ &\Rightarrow x\ \ker F\ g\omega \\ &\Rightarrow x \in [g\omega]_{\ker F}. \end{aligned}$$

Conversely,

$$x \in [g\omega]_{\ker F} \Rightarrow x\ \ker F\ g\omega$$

$$\Rightarrow g^{-1}x \ \ker F \ \omega$$
$$\Rightarrow g^{-1}x \in [\omega]_{\ker F}$$
$$\Rightarrow x = g(g^{-1}x) \in g[\omega]_{\ker F}.$$

(ii) We have

$$\begin{aligned} F^*(g[\omega]_{\ker F}) &= F^*([g\omega]_{\ker F}) \\ &= \{[\omega']_{\ker F} \mid \omega' \in F(g\omega)\} \\ &= \{[\omega']_{\ker F} \mid \omega' \in gF(\omega)\} \\ &= \{[\omega']_{\ker F} \mid g^{-1}\omega' \in F(\omega)\} \\ &= \{[gt]_{\ker F} \mid t \in F(\omega)\} \\ &= g\{[t]_{\ker F} \mid t \in F(\omega)\} \\ &= gF^*([\omega]_{\ker F}) \end{aligned}$$

for all $g \in G$ and $\omega \in \Omega$.

The proof is complete. □

Remark 4.7. *For a soft set $F_1 : \Omega \to P(\Omega)$, consider the soft set $F_2 : \Omega \to P(\Omega)$ defined by $F_2(\omega) = [\omega]_{\ker F_1}$, then*

$$\Omega \underset{F_2}{\overset{F_1}{\rightrightarrows}} P(\Omega)$$

is an equalizer situation in $S-Act$. One can easily check that the equalizer of F_1 and F_2, represented as $Eq(F_1,F_2)$, is the pair (E,inc) where $E = \{\omega \in \Omega \mid F_1(\omega) = [\omega]_{\ker F_1}\} \subseteq \Omega$ is nonempty.

Let us use the notation $S(G \mid \Omega)$ for the set of those soft sets such as $F : \Omega \to P(\Omega)$ for which the action $(G \mid \Omega)$ is an F-soft action and $\mathrm{Con}(G \mid \Omega)$ for the set of all congruence relations on Ω. Obviously, for each group action $(G \mid \Omega)$, the identity soft set belongs to $S(G \mid \Omega)$ and the identity relation on Ω (which is denoted by Δ_Ω) belongs to $\mathrm{Con}(G \mid \Omega)$. Consequently, $S(G \mid \Omega)$ and $\mathrm{Con}(G \mid \Omega)$ are both nonempty.

Proposition 4.8. *Let G acts on a nonempty set Ω. Then $\mathrm{Con}(G \mid \Omega)$ is equipotent to the subset H of $S(G \mid \Omega)$ consisting those soft sets $F \in S(G \mid \Omega)$ satisfying the condition $F(\omega) \times F(\omega) \subseteq \ker F$, and $\omega \in F(\omega)$ for all $\omega \in \Omega$.*

Proof. Let $F \in S(G \mid \Omega)$. First we show that $F \in H$ if and only if F satisfies the condition "$(\omega,\omega') \in \ker F$ if and only if $\omega' \in F(\omega)$", for all $\omega,\omega' \in \Omega$.

Let $F \in H$ and $\omega,\omega' \in \Omega$. If $(\omega,\omega') \in \ker F$, then $F(\omega) = F(\omega')$. Also, $\omega' \in F(\omega')$ by the assumption. Thus, $\omega' \in F(\omega)$. Conversely, let $\omega' \in F(\omega)$. Since $\omega \in F(\omega)$ we get $(\omega,\omega') \in F(\omega) \times F(\omega)$. Consequently, $(\omega,\omega') \in \ker F$ by hypothesis.

For the converse, suppose for $\omega,\omega' \in \Omega, (\omega,\omega') \in \ker F$ if and only if $\omega' \in F(\omega)$. We show that $F \in H$. Let $\omega \in \Omega$. First, $F(\omega) = F(\omega)$ implies that $(\omega,\omega) \in \ker F$. So, $\omega \in F(\omega)$. Moreover, if

$(\omega_1,\omega_2)\in F(\omega)\times F(\omega)$, then $\omega_1\in F(\omega)$ and $\omega_2\in F(\omega)$. Using the assumption, $(\omega,\omega_1),(\omega,\omega_2)\in \ker F$ and hence $(\omega_1,\omega_2)\in\ker F$. Thus, $F(\omega)\times F(\omega)\subseteq \ker F$ and then $F\in H$.

