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Research Article

The m-Clique Load and the m-Clique Sequence of Graphs

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Abstract. This paper introduces the concepts of the m-clique load, the m-clique sequence and the m-clique density of graphs. The number of distinct maximum cliques over all maximal cliques is called the m-clique load of G and denoted, $\diamond(G)$. The m-clique sequence denoted, $\diamond$-sequence of a graph G with $\epsilon(G) \geq 1$ is the sequence with entries representing the number of maximal cliques of same order found in G, in descending order. A finite sequence of positive integers each indexed with a distinct positive integer subscript which is c-graphical, is characterised. The m-clique density of a graph G denoted, $p_{c_i}(G)$ is the probability of uniformly at random, choosing a maximal clique K_{c_i}, $1 \leq c_i \leq \nu(G)$. Introductory results for certain graph classes and power graphs of balanced caterpillars, $C^{\mathfrak{L}}_{P_n}$ are also presented.

Keywords. m-clique, m-clique load, m-clique sequence, m-clique density

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1. Introduction

For general notation and concepts in graph theory, we refer to [3, 5]. Unless mentioned otherwise, a graph $G = G(V,E)$ on $\nu(G)$ vertices with $\epsilon(G)$ edges will be a finite undirected and connected simple graph. A clique of a graph G is a complete subgraph of G. A maximal clique of G is a clique Q for which a neighbor u of any vertex $v \in V(Q)$, $u \notin V(Q)$ is not adjacent to all vertices in $V(Q)$. We denote a maximal clique as m-clique. Since the m-cliques of a graph may differ in order, we define the *m-clique load* denoted, $\diamond(G)$ to be the number of distinct m-cliques over all m-cliques of G. Clearly an acyclic graph (a tree) H on $n \geq 2$ vertices has, $\diamond(H) = \epsilon(H)$. Equally clear is that for complete graphs K_n, $n \geq 1$ we have $\diamond(K_n) = 1$.

Since the m-cliques of a graph G may differ in order, and the number of distinct m-cliques of the same order may differ from that of a different order. Therefore, there is no relationship between $\diamond(G)$ and $\diamond(G')$ with G' a subgraph of G.

Distinct m-cliques may have vertices or edges in common. The latter m-cliques are called *intersecting m-cliques*. The intersection of m-cliques are taken as the clique common to the m-cliques. For example if two distinct m-cliques share a clique which is a triangle C_3 on vertices v_i, v_j, v_k, then the edges v_iv_j, v_iv_k, v_jv_k are *inherently* common edges and therefore, not distinguished as common edges. Similarly the vertices v_i, v_j, v_k are *inherently* common vertices and are therefore, not distinguished as common vertices.

Call the vertices and the edges of a graph G the *primitive elements* of G. Then the probability of uniformly at random, choosing either a vertex or an edge is given by $p_{v\in V(G)}(G) = \frac{\nu(G)}{\nu(G)+\epsilon(G)}$ and $p_{e\in E(G)}(G) = \frac{\epsilon(G)}{\nu(G)+\epsilon(G)}$, respectively. Similarly the m–clique density denoted, $p_{c_i}(G)$ is the probability of uniformly at random, choosing a maximal clique K_{c_i}, $1 \leq c_i \leq \nu(G)$. Therefore, the m-clique density of a graph is not necessary a unique value since it depends on the order of the clique under consideration.

The motivation for studying these concepts is their application in the theory of proper colouring of graphs, and therefore finding further results relating to the chromatic number, the b-chromatic number and the Thue chromatic number of graphs. Finding maximal independent sets, maximal independent sequences in complement graphs is another application. Following from Definition 3.1 in Section 3 it is easy to see that for the number of maximum independent sets in G denoted, $\boxdot(G)$, we have $\diamond(G) = \boxdot(G^c)$. Furthermore, $s^{\diamond}(G) = s^{\boxdot}(G^c)$. These observations open scope for further research.

Determining cliques in general in a graph enjoyed extensive research. We refer to *A fast algorithm to find a maximum clique* by Östergård [12]. Techniques to determine cliques in *soft graphs* by Hoede [9], refers. Also, *detecting large cliques* by Andreev et al. [2] and *the number of cliques in dense graphs*, Hedman [7], refer. It is possible to find all cliques by using the *Bron-Kerbosch algorithm* [4]. The aim of this paper is to find introductory results for the m-clique load and the $\diamond$-sequence of some graphs. Hence, graphs with easily detectable maximal cliques will be studied.

2. The Clique Load

Intersecting m-cliques lead to perhaps an obvious, but important introductory result.

Proposition 2.1. *If two or more distinct m-cliques of G share $k \geq 1$ common vertices $v_1, v_2, v_3, \ldots, v_k$, then the induced graph, $\langle v_1, v_2, v_3, \ldots, v_k \rangle$ is a clique of G.*

Proof. *Case* (i): If $k = 1$, let the common vertex be v. Since, $K_1 \simeq \langle v \rangle$ is complete, the result follows.

Case (ii): Let $k \geq 2$. Consider any two vertices v_1, v_2 common to two or more m-cliques of G. Then the edge v_1v_2 is common to these m-cliques. Hence, for all mutually distinct pairs of vertices $v_i, v_j \in \{v_m : 1 \leq m \leq k\}$ the edges $v_iv_j \in E(G)$ are common to these m-cliques. Thus, G being a simple graph it follows that $\langle v_1, v_2, v_3, \ldots, v_k \rangle$ is a simple subgraph and complete say, $K_n, n \geq 1$ and therefore a clique of G. □

The next corollary follows immediately from Proposition 2.1.

