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(*Invited paper*)

General Proximal Point Algorithmic Models and Nonlinear Variational Inclusions Involving RMM Mappings

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Abstract. The proximal point algorithms based on relative A-maximal monotonicity (RMM) is introduced, and then it is applied to the approximation solvability of a general class of nonlinear inclusion problems using the generalized resolvent operator technique. This algorithm seems to be more application-oriented to solving nonlinear inclusion problems. Furthermore, the obtained result could be applied to generalize the Douglas-Rachford splitting method to the case of RMM mapping based on the generalized proximal point algorithm.

1. Introduction

Consider a real Hilbert space X with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. Here we are concerned with a general class of nonlinear variational inclusion problems: determine a solution to

$$0 \in M(x), \tag{1.1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

Recently, the Verma [24] investigated the solvability of a generalized class of variational inclusion systems involving RMM, RMRM, PSM and cocoercive mappings. These notions, especially RMM and RMRM generalize most of the existing concepts of general maximal monotonicity in literature. On the other hand, these notions do have significant applications to proximal point algorithms and its variants, especially introduced and studied by Eckstein and Bertsekas [1], that was based on the work of Rockafellar [10], while solving inclusion problems of the form (1.1). Furthermore, Verma [19] applied the relative A-maximal monotonicity (RMM), in the context of investigating the approximation solvability of an inclusion problem of the form (1.1) relating to sensitivity analysis.

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In this communication, we develop a general framework for the generalized proximal point algorithm in light of the notion of the relative A-maximal monotonicity (RMM) of the set-valued map M, that encompasses most of proximal point algorithms applied to the context of solving general inclusion problems (1.1) in literature. Verma [17] introduced and studied the notion of A-maximal monotonicity, while examining the approximation solvability of inclusion problems of the form (1.1) that may have applications to problems arising from mathematical economics, optimization and control theory, operations research, mathematical finance, mathematical programming, and decision and management sciences. The notion of A-maximal monotonicity generalizes the existing general theory of maximal monotone mappings, including the H-maximal monotone mappings [4]. For more literature, we recommend the reader [1]-[27].

2. General Relative A-Maximal Monotonicity

In this section we present some basic properties on relative A-maximal monotonicity (RMM) and related results. Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, that is, the set $\{(x, y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x, y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x, y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$D(M) = \{x \in X : \exists\, y \in X : (x, y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

The inverse M^{-1} of M is $\{(y, x) : (x, y) \in M\}$.

Definition 2.1. Let $A : X \to X$ be a single-valued mapping, and $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) Monotone if

$$\langle u^* - v^*, u - v\rangle \geq 0 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(ii) (r)-strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(iii) (r)-expanding if there exists a positive constant r such that

$$\|u^* - v^*\| \geq r\|u - v\| \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(iv) (m)-relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(v) (c)-cocoercive if there exists a positive constant c such that

$$\langle u^* - v^*, u - v\rangle \geq c\|u^* - v^*\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(vi) Cocoercive if we have

$$\langle u^* - v^*, u - v\rangle \geq \|u^* - v^*\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(vii) (d)-relaxed cocoercive if there exists a positive constant d such that

$$\langle u^* - v^*, u - v\rangle \geq -d\|u^* - v^*\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(viii) Nonexpansive if

$$\|u^* - v^*\| \leq \|u - v\| \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(ix) Firmly nonexpansive if

$$\|u^* - v^*\|^2 \leq \langle u^* - v^*, u - v\rangle \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(x) (c)-Firmly nonexpansive if there is a positive constant c such that

$$\|u^* - v^*\|^2 \leq c\langle u^* - v^*, u - v\rangle \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(xi) Hypermonotone if

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(xii) (r)-Strongly hypermonotone if there exists a positive constant r such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq r\|u - v\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(xiii) (m)-relaxed hypermonotone if there exists a positive constant m such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq (-m)\|u - v\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(xiv) (c)-hypercocoercive if there exists a positive constant c such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq c\|u^* - v^*\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(xv) Hypercocoercive if we have

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq \|u^* - v^*\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(xvi) (d)-relaxed hypercocoercive if there exists a positive constant d such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq -d\|u^* - v^*\|^2 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(xvii) Firmly hypernonexpansive if

$$\|u^* - v^*\|^2 \leq \langle u^* - v^*, A(u) - A(v)\rangle \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

(xviii) (c)-Firmly hypernonexpansive if there is a positive constant c such that

$$\|u^* - v^*\|^2 \leq c\langle u^* - v^*, A(u) - A(v)\rangle \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M).$$

Definition 2.2. The map $M : X \to 2^X$ is said to be maximal monotone if

(i) M is monotone, that is,

$$\langle u^* - v^*, u - v\rangle \geq 0 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M),$$

and

(ii) it follows from $(u, u^*) \in X \times X$ and

$$\langle u^* - v^*, u - v\rangle \geq 0 \ \forall\ (v, v^*) \in \text{graph}(M)$$

that $(u, u^*) \in \text{graph}(M)$.