Now, define the mapping $\alpha : H\to \mathrm{Con}(G\mid\Omega)$ by $\alpha(F)=\ker F$ for each $F\in H$. Let $F_1,F_2\in H$ and $\alpha(F_1)=\alpha(F_2)$. Then $\ker F_1=\ker F_2$. We claim that $F_1=F_2$. Take any $\omega\in\Omega$. It must be shown that $F_1(\omega)=F_2(\omega)$. For every $\omega'\in F_1(\omega)$, we have $(\omega,\omega')\in\ker F_1=\ker F_2$ and so $(\omega,\omega')\in\ker F_2$. This gives that $\omega'\in F_2(\omega)$ whence $F_1(\omega)\subseteq F_2(\omega)$. By the same way, $F_2(\omega)\subseteq F_1(\omega)$. Hence, $F_1=F_2$ which means that α is one to one. Finally, we prove that α is onto. For this, take any $\rho\in\mathrm{Con}(G\mid\Omega)$. Consider the (natural) soft set $\Pi_\rho:\Omega\to\Omega/\rho\subseteq P(\Omega)$ defined by $\Pi_\rho(\omega)=[\omega]_\rho$ for all $\omega\in\Omega$. Using Lemma **??**, for each $g\in G$ and $\omega\in\Omega, \Pi_\rho(g\omega)=[g\omega]_\rho=g[\omega]_\rho=g\Pi_\rho(\omega)$. Therefore, $(G\mid\Omega)$ is a Π_ρ-soft action. On the other hand, for each $\omega\in\Omega, \Pi_\rho(\omega)\times\Pi_\rho(\omega)=[\omega]_\rho\times[\omega]_\rho\subseteq\rho=\ker\Pi_\rho$ and $\omega\in[\omega]_\rho=\Pi_\rho(\omega)$. Hence, $\Pi_\rho\in H$ and $\alpha(\Pi_\rho)=\ker\Pi_\rho=\rho$, as desired. Consequently, α is a bijection and then $\mathrm{Con}(G\mid\Omega)$ is equivalent to H. □

Using Theorem 4.8, the following is immediate:

Corollary 4.9. *Let* $(G\mid\Omega)$. *Then* $|\mathrm{Con}(G\mid\Omega)|\le|S(G\mid\Omega)|$.

5. Conclusion

Certain concrete examples justifying the contents of sections 3 and 4.

Example 5.1. Consider $X=D_6=\langle x_1,x_2\mid x_1^3=1,x_2^2=1,(x_1x_2)^2=1\rangle$ the dihedral group and let $A=\{x_1,x_2\}$. Then, $(F,A)=\{(\langle x_1\rangle,x_1),(\langle x_2\rangle,x_2)\}$ is a soft cyclic group.

Example 5.2. In Example 5.3, if $(H,B)=\{(\{a\},2),(\{a^2\},3)\}$ then $\langle(H,B)\rangle=15$.

Example 5.3. Let $A=\{1,2,3\}$, $G=\langle a\mid a^4=1\rangle=\{1,a,a^2,a^3\}$. Also, assume that $H_1=\langle 1\rangle$, $H_2=\langle a\rangle=G$, $H_3=\langle a^2\rangle$. Moreover, let $(F,A)=\{(\langle 1\rangle,1),(\langle a\rangle,2),(\langle a^2\rangle,3)\}$. By the Theorem 3.2 notations, $m_1=1$, $m_2=3$, $m_3=2$; thus (F,A) has

$$(1+3+2)+((1\times 2)+(1\times 3)+(2\times 3))+(1\times 2\times 3)=23$$

soft subgroups which are listed below:

$$1=\{(\langle 1\rangle,1)\},$$
$$2=\{(\langle 1\rangle,2)\},$$
$$3=\{(\langle a\rangle,2)\},$$
$$4=\{\langle a^2\rangle,2\},$$
$$5=\{(\langle 1\rangle,3)\},$$
$$6=\{(\langle a^2\rangle,3)\},$$
$$7=\{(\langle 1\rangle,1),(\langle 1\rangle,2)\},$$
$$8=\{(\langle 1\rangle,1),(\langle a\rangle,2)\},$$
$$9=\{(\langle 1\rangle,1),(\langle a^2\rangle,2)\},$$