Corollary 2.2. *If two or more distinct m-cliques of G share* $k \geq 1$ *common vertices* $v_1, v_2, v_3, \ldots, v_k$, *then the m-cliques share* $\frac{1}{2}k(k-1)$ *edges.*

Proof. From Proposition 2.1 it follows that the induced graph $\langle v_1, v_2, v_3, \ldots, v_k \rangle$ is a clique. Since a clique is a complete graph, the number of edges shared is $\frac{1}{2}k(k-1)$. □

Lemma 2.3. *The only m-clique say, Q of order* $n \geq 2$ *for which the count,* (1 m-*clique*) $= \epsilon(Q) = 1$, *is the m-clique* K_2 (*or* P_2).

Proof. Trivial. □

When a cluster of two cycles (not necessarily of the same order) are allowed to merge at least one edge to share at least one common edge, the new graph is called a *C-gridlike cluster*. Two or more *C-gridlike clusters* are allowed to merge similarly to get an expanded *C-gridlike cluster*. When a cluster of two or more cycles are all allowed to merge a vertex to share a common vertex, the new graph is called a *C-cloverlike cluster*. When a cluster of two or more cycles are allowed to all merge an edge for all to share a common edge, the new graph is called a *C-booklike cluster*. When two cycles $C_n \subseteq C$-element, $C_m \subseteq C$-element are joined by a path $ve_0w_1e_1w_2e_2w_3 \ldots e_{m-1}w_me_mu$, $v \in V(C_n)$, $u \in V(C_m)$ we say they are adjacent. We now define the classes of cyclic (C-like) and acyclic (C-treelike) graphs G^*.

Definition 2.1. If a graph G^* is the composition of adjacent C-elements and G^* has a cycle between at least two C-elements, G^* is cyclic (or C-like), else it is acyclic (or C-treelike).

Observation 2.2. C-like and C-treelike graphs can be decomposed into cycles $C_i, C_j, C_k, \ldots, C_m$ and paths $P_g, P_h, P_l, \ldots, P_t$ such that:

$$\epsilon(G^*) = \sum_{\text{all-cycles}} \epsilon(C_\ell) + \sum_{\text{all-paths}} \epsilon(P_{\ell'}), \qquad \ell \in \{i, j, k, \ldots, m\} \text{ and } \ell' \in \{g, h, l, \ldots, t\}.$$

The decomposition may not be unique, and may not be isomorphic to the initial composition C-elements. Formalising this observation remains open.

Proposition 2.4. *The only graphs G of order* $n \geq 2$ *with* $\diamond(G) = \epsilon(G)$ *are bipartite graphs, odd cycles, C-like and C-treelike graphs.*

Proof. (i) For a bipartite graph G the maximal m-cliques are all K_2 (or P_2) hence, $\diamond(G) = \epsilon(G)$. For any other graph H containing at least a triangle C_3, at least 3 edges as discounted for the count of (1 m-clique). A larger m-clique implies the existence of a triangle, hence an odd cycle in H. Therefore, H is not bipartite. The result applies to even cycles and paths as well.

(ii) Clearly the only m-cliques in an odd cycle C_n, $n \geq 3$ are K_2 (or P_2) hence, $\diamond(C_n)) = \epsilon(C_n)$.

(iii) From Observation 2.2 it follows that C-like and C-treelike graphs G^* can be decomposed into cycles $C_i, C_j, C_k, \dots, C_m$ and maximal paths $P_g, P_h, P_l, \dots, P_t$ such that:

$$\epsilon(G^*) = \sum_{\text{all-cycles}} \epsilon(C_\ell) + \sum_{\text{all-paths}} \epsilon(P_{\ell'}),\ \ell \in \{i,j,k,\dots,m\} \text{ and } \ell' \in \{g,h,l,\dots,t\}.$$

Following from (i) and (ii) it follow that for all paths and all cycles in the decomposition,

$$\diamond(G^*) = \sum_{\text{all-cycles}} \diamond(C_\ell) + \sum_{\text{all-paths}} \diamond(P_{\ell'}),\quad \ell \in \{i,j,k,\dots,m\} \text{ and } \ell' \in \{g,h,l,\dots,t\}.$$

Therefore,

$$\diamond(G^*) = \epsilon(G^*).$$ □

If the condition of connectedness is relaxed and $\mathfrak{N}_{0,n}$ denotes the null graph (edgeless graph) on n vertices, then for any graph G on n vertices, we have $\diamond(G) \leq \diamond(\mathfrak{N}_{0,n}) = n$.

Theorem 2.5. *For any connected graph G of order $n \geq 2$, $\diamond(G) \leq \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor$.*

Proof. Clearly, for any connected graph of order $n \geq 2$, the complete bipartite graph $H = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ is the unique connected triangle free graph with maximum number of edges. Hence, the largest m-cliques found in H are K_2 (or P_2). Therefore, $\diamond(G) \leq \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor$. □

Corollary 2.6. *A connected graph G of order $n \geq 2$ and $\epsilon(G)$ edges which has no odd cycle has, $\diamond(G) = \epsilon(G)$.*

Proof. It is known that a graph G is bipartite if and only G contains no odd cycle. Therefore, G has no triangles, K_3 (or C_3) hence, the largest m-cliques are K_2 (or P_2). So, $\diamond(G) = \epsilon(G)$. □

2.1 Line Graph of Some Graphs

Consider the line graph $L(G)$ of the graph G. We have the next result.

Proposition 2.7. *If a graph G of order $n \geq 4$ has ℓ vertices of maximum degree $\Delta(G)$, then $\diamond(L(G)) = \ell$ and these maximum m-cliques are $K_{\Delta(G)}$.*

Proof. Because $L(K_1)$ is empty, $L(K_2) = L(P_2) = K_1$ and $L(C_3) = C_3$ these graphs are excluded hence the bound $n \geq 4$.