Definition 2.3. Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be *maximal hypermonotone* (MHM) if

(i) M is hypermonotone, that is,

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \ \forall\ (u, u^*), (v, v^*) \in \text{graph}(M),$$

and it follows from

(ii) $(u,u^*) \in X \times X$ and

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \ \forall \ (v,v^*) \in \text{graph}(M)$$

that $(u,u^*) \in \text{graph}(M)$.

Definition 2.4 ([19])**.** Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be relative A-maximal monotone (RMM) if:

(i) M is hypermonotone, that is,

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \ \forall \ (u,u^*), (v,v^*) \in \text{graph}(M).$$

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Example 2.1. We consider an example where M is hypermonotone but not monotone. Let $X = (-\infty, +\infty)$, $M(x) = -x$ and $A(x) = -\frac{1}{2}x$ for all $x \in X$. Then it is easy to check that M is hypermonotone but not monotone.

Definition 2.5 ([19])**.** Let $A : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an RMM mapping. Then the generalized resolvent operator $R^M_{\rho,A} : X \to X$ is defined by

$$R^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Proposition 2.1. *Let $A : X \to X$ be an (r)-strongly monotone single-valued mapping and let $M : X \to 2^X$ be an RMM mapping. Then $(A+\rho M)$ is maximal hypermonotone for $\rho > 0$.*

Proof. In light of Definition 2.3, $A + \rho M$ is hypermonotone for $\rho > 0$, since A is (r)-strongly monotone and M is hypermonotone. All be need to show at this stage is for $(u,u^*) \in X \times X$ and

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \ \forall \ (v,v^*) \in \text{graph}(A + \rho M),$$

we have $(u,u^*) \in \text{graph}(A + \rho M)$.

To achieve this, assume that there exists some $(u_0, u_0^*) \notin \text{graph}(A + \rho M)$ such that

$$\langle u_0^* - v^*, A(u_0) - A(v)\rangle \geq 0 \ \forall \ (v,v^*) \in \text{graph}(A + \rho M).$$

Since M is RMM (and hence $(A + \rho M)X = X$ for $\rho > 0$), there exists an $(u_1, u_1^*) \in \text{graph}(A + \rho M)$ such that $A(u_0) + \rho u_0^* = A(u_1) + \rho u_1^*$, and as a result, we have

$$\rho \langle u_0^* - u_1^*, A(u_0) - A(u_1)\rangle = -\langle A(u_0) - A(u_1), A(u_0) - A(u_1)\rangle \geq 0.$$

Now aplying the (r)-strong monotonicity of A (and hence $\|A(x) - A(y) \geq r\|x - y\|$), we infer that $u_0 = u_1$, and as a result, we conclude $u_0^* = u_1^*$, a contradiction to $(u_0, u_0^*) \notin \text{graph}(A + \rho M)$. □

Example 2.2. Let $A : X \to X$ be (r)-strongly monotone. Let $f : X \to R$ be a locally Lipschitz function such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \ \forall \ (u, u^*), (v, v^*) \in \text{graph}(\partial f),$$

and $R(A + \partial f) = X$. Then clearly $A + \partial f$ is strongly hypermonotone, and for $(u, u^*) \in X \times X$ and

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \ \forall \ (v, v^*) \in \text{graph}(A + \partial f),$$

we have $(u, u^*) \in \text{graph}(A + \partial f)$. Thus, $A + \partial f$ is maximal hypermonotone.

Proposition 2.2 ([19]). *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be RMM. Then the generalized resolvent operator associated with M and defined by*

$$R^M_{\rho,A}(u) = (A + \rho M)^{-1}(u) \ \forall \ u \in X,$$

is $(\frac{1}{r})$-Lipschitz continuous.