$$10 = \{(\langle 1\rangle, 1), (\langle 1\rangle, 3)\},$$
$$11 = \{(\langle 1\rangle, 1), (\langle a^2\rangle, 3)\},$$
$$12 = \{(\langle 1\rangle, 2), (\langle 1\rangle, 3)\},$$
$$13 = \{(\langle 1\rangle, 2), (\langle a^2\rangle, 3)\},$$
$$14 = \{(\langle a\rangle, 2), (\langle 1\rangle, 3)\},$$
$$15 = \{(\langle a\rangle, 2), (\langle a^2\rangle, 3)\},$$
$$16 = \{(\langle a^2\rangle, 2), (\langle 1\rangle, 3)\},$$
$$17 = \{(\langle a^2\rangle, 2), (\langle a^2\rangle, 3)\},$$
$$18 = \{(\langle 1\rangle, 1), (\langle 1\rangle, 2), (\langle 1\rangle, 3)\},$$
$$19 = \{(\langle 1\rangle, 1), (\langle 1\rangle, 2), (\langle a^2\rangle, 3)\},$$
$$20 = \{(\langle 1\rangle, 1), (\langle a\rangle, 2), (\langle 1\rangle, 3)\},$$
$$21 = \{(\langle 1\rangle, 1), (\langle a\rangle, 2), (\langle a^2\rangle, 3)\},$$
$$22 = \{(\langle 1\rangle, 1), (\langle a^2\rangle, 2), (\langle 1\rangle, 3)\},$$
$$23 = \{(\langle 1\rangle, 1), (\langle a^2\rangle, 2), (\langle a^2\rangle, 3)\}.$$

Example 5.4. Let G,K are two cyclic groups of the same order 6 generated by a,b, respectively. Assume that $f : G \to K$ defined by $f(a) = b$ and $A = \{1,2,3\}$. Let $(F,A) = \{(\langle 1_G\rangle, 1), (\langle a^2\rangle, 2), (\langle a^3\rangle, 3)\}$, $(H,A) = \{(\langle 1_K\rangle, 1), (\langle b^2\rangle, 2), (\langle b^3\rangle, 3)\}$, then $\theta = (f, 1_A)$ is a soft isomorphism, thus $(F,A) \sim (H,A)$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] H. Aktas and N. Cogman, Soft sets and soft groups, *Information Sciences* **177** (2007), 2726–2735.

[2] K. Basu, R. Deb and P.K. Pattanaik, *Soft sets: An ordinal formulation of vagueness with some applications to the theory of choice*, *Fuzzy Sets and Systems* **45** (1992), 45–58.

[3] H. Aktas and S. Ozlu, Cyclic soft groups and their applications on groups, *The Scientific World Journal* **2014**, Article ID 437324.

[4] M. Aslam and S.M. Qurashi, Some contribution to soft groups, *Annals of Fuzzy Mathematics and Informatics* **4** (1) (2012), 177–195.

[5] R. Lin and J. Wang, Soft homomorphism properties on soft groups and soft normal groups, *IEEE International Conference on Granular Computing* (2011), 851–854..

[6] P.K. Maji, R. Biswas and A.R. Roy, Fuzzy soft sets, *The Journal of Fuzzy Mathematics* 9 (3) (2001), 677–692.

[7] P.K. Maji, R. Biswas and A.R. Roy, Soft set theory, *Computers and Mathematics with Applications* **45** (2003), 555–562.

[8] D. Molodtsov, Soft set theory - First results, *Computers and Mathematics with Applications* **37** (1999), 19–31.

[9] S.K. Nazmul and S.K. Samanta, Soft topological soft groups, *Mathematical Sciences* **6** (66) (2012), 1–10.

[10] A. Sezgin and A.O. Atagun, Soft groups and normalistic soft groups, *Computer and mathematics with Applications* **62** (2011), 685–698.

[11] X. Yin and Z. Liao, Study on soft groups, *Journal of Computers* **8** (4) (2013), 960–967.

[12] M. Zhou, S. Li and M. Akram, Categorical properties of soft sets, *The scientific World Journal* **2014** (2014), Article ID 783056.