Consider a vertex v of G with $d_G(v) = \Delta(G)$ and label the edges incident with v to be, $e_1, e_2, e_3, \dots, e_{\Delta(G)}$ and the corresponding line graph vertices to be, $u_{e_1}, u_{e_2}, u_{e_3}, \dots, u_{e_{\Delta(G)}}$. From the definition of the line graph $L(G)$, the induced subgraph $D = \langle u_{e_1}, u_{e_2}, u_{e_3}, \dots, u_{e_{\Delta(G)}} \rangle$ of $L(G)$ is maximal complete, hence a m-clique, $K_{\Delta(G)}$. Assume there exist a larger m-clique D^* of $L(G)$. Hence $|V(D^*)| \geq |V(D)| + 1$. It implies that a vertex u exists in G with

$\frac{1}{2}|V(D^*)|(|V(D^*)|-1)| \geq \frac{1}{2}|V(D)|(|V(D)|-1)$ edges incident with u in G. Hence $d_G(u) > d_G(v)$, which contradicts that $d_G(v) = \Delta(G)$ so such larger m-clique does not exist. Therefore, $\diamond(L(G)) = \ell$ and these maximum m-cliques are $K_{\Delta(G)}$. □

Proposition 2.8. *For:*

(i) P_n, $n \geq 3$, *we have* $\diamond(L(P_n)) = n-1$.

(ii) C_n, $n \geq 3$, *we have* $\diamond(L(C_n)) = n$.

(iii) K_3, $\diamond(L(K_3)) = 1$, *otherwise* $\diamond(L(K_n)) = n$, $n \geq 4$.

(iv) *A star* $K_{1,n}$, $n \geq 3$, *we have* $\diamond(L(K_{1,n})) = 1$.

(v) *A wheel* $W_{1,n}$, $n \geq 4$ *we have* $\diamond(W_{1,n}) = n$ *and* $\diamond(L(W_{1,n})) = 1$ *with maximum* m*-clique*, K_n.

(vi) *A* m*-regular graph* G, $m \geq 3$, *we have* $\diamond(L(G)) = \nu(G)$.

(vii) *A graph* G *with non-intersecting maximum* m*-cliques*, K_m. *If all vertices* v *in the maximum* m*-cliques of* G *have degree equal to* $\Delta(G)$, *then* $\diamond(L(G)) = m \cdot \diamond(G)$.

Proof. (i) and (ii): Because $L(P_n) = P_{n-1}$ and $L(C_n) = C_n$, the results are obvious.

(iii): Since $L(K_3) = K_3$ the first part follows trivially. For K_n, $n \geq 4$ only the edges incident with vertex v induce a maximal complete graph K_{n-1} in $L(K_n)$. Hence, $L(K_n)$ has n of these maximum m-cliques. So $\diamond(L(K_n)) = n$.

(iv): Since, $L(K_{1,n})$, $n \geq 3$ is maximum complete it follows that $\diamond(L(K_{1,n})) = 1$.

(v): The central vertex of the wheel together with any pair of adjacent vertices on the cycle induce a triangle C_3. Since the cycle C_n has n edges exactly n such distinct of adjacent pairs of vertices exist. Thus the first part of the result follows. The second part follows from (ii).

(vi): Similar to a star $K_{1,m}$, the edges incident with a vertex $v \in V(G)$, induce a maximal complete graph K_m in $L(G)$. Hence, $\nu(G)$ such maximal complete graphs are induced in $L(G)$. Therefore, $\diamond(L(G)) = \nu(G)$.

(vii): Since all edges incident with a vertex in a maximum m-clique are of regular and maximum *edge-degree* the result follows immediately. □

In [1] Abdo and Dimitrov defined the *subdivision graph* of a graph G denoted $S(G)$ to be the graph obtained by subdividing each edge e_i of G with an additional vertex v_{e_i}. Clearly the subdivision graph destroys completeness so, $\diamond(S(G)) = 2\epsilon(G)$.

Proposition 2.9. *Consider a graph* G *with non-intersecting maximum* m*-cliques*, K_m. *If all vertices* v *in the maximum* m*-cliques of* G *have degree equal to* $\Delta(G)$, *then* $\diamond(S(G)) = \diamond(L(G)) = m \cdot \diamond(G)$.

Proof. The part, $\diamond(L(G)) = m \cdot \diamond(G)$ follows from Proposition 2.8(vii). Since all edges incident with a vertex in a maximum m-clique of G are of regular and maximum *edge-degree* and remains such in $S(G)$ the result follows immediately. □

We recall that a *triangle parallel graph* of a graph G denoted by $P(G)$, is defined to be the graph obtained by replacing each edge of G with a cycle C_3. So, for an acyclic graph H it is evident that $\diamond(P(H)) = \epsilon(H)$.

Theorem 2.10. *Let T be a tree of order at least 3. If T have $m \geq 1$ vertices with degree equal to $\Delta(T)$, then $\diamond(L(P(T))) = m$ with largest cliques, $K_{2\Delta(T)}$.*

Proof. From the definition of $P(G)$ all vertex degrees doubled in count. Hence $P(T)$ has m degrees of equal degree, equal to $\Delta(P(T)) \geq 4$. Hence, the first part of the result follows from Proposition 2.8(iv). Label the vertices in T with equal degree, equal to $\Delta(T)$, $v_1, v_2, v_3, \ldots, v_m$, respectively. Clearly in $P(T)$ we have $d_{P(T)}(v_i) = 2 \cdot d_T(v_i)\ \forall\ 1 \leq i \leq m$. So the maximum m-clique induced in $L(P(T))$ by the edges incident with v_i, is given by $K_{2\Delta(T)}$. □

2.2 Jaco Graphs

Despite earlier definitions in respect of the family of Jaco graphs [8], the definitions found in [10] serve as the unifying definitions. For ease of reference some important definitions are repeated here. Note that a Jaco graph is a directed graph.