Proposition 2.3 ([19]). *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be RMM. Then the generalized resolvent operator associated with M and defined by*

$$R^M_{\rho,A}(u) = (A + \rho M)^{-1}(u) \ \forall \ u \in X,$$

satisfies

$$\langle u - v, A(R^M_{\rho,A}(u)) - A(R^M_{\rho,A}(v))\rangle \geq \|A(R^M_{\rho,A}(u)) - A(R^M_{\rho,A}(v))\|^2.$$

Proposition 2.4 ([19]). *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be RMM. Then the generalized resolvent operator associated with M and defined by*

$$R^M_{\rho,A}(u) = (A + \rho M)^{-1}(u) \ \forall \ u \in X,$$

satisfies

$$\begin{aligned}&\langle A(u) - A(v), A(R^M_{\rho,A}(A(u))) - A(R^M_{\rho,A}(A(v)))\rangle \\ &\quad \geq \ \|A(R^M_{\rho,A}(A(u))) - A(R^M_{\rho,A}(A(v)))\|^2.\end{aligned}$$

When $A = I$ in Proposition 2.4, we have the well-known result in literature.

Proposition 2.5. *Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal monotone. Then the resolvent operator associated with M and defined by*

$$R^M_{\rho}(u) = (I + \rho M)^{-1}(u) \ \forall \ u \in X,$$

satisfies

$$\langle u - v, R^M_{\rho}(u) - R^M_{\rho}(v)\rangle \geq \|R^M_{\rho}(u) - R^M_{\rho}(v)\|^2,$$

that is, the resolvent operator R^M_{ρ} is firmly nonexpansive.

3. **Generalized Proximal Point Algorithm**

This section deals with an introduction of a generalized version of the proximal point algorithm and its applications to approximation solvability of the inclusion problem (1.1) based on the relative A-maximal monotonicity (RMM).

Proposition 3.1 ([19]). *Let $J_k = A - A\circ J^M_{\rho,A}\circ A$. If, in addition,*

$$\langle A(u)-A(v), A(J^M_{\rho,A}(A(u))) - A(J^M_{\rho,A}(A(v)))\rangle \\ \geq \|A(J^M_{\rho,A}(A(u))) - A(J^M_{\rho,A}(A(v)))\|^2.$$

then

$$\|A(J^M_{\rho,A}(A(u))) - A(J^M_{\rho,A}(A(v)))\|^2 + \|J_k(u)-J_k(v)\|^2 \\ \leq \|A(u)-A(v)\|^2\ \forall\ u,v\in X. \tag{3.1}$$

Theorem 3.1. *Let X be a real Hilbert space, let $A: X\to X$ be (r)-strongly monotone, and let $M: X\to 2^X$ be RMM. Then the following statements are equivalent:*

(i) *An element $u\in X$ is a solution to* (1.1).

(ii) *For an $u\in X$, we have*

$$u = R^M_{\rho,A}(A(u)).$$

where $R^M_{\rho,A}(u) = (A+\rho M)^{-1}(u)$.

In the following theorem, we apply the generalized proximal point algorithm to approximating the solution of (1.1), and as a result, we establish the weak convergence.

Theorem 3.2. *Let X be a real Hilbert space, let $A: X\to X$ be (r)-strongly monotone and weakly continuous, and let $M: X\to 2^X$ be RMM. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the generalized proximal point algorithm*

$$A(x^{k+1}) = (1-\alpha_k)A(x^k) + \alpha_k y^k\ \forall\ k\geq 0, \tag{3.2}$$

and y^k satisfies

$$\|y^k - A(R^M_{\rho_k,A}(A(x^k)))\| \leq \epsilon_k,$$

where $R^M_{\rho_k,A} = (A+\rho_k M)^{-1}$, and $\{\epsilon_k\},\{\alpha_k\},\{\rho_k\}\subseteq[0,\infty)$ are scalar sequences.

Suppose that $\{x^k\}$ is bounded in the sense that there exists at least one solution to $0\in M(x)$. Then the sequence $\{x^k\}$ converges weakly to a unique solution x^* of (1.1) with $\sum_{k=0}^{\infty}\epsilon_k<\infty$, $\inf\alpha_k>0$, $\sup\alpha_k<2$, $\alpha=\limsup_{k\to\infty}\alpha_k$, and $\inf\rho_k>0$.

Proof. Let x^* be a zero of M. We infer from Theorem 3.1 that any solution to (1.1) is a fixed point of $J^M_{\rho_k,A}\circ A$. Thus, $R^M_{\rho_k,A}(A(x^*)) = x^*$ and for $J_k = A - A\circ J^M_{\rho,A}\circ A$, we need to show that

$$J_k(x^k) = A(x^k) - A(R^M_{\rho,A}(A(x^k)))\to 0.$$

Now we begin with the result, that follows, in light of Proposition 2.4 that

$$\langle A(x^k) - A(x^*), J_k(x^k) - J_k(x^*)\rangle \geq \|J_k(x^k) - J_k(x^*)\|^2, \tag{3.3}$$

that is, J_k is firmly nonexpansive with respect to A.