Definition 2.3 ([10])**.** The family of finite linear Jaco graphs denoted by $\{J_n(f(x)) : f(x) = mx + c; x \in \mathbb{N}$ and $m, c \in \mathbb{N}_0\}$ is the defined by $V(J_n(f(x))) = \{v_i : i \in \mathbb{N}, i \leq n\}$, $A(J_n(f(x))) \subseteq \{(v_i, v_j) | i, j \in \mathbb{N}, i < j \leq n\}$ and $(v_i, v_j) \in A(J_n(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

Definition 2.4 ([10])**.** Vertices with degree $\Delta(J_n(f(x)))$ is called Jaconian vertices and the set of vertices with maximum degree is called the Jaconian set of the linear Jaco graph $J_n(f(x))$, and denoted, $\mathbb{J}(J_n(f(x)))$ or, $\mathbb{J}_n(f(x))$ for brevity.

Definition 2.5 ([10])**.** The lowest numbered (indexed) Jaconian vertex is called the prime Jaconian vertex of a linear Jaco graph.

Definition 2.6 ([10])**.** If v_i is the prime Jaconian vertex, the complete subgraph on vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ is called the Hope subgraph of a linear Jaco graph and denoted, $\mathbb{H}(J_n(f(x)))$ or, $\mathbb{H}_n(f(x))$ for brevity.

For now we will only consider the special case, $f(x) = x$. The next lemma holds.

Lemma 2.11 ([10])**.** *For the function $f(x) = x$ we have for $J_n(x)$ that:*

(i) $d^+(v_n) + d^-(v_n) = n$.

(ii) $d^-(v_{n+1}) \in \{d^-(v_n), d^-(v_n) + 1\}$.

(iii) *If $(v_i, v_k) \in A(J_\infty(x))$ and $i < j < k$, then $(v_j, v_k) \in A(J_\infty(x))$.*

(iv) $d^+(v_n) = a_n$, $n \geq 2$.

Theorem 2.12. *The underlying Jaco graph $J_n^*(x)$, has $\diamond(J_n^*(x)) \leq 3$.*

Proof. Utilising Lemma 2.11 the table below (also known as the Fisher table [8, 10])[1], can easily be constructed iteratively.

$\phi(v_i) \to i \in \mathbb{N}$	$d^-(v_i) = \nu(\mathbb{H}_{i-1})$	$d^+(v_i) = i - d^-(v_i)$	$\mathbb{J}(J_i(x))$	$\Delta(J_i(x))$
$1 = f_2$	0	1	$\{v_1\}$	0
$2 = f_3$	1	1	$\{v_1, v_2\}$	1
$3 = f_4$	1	2	$\{v_2\}$	2
4	1	3	$\{v_2, v_3\}$	2
$5 = f_5$	2	3	$\{v_3\}$	3
6	2	4	$\{v_3, v_4, v_5\}$	3
7	3	4	$\{v_4, v_5\}$	4
$8 = f_6$	3	5	$\{v_5\}$	5
9	3	6	$\{v_5, v_6, v_7\}$	5
10	4	6	$\{v_6, v_7\}$	6
11	4	7	$\{v_7\}$	7
12	4	8	$\{v_7, v_8\}$	7
$13 = f_7$	5	8	$\{v_8\}$	8
14	5	9	$\{v_8, v_9, v_{10}\}$	8
15	6	9	$\{v_9, v_{10}\}$	9
16	6	10	$\{v_{10}\}$	10
17	6	11	$\{v_{10}, v_{11}\}$	10
18	7	11	$\{v_{11}\}$	11
19	7	12	$\{v_{11}, v_{12}, v_{13}\}$	11
20	8	12	$\{v_{12}, v_{13}\}$	12
$21 = f_8$	8	13	$\{v_{13}\}$	13
22	8	14	$\{v_{13}, v_{14}, v_{15}\}$	13
23	9	14	$\{v_{14}, v_{15}\}$	14
24	9	15	$\{v_{15}\}$	15
25	9	16	$\{v_{15}, v_{16}\}$	15
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now consider any $n \in \mathbb{N}$. Then in $J_n^*(x)$ a maximum m-clique, $K_{d^-(v_n)+1}$ is given by

$$\langle v_{n-d^-(v_n)}, v_{(n-d^-(v_n))+1}, v_{(n-d^-(v_n))+2}, \ldots, v_n \rangle.$$

In [9] it has been shown that a value $d^-(v_n)$ can repeat at most 3 times. Hence, $\diamond(J_n^*(x)) \leq 3$ follows. □

Conjecture 2.2.1. *For $J_n^*(f(x))$, $f(x) = mx + c$, $m \geq 1$, $c \geq 1$ we have: $\diamond(J_n^*(f(x))) \leq 2$.*

[1]Named after Dr Paul Fisher, Department of Mathematics, University of Botswana, who described the Fisher algorithm informally to the first author in personal correspondence.

2.3 Double Graph

We recall that the notion of the *double graph* of a graph is defined as follows. Consider a graph G of order $n \geq 2$ and label the vertices $v_1, v_2, v_3, \dots, v_n$. Copy G denoted G' and label the corresponding *mirror* vertices $u_1, u_2, u_3, \dots, u_n$. The double graph of G is the graph $G_{\mathscr{D}}$ defined as $G_{\mathscr{D}}(V,E)$, $V(G_{\mathscr{D}}) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$, $E(G_{\mathscr{D}}) = E(G) \cup E(G') \cup \{v_i u_j :$ if and only if edge $v_i v_j \in E(G)\}$.

Theorem 2.13. *For any graph G of order $n \geq 3$, $\diamond(L(G_{\mathscr{D}})) = 2 \cdot \diamond(L(G))$.*

Proof. We construct the proof as follows. First consider the union $H = G \cup G$. Clearly if graph G has ℓ vertices of degree equal to $\Delta(G)$ then H has 2ℓ such vertices. Consider any vertex $v_i \in V(G)$, $\deg_G(v_1) = \Delta(G)$ and label its neighboring vertices $v_{i,1}, v_{i,2}, \dots, v_{i,\Delta(G)}$. So in the copy graph the mirror vertices u_i and $u_{i,1}, u_{i,2}, \dots, u_{i'\Delta(G)}$ exist. Adding the defined edges to obtain $G_{\mathscr{D}}$ results in 2ℓ vertices with degree equal to $\Delta(G_{\mathscr{D}}) = 2\Delta(G)$. Following from Proposition 2.9 it follows that $\diamond(L(G_{\mathscr{D}})) = 2 \cdot \diamond(L(G))$. □

2.4 Set-graphs

The notion of a set-graph was introduced in [9]. For ease of reference the definition of a set graph is repeated here.

Definition 2.7 ([9]). Let $A^{(n)} = \{a_1, a_2, a_3, \dots, a_n\}$, $n \in \mathbb{N}$ be a non-empty set and the i-th s-element subset of $A^{(n)}$ be denoted by $A^{(n)}_{s,i}$. Now consider $\mathscr{S} = \{A^{(n)}_{s,i} : A^{(n)}_{s,i} \subseteq A^{(n)}, A^{(n)}_{s,i} \neq \emptyset\}$. The set graph corresponding to set $A^{(n)}$, denoted $G_{A^{(n)}}$, is defined to be the graph with $V(G_{A^{(n)}}) = \{v_{s,i} : A^{(n)}_{s,i} \in \mathscr{S}\}$ and $E(G_{A^{(n)}}) = \{v_{s,i} v_{t,j} : A^{(n)}_{s,i} \cap A^{(n)}_{t,j} \neq \emptyset\}$, where $s \neq t$ or $i \neq j$.

Note that in [9] the result that the set-graph $G_{A^{(n)}}$, $n \geq 2$ has $2n-2$ largest complete subgraphs, $K_{2^{n-1}}$ was proven. It implies that $\diamond(G_{A^{(n)}}) = 2n-2$, $n \geq 2$.

Corollary 2.14. *For a set-graph $G_{A^{(n)}}$, $n \geq 3$ we recursively have: $\diamond(G_{A^{(n)}}) = \diamond(G_{A^{(n-1)}}) + 2$ and if subgraphs D, D' are maximum m-cliques of $G_{A^{(n)}}$ and $G_{A^{(n-1)}}$ respectively, then $|V(D)| = 2|V(D')|$.*

Proof. The corollary is a direct consequence of the proof of Proposition 2.11 in [9]. □

We note that $\diamond(G_{A^{(n)}})$, $n \geq 3$ is always even. So if $\diamond(G)$ is odd, then G is non-isomorphic to a set-graph.

2.5 k-th Power Graph of Caterpillars

We recall that a *caterpillar* is a graph obtained from a path P_n, $n \geq 1$ with vertex set $V(P_n) = \{u_i : 0 \leq i \leq n-1\}$, with arbitrary non-negative integers of leafs say, $l_0, l_1, l_2, \dots, l_{n-1}$ attached to the corresponding indexed vertices. In popular literature the leafs are called *thorns* and we call the clusters of thorns l_i, $0 \leq i \leq n-1$, *trods*. The path P_n, $n \geq 1$ is called the *spine* or *stalk* of the caterpillar. Clearly P_n is a unique dominating path of the caterpillar. Let $\mathfrak{L} = \{l_i : 0 \leq l \leq n-1\}$. We denote the corresponding caterpillar $C^{\mathfrak{L}}_{P_n}$. From Corollary 2.6 it follows immediately that, $\diamond(C^{\mathfrak{L}}_{P_n}) = (n-1) + \sum_{i=1}^{n} \ell_i$. A caterpillar $C^{\mathfrak{L}}_{P_n}$ for which $\ell_i = \ell_j$, $0 \le i, j \le n-1$, is called a *balanced caterpillar*.

Theorem 2.15. *For the k-th power graph, $2 \le k \le n+1$ of a balanced caterpillar $C^{\mathfrak{L}}_{P_n}$ we have:*

(i) $\diamond(C^{\mathfrak{L}^k}_{P_n}) = n - i$, $i = 2j, 1 \le j \le \lfloor \frac{n}{2} \rfloor$, *and* $2 \le k \le \lfloor \frac{n}{2} \rfloor + 1$,

(ii) $\diamond(C^{\mathfrak{L}^k}_{P_n}) = 1$, $\lceil \frac{n}{2} \rceil + 1 \le k \le n$.

Proof. (i): Label the leafs ℓ_i to be $v_{i,1}, v_{i,2}, \dots, v_{i,\ell}$. Assume the vertices of the spine P_n is labeled from left-to-right. Clearly, in $C^{\mathfrak{L}^2}_{P_n}$ a leaf $v_{1,m}$, $1 \le m \le \ell$ only has a furthest connectivity reach to the right, to connect with u_1. By symmetry a leaf $v_{(n-1),m}$, $1 \le m \le \ell$ only has a furthest connectivity reach to the left, to connect with u_{n-2}. All the other leafs have both left-to-right and right-to-left connectivity reach. So the largest cliques in $C^{\mathfrak{L}^2}_{P_n}$ are those containing a leafs, $v_{i,j}$, $1 \le i \le n-2$ and $j \in \{1, 2, 3, \dots, \ell\}$. There are exactly $n-2$ such maximum m-cliques. Hence, $n-2$, $2 = 2 \cdot 1$, $1 \le 1 \le \lfloor \frac{n}{2} \rfloor + 1$ such cliques. The result follows through immediate induction.