Next we start the main part of the proof by expressing (for all $k \geq 0$),

$$\begin{aligned} A(z^{k+1}) &= (1-\alpha_k)A(x^k) + \alpha_k A(R^M_{\rho_k,A}(A(x^k))) \\ &= (A - \alpha_k J_k)(x^k). \end{aligned}$$

For all $k \geq 0$, using (3.3), we have

$$\begin{aligned} &\|A(z^{k+1}) - A(x^*)\|^2 \\ &= \|A(x^k) - \alpha_k J_k(x^k) - A(x^*)\|^2 \\ &= \|A(x^k) - A(x^*)\|^2 - 2\alpha_k \langle A(x^k) - A(x^*), J_k(x^k)\rangle + \alpha_k^2 \|J_k(x^k)\|^2 \\ &\leq \|A(x^k) - A(x^*)\|^2 - 2\alpha_k \|J_k(x^k) - J_k(x^*)\|^2] + \alpha_k^2 \|J_k(x^k)\|^2 \\ &= \|A(x^k) - A(x^*)\|^2 - [2\alpha_k - \alpha_k^2]\|J_k(x^k)\|^2. \end{aligned}$$

Since $\alpha_k[2-\alpha_k] > 0$, we have

$$\|A(z^{k+1}) - A(x^*)\| \leq \|A(x^k) - A(x^*)\|. \tag{3.4}$$

Since $A(x^{k+1}) = (1-\alpha_k)A(x^k) + \alpha_k y^k$, we have

$$A(x^{k+1}) - A(x^k) = \alpha_k(y^k - A(x^k)).$$

It follows that

$$\begin{aligned} &\|A(x^{k+1}) - A(z^{k+1})\| \\ &= \|(1-\alpha_k)A(x^k) + \alpha_k y^k - [(1-\alpha_k)A(x^k) + \alpha_k R^M_{\rho_k,A}(A(x^k))]\| \\ &= \|\alpha_k(y^k - R^M_{\rho,A}(A(x^k)))\| \\ &\leq \alpha_k \epsilon_k. \end{aligned}$$

Next, we estimate using the above arguments

$$\begin{aligned} \|A(x^{k+1}) - A(x^*)\| &\leq \|A(z^{k+1}) - A(x^*)\| + \|A(x^{k+1}) - A(z^{k+1})\| \\ &\leq \|A(z^{k+1}) - A(x^*)\| + \alpha_k \epsilon_k \\ &\leq \|A(x^k) - A(x^*)\| + \alpha_k \epsilon_k. \end{aligned} \tag{3.5}$$

Therefore, we have

$$\|A(x^{k+1}) - A(x^*)\| \leq \|A(x^k) - A(x^*)\| + \alpha_k \epsilon_k. \tag{3.6}$$

Combining (3.6) for $k \geq 0$, we have

$$\|A(x^{k+1}) - A(x^*)\| \leq \|A(x^0) - A(x^*)\| + \sum_{j=0}^{k} \alpha_j \epsilon_j$$

$$\leq \ \|A(x^0) - A(x^*)\| + 2\sum_{k=0}^{\infty} \epsilon_k. \tag{3.7}$$

Since A is (r)-strongly monotone (and hence, $\|A(u) - A(v) \geq r\|u - v\|$), we have

$$\|x^{k+1} - x^*\| \leq \frac{1}{r}\left[\|A(x^0) - A(x^*)\| + 2\sum_{k=0}^{\infty} \epsilon_k\right]. \tag{3.8}$$

We infer now that the sequence $\{x^k\}$ is bounded.

Using (3.7), we further derive the estimate leading to $J_k(x^k) \to 0$.

$$\begin{aligned}
&\|A(x^{k+1}) - A(x^*)\|^2 \\
&\quad = \ \|A(z^{k+1}) - A(x^*) + A(x^{k+1}) - A(z^{k+1})\|^2 \\
&\quad \leq \ \|A(z^{k+1}) - A(x^*)\|^2 + 2\langle A(z^{k+1}) - A(x^*), A(x^{k+1}) - A(z^{k+1})\rangle \\
&\qquad + \|A(x^{k+1}) - A(z^{k+1})\|^2 \\
&\quad \leq \ \|A(x^k) - A(x^*)\|^2 - \alpha_k[2 - \alpha_k]\|J_k(x^k)\|^2 \\
&\qquad + 2\alpha_k\epsilon_k(\|A(x^k) - A(x^*)\| + 2\alpha_k\epsilon_k) + \alpha_k^2\epsilon_k^2.
\end{aligned}$$

Since $\{\epsilon_k\}$ is summable (and hence $\{\epsilon_k^2\}$ is summable) and $\sum_{k=0}^{\infty} \alpha_k < \infty$), it implies, for all k, that

$$\begin{aligned}
\|A(x^{k+1} - A(x^*)\|^2 \ \leq \ & \|A(x^0) - A(x^*)\|^2 \\
& + 2\alpha_k\epsilon_k\left(\|A(x^0) - A(x^*)\| + 2\sum_{j=0}^{k} \alpha_j\epsilon_j\right) \\
& + \sum_{j=0}^{k} \alpha_j^2\epsilon_j^2 - \alpha_k[2 - \alpha_k]\sum_{j=0}^{k} \|J_j(x^j)\|^2.
\end{aligned}$$

It follows that $\sum_{j=0}^{\infty} \|J_j(x^j)\|^2 < \infty$ implies $J_k(x^k) \to 0$ *as* $k \to \infty$.

Now, by the Generalized Representation Lemma, for all k, there is a unique point $(u^k, v^k) \in M$ such that $A(u^k) + \rho_k v^k = A(x^k)$. Since $J_k(x^k) \to 0$ and $u^k = (R^M_{\rho_k,A} \circ A)(x^k)$, it implies $A(x^k) - A(u^k) \to 0$. Furthermore, $\rho_k v^k = A(x^k) - A(u^k)$, and hence $v^k = \rho_k^{-1} J_k(x^k) \to 0$, where ρ_k is bounded away from zero. As $\{x^k\}$ is bounded in light of (3.8), it must have a weak cluster point, say x'. Suppose that $\{x^k(j)\}_{j=0}^{\infty}$ be a subsequence such that $x^{k(j)} \xrightarrow{w} x'$.

Since $A(x^k) - A(u^k) \to 0$ and A is (r)-strongly monotone (and hence, A is (r)-expanding, that is, $\|A(x^k) - A(u^k)\| \geq r\|x^k - u^k\|$), it follows that $x^k - u^k \to 0$. Given that A is weakly continuous, we have $A(u^{k(j)}) \xrightarrow{w} A(x')$. Finally, consider a point $(u, v) \in M$. Then relative A-maximal monotonicity of M ensures that

$$\langle A(u) - A(u^k), v - v^k\rangle \geq 0 \ \forall\ k \geq 0.$$

Thus, we have

$$\langle A(u) - A(x'), v - 0\rangle \geq 0.$$

Since M is relative A-maximal monotone, and (u,v) is arbitrary, we have $(x',0) \in M$. As a result, x' is a solution to (1.1). Moreover, the uniqueness of the solution easily follows. □

(ii) For $A = I$ (identity), Theorem 3.2 reduces to ([1, Theorem 3]).

Theorem 3.3. *Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal monotone. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the generalized proximal point algorithm*

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^k \ \forall\ k \geq 0, \tag{3.9}$$

and y^k satisfies

$$\|y^k - J^M_{\rho_k,A}(x^k)\| \leq \epsilon_k\,,$$

where $J^M_{\rho_k} = (I+\rho_k M)^{-1}$, and $\{\epsilon_k\},\{\alpha_k\},\{\rho_k\} \subseteq [0,\infty)$ are scalar sequences.

Then the sequence $\{x^k\}$ converges weakly to a unique solution x^* of (1.1) with where $\sum_{k=0}^{\infty}\epsilon_k < \infty$, $\inf \alpha_k > 0$, $\sup \alpha_k < 2$, $\alpha = \limsup_{k\to\infty}\alpha_k$, and $\inf \rho_k > 0$.

4. Concluding Remark

In literature [26, 27], the Yosida regularization/approximation has been applied in the context of solving evolution equations as well as evolution inclusions in Hilbert and Banach space settings, where the Yosida regularization and Yosida approximation, respectively, of the form

$$M_{\rho,\mathrm{regular}} = M(I+\rho M)^{-1},$$

and

$$M_{\rho,\mathrm{app}} = \rho^{-1}(I-(I+\rho M)^{-1}) \quad \text{for } \rho > 0,$$

are considered. As a matter of fact, both are mutually equivalent in nature and to applications.

Based on our construction in Proposition 3.1, we generalized the Yosida approxi-mation to the case of A-maximal monotonicity as

$$M_{\rho,\mathrm{app}} = \rho^{-1}(A - A(A+\rho M)^{-1}A) \quad \text{for } \rho > 0,$$

and applied to the solvability of a class of first-order evolution inclusions empowered by the maximal accretivity/maximal monotonicity in a forthcoming communication.

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