(ii): At the critical power $k = \lceil \frac{n}{2} \rceil + 1$ only one maximum m-clique exists. Thereafter, as k increases only the order of the clique increases. Since a power graph is only defined for $k \le diam(G)$, Case (ii) follows immediately. □

3. m-Clique Sequence of Graph

We introduce the notion of the *m-clique sequence* or $\diamond$-sequence of a graph G with $\epsilon(G) \ge 1$. The $\diamond$-sequence of G is denoted, $s^{\diamond}(G)$. Essentially the next definition formalises the notion that the m-clique sequence of a graph G is the sequence with entries representing the number of m-cliques of same order found in G, in descending order. Furthermore, each entry has the corresponding m-clique order as subscript.

Definition 3.1. The $\diamond$-sequence of a graph G with $\epsilon(G) \ge 1$ is the sequence, $s^{\diamond}(G) = (a_{i,c_i} : 1 \le i \le t_G)$, c_i the m-clique order, a_i is the number of distinct i-th m-cliques (in terms of order) imbedded in G and t_G is the number of different m-cliques found in G.

The value t_G can also be written as $t_G = |s^{\diamond}(G)|$. Noting that an edge and an isolated vertex are inherently a maximum complete subgraph, it follows that for K_1 (or P_1), $s^{\diamond}(K_1) = (1_1)$. and for K_2 (or P_2), $s^{\diamond}(K_2) = (1_2)$. The *sequence simplication law* allows that entries (or elements) of a $\diamond$-sequence with equal order subscripts, be added. The application of the law is denoted by, $simp(a_{i,c_i} : 1 \le i \le t_G)$. Similar to the union operation found in set theory, a *u-plus* operation is valid for $\diamond$-sequences. We define $s^{\diamond}(G) \uplus s^{\diamond}(H) = simp(s^{\diamond}(G) \cup s^{\diamond}(H))$.

Perhaps an improved illustrative application of Definition 3.1 and the *u-plus* operation is found in the next result.

Theorem 3.1. *For the join of two graphs G and H we have that:*

(i) $\diamond(G + H) = \diamond(G) \cdot \diamond(H)$.

(ii) $|s^{\diamond}(G+H)| \le |s^{\diamond}(G)| \cdot |s^{\diamond}(H)|$.

Proof. (i): Let $s^{\diamond}(G) = (a_{i,c_i} : 1 \le i \le t_G)$ and $s^{\diamond}(H) = (b_{i,c_i} : 1 \le i \le t_H)$. Consider maximum m-cliques K_p, K_q of graphs G and H, respectively. Clearly the m-clique $K_p + K_q = K_{(p+q)}$ is a maximum m-clique of $G+H$. So, the number of maximum m-cliques of $G+H$ is given $a_{1,p} \cdot b_{1,q} = (a \cdot b)_{1,(p+q)} = \diamond(G) \cdot \diamond(H)$.

(ii): Determine the sequences $(a_{i,c_i} \cdot b_{j,c_j} : 1 \le j \le t_H)$ for $1 \le i \le t_G$. Now construct the sequence $\biguplus_{i=1}^{t_G} (a_{i,c_i} \cdot b_{j,c_j} : 1 \le j \le t_H)$. Since it is possible for some $c_i \ne c_k$ and $c_j \ne c_m$, then $c_i + c_j = c_k + c_m$, it follows that $\left| simp\left(\biguplus_{i=1}^{t_G} (a_{i.c_i} \cdot b_{j,c_j} : 1 \le j \le t_H) \right) \right| \le \left| \biguplus_{i=1}^{t_G} (a_{i.c_i} \cdot b_{j,c_j} : 1 \le j \le t_H) \right|$. Hence, $|s^{\diamond}(G+H)| \le |s^{\diamond}(G)| \cdot |s^{\diamond}(H)|$. □

From the proof of Theorem 3.1(i) it follows that the maximum m-clique of $G+H$ is $K_{(p+q)}$ and similarly, the minimum (smallest) m-clique of $G+H$ is $K_{(c_{t_G} + c_{t_H})}$.

Lemma 3.2. *From the Fisher table (Theorem 2.12), $s^{\diamond}(J_n^*(x))$ can be determined iteratively.*

Example 3.1. For $J_{10}(x)$ we have $s^{\diamond}(J_{10}^*(x)) = (1_5, 2_4, 1_3, 2_1)$ and for $J_{12}^*(x)$ we have $s^{\diamond}(J_{12}^*(x)) = (3_5, 2_4, 1_3, 2_1)$.

3.1 Characterization of m-Clique Sequences

Graphic sequences have been well studied. First work was done to characterize a degree sequence. That is, to determine whether or not a sequence of positive integers can be the degree sequence of some graph. A similar question now arises in respect of m-clique sequences. A finite sequence of positive integers each indexed with a distinct positive integer subscript is *clique graphical or c-graphical* if it is the m-clique sequence of some graph. Since, $c_i \in \mathbb{N}$, $c_t = 1$ is possible, a disconnected graph embodiment is possible as well. A graph embodiment G corresponding to some mathematical object is said to be *maximal connected* if the number of components of G is a minimum.

Theorem 3.3. *Any finite sequence of positive integers each indexed with a distinct positive integer subscript is maximal connected c-graphical.*

Proof. Consider any finite sequence $(a_{i,c_i} : 1 \le i \le t)$, t the number of entries (elements) in the sequence and $a_i, c_i \in \mathbb{N}$, $c_i \ne c_j$.

Case (i) (a): If $t = 1$ and $a_1 = k$, $c_1 = 1$, then the null graph (edgeless) $\mathfrak{N}_{0,k}$ is the corresponding maximal connected graph.

(b): If $t = 1$ and $c_1 \ge 2$, construct $G_1^* = Q_1$-cloverlike from a_1 copies of K_{c_1}. Clearly $s^{\diamond}(G_1^*) = (a_{1,c_1})$.

Case (ii) (a): If $t \ge 2$ and $c_t \ge 2$ choose any two entries a_{i,c_i}, a_{j,c_j}, $c_i > c_j$ and consider a_i copies of $Q_1 = K_{c_i}$ and a_j copies of $Q_2 = K_{c_j}$, respectively. Construct a $G_1^* = Q_1$-cloverlike graph and a $G_2^* = Q_2$-cloverlike graph. Now construct a graph H_2^* by merging any vertex $v \in V(G_1^*)$ with any vertex $u \in V(G_2^*)$. Clearly, $s^\diamond(H_2^*) = (a_{i,c_i}, a_{j,c_j})$. It follows that a $G_3^* = Q_3$-cloverlike graph can be constructed from a_k copies of K_{c_k}. Now the graph H_3^* can be constructed by merging any vertex $v \in V(H_2^*)$ and $u \in V(G_3^*)$. For H_3^* the $\diamond$-sequence is $s^\diamond(H_3^*) = (a_{i,c_i}, a_{j,c_j}, a_{k,c_k})$, or $(a_{i,c_i}, a_{k,c_k}, a_{j,c_j})$, or $(a_{k,c_k}, a_{i,c_i}, a_{j,c_j})$. Because the sequence is finite it is always possible to iteratively choose an entry once, one at a time to construct the graph H_t^* for which $s^\diamond(H_t^*)$ corresponds to the appropriate re-ordering of the entries of the initial sequence. This is possible in t steps.

(b): If $c_t = 1$ construct a minimal connected graph G' as in Case (ii)(a) for $(a_{i,c_i} : 1 \le i \le t-1)$. Then $G = G' \cup \mathfrak{N}_{0,a_t}$ is a corresponding maximal connected graph. □

Clearly a graph for which the m-clique sequence corresponds to a given sequence of positive integers each indexed with a positive integer subscript is not unique. We now determine such a graph with minimum edges. We introduce a concept for intersecting set-objects similar to that found in set theory. Let set A_1, A_2 each be a copy of the same set-object A. Construct a set-object $B = \{\text{object-elements of } A\}$, $|B| \le |A|$. Set-object B is also denoted $B = A_1 \cap^+ A_2$. If $B = A_1 \cap^+ A_2 = A$ we say it is a complete intersection. If $|B| = |A_1 \cap^+ A_2| < |A|$ we say it is an incomplete intersection. Applying these concepts to graphs we find for two copies of a cycle $A_1 = C_n$, $A_2 = C_n$ that a complete intersection is given by $B = A_1 \cap^+ A_2 = C_n$. On the other hand, a maximal incomplete intersection is given by $B = A_1 \cap^+ A_2 = \{v_1, v_2, \dots, v_{n-1}, v_1v_2, v_2v_3, \dots, v_{n-2}v_{n-1}\}$. Hence, two copies of the vertex v_n and two copies of the edges $v_{n-1}v_n$, v_1v_n are excluded. Similarly, take a copy of the complete graph K_{n-1}. Add $m \in \mathbb{N}$ vertices $v'_1, v'_2, v'_3, \dots, v'_m$ and connect each vertex completely to K_{n-1}, only. The new graph denoted, $K_n^{\cap^+}$ has $\diamond(K_n^{\cap^+}) = m$, $s^\diamond(K_n^{\cap^+}) = (m_n)$ and the maximal incomplete intersection corresponding to $\bigcap^+_{\forall K_n} K_n = K_{n-1}$. This observation of a maximal incomplete intersection of graphs provides the basis for the *Minimal Graphical Embodiment Algorithm for a Sequence*.

3.2 Minimal Graphical Embodiment Algorithm for a Sequence (MGEAS)

Let the finite sequence be $(a_{i,c_i} : 1 \le i \le t)$, t the number of entries (elements) in the sequence and $a_i, c_i \in \mathbb{N}$, $c_i \ne c_j$.

Step 0: Re-order the entries to obtain the sequence $(b_{i,c_i} : 1 \le i \le t)$, t the number of entries (elements) in the sequence and $b_i, c_i \in \mathbb{N}$, $c_i > c_{i+1}$. Set $G_0^* = K_1$. Go to step 1.

Step 1: Let $i = 1$, set $j = i$. Go to step 2.

Step 2: If $j > t$, go to step 6, else go to step 3.

Step 3: Consider a_j copies of $Q_j = K_{c_j}$ and construct $G_j^* = Q_j^{\cap^+}$. Go to step 4.

Step 4: Join G_{j-1}^*, G_j^* by merging any two vertices $u \in V(G_{j-1}^*)$ and $v \in V(G_j^*)$. Go to step 5.

Step 5: Set $i = j + 1$, then set $j = i$. Go to step 2.

Step 6: Exit.

Theorem 3.4 (Ramokgopa's theorem). [2] *For a finite sequence* $(a_{i,c_i} : 1 \le i \le t)$, *t the number of entries (elements) in the sequence and* $a_i, c_i \in \mathbb{N}$, $c_i > c_{i+1}$ *the graph* G_t^* *obtained from the MGEAS, has:*

(i) $\diamond(G_t^*) = a_1$.

(ii) $s^{\diamond}(G_t^*) = (a_{i,c_i} : 1 \le i \le t)$.

(iii) $\epsilon(G_t^*) = \min\{\epsilon(G) : s^{\diamond}(G) = (a_{i,c_i} : 1 \le i \le t)\}$.

(iv) $\epsilon(G_t^*) = \sum\limits_{i=1}^{t} \big(\frac{1}{2}(c_i - 1)(c_i - 2) + a_i(c_i - 1)\big)$.

Proof. (i): Since $c_1 > c_i$, $2 \le i \le t$, a maximum clique of G_t^* is K_{c_1}. Following from MGEAS, exactly a_1 maximum cliques exist hence, $\diamond(G_t^*) = a_1$.

(ii): Following from Step 3 each $G_j^* = Q_j^{\cap^+}$ corresponds to a_j maximum cliques, K_{c_j} of G_j^* which is equivalent to a_j maximal cliques of G_t^*. The latter is true because the merging of any two vertices (step 4) does not increase or decrease the order of maximal cliques. Through immediate induction the result follows.

(iii) and (iv): Consider any entry a_{i,c_i}. The complete graph K_{c_i-1} has a minimum of $\frac{1}{2}(c_j-1)(c_j-2)$ edges. Add the vertices $v'_1, v'_2, v'_3, \dots, v'_{a_i}$ and construct $K_{a_i}^{\cap^+}$. Hence, the complete connection of any vertex v'_k, $1 \le k \le a_i$ added exactly, hence minimum, $c_i - 1$ edges. Therefore $\epsilon(K_{c_i}^{\cap^+}) = \frac{1}{2}(c_i - 1)(c_i - 2) + a_i(c_i - 1)$ edges. Because the merging of any two vertices (step 4) does not increase or decrease the order of maximal cliques the total number of edges of G_t^* is a minimum. Therefore, $\epsilon(G_t^*) = \min\{\epsilon(G) : s^{\diamond}(G) = (a_{i,c_i} : 1 \le i \le t)\}$. Finally, $\epsilon(G_t^*) = \sum\limits_{i=1}^{t} \big(\frac{1}{2}(c_i - 1)(c_i - 2) + a_i(c_i - 1)\big)$. □

Consider any path $P_n = v_1e_1v_2e_2v_3e_3 \dots e_{n-1}v_n$, $n \ge 2$. Substitute each edge e_i, $1 \le i \le n-1$ with a complete graph K_{n_i}, $v_i, v_{i+1} \in V(K_{n_i})$. We say that the complete graphs K_{n_i}, $1 \le i \le n-1$ have been joined *path-like* or, p-like for brevity.

Proposition 3.5. *For a finite sequence* $(a_{i,c_i} : 1 \le i \le t)$, *t the number of entries (elements) in the sequence and* $a_i, c_i \in \mathbb{N}$, $c_i > c_{i+1}$ *the graph* $G_t^{(p)}$ *obtained by joining all maximal cliques* K_{c_i}, *(a_i copies) for* $1 \le i \le t$, *p-like then* $\epsilon(G_t^{(p)}) = \max\{\epsilon(G) : s^{\diamond}(G) = (a_{i,c_i} : 1 \le i \le t)\} = \sum\limits_{i=1}^{t} \frac{1}{2} a_i \cdot c_i(c_i - 1)$.

Proof. Similar reasoning to that found in the proof of Ramokgopa's theorem (Theorem 3.4) yields the result. □

Other graphs are in existence as well and if a minimal graphical embodiment is not required, the following results holds.

Proposition 3.6. *For a finite sequence* $(a_{i,c_i} : 1 \le i \le t)$, *t the number of entries (elements) in the sequence and* $a_i, c_i \in \mathbb{N}$, $c_i > c_{i+1}$, *a corresponding maximal connected graph embodiment has*

[2] The first author wishes to dedicate this theorem to Cllr Kgosientso Ramokgopa, the Executive Mayor of the City of Tshwane to thank him for his brilliant innovation leadership during his term in Office.

order:

$$\nu(G) \le \begin{cases} \sum\limits_{i=1}^{t} a_i - 1, & \text{if } c_t \ge 2,\\ \sum\limits_{i=1}^{t-1} a_i + (a_t - 1), & \text{if } c_t = 1. \end{cases}$$

Proof. (i): If $c_t \ge 2$ consider a_i, $1 \le i \le t$ copies of each m-clique K_{a_i}, and connect them p-like by merging a distinct pair of vertices between the complete graphs. Since a *vertex count* of 1 is lost with each merge, the result, $\max\{\nu(G)\} = \sum\limits_{i=1}^{t} a_i - 1$, follows through immediate induction.

(ii): If $c_t = 1$, repeat the construction in (i) in respect of $(a_{i,c_i} : 1 \le i \le t-1)$ to obtain the corresponding connected graph G' of maximum order. Hence, $G = G' \cup \mathfrak{N}_{0,a_t}$ is the maximal connected graphical embodiment with $\max\{\nu(G)\} = \sum\limits_{i=1}^{t-1} a_i + (a_t - 1)$. □

4. Maximal Clique Density of Certain Graphs

If the order of a maximum clique found in G is $c_1 = s$, then the maximum m-clique density of graph G is given by $p_s(G) = \frac{a_1}{\binom{\nu(G)}{s}} = \frac{\diamond(G)}{\binom{\nu(G)}{s}}$. It is important to note that for a complete graph K_n we have:

$$p_s(K_n) = \begin{cases} 0, & 1 \le s \le n-1 \text{ or } s > n,\\ 1, & s = n. \end{cases}$$

To the contrary if one wishes to choose s vertices from $V(K_n)$ so that the vertices induce a complete graph the probability denoted $p'_s(K_n)$ is given by:

$$p'_s(K_n) = \begin{cases} 0, & s > n,\\ 1, & 1 \le s \le n. \end{cases}$$

Through immediate induction it follows that in general, $p_s(G) \le p'_s(G)$. If $c_1 \le \ell \le c_{t_G}$ is chosen uniformly at random, then $p_\ell(G) = \frac{1}{c_{t_G}} \cdot \frac{a_i}{\binom{n}{\ell}}$, $\ell = c_i$. Assume that the integrity of technology units is measured at the level of the integrity of maximal cliques in the technology configuration. Assume the integrity failure is signaled by any order of a maximal clique containing the failed technology unit. Then the results can be applied to optimal fault detection.

Example 4.1. Consider a wheel $W_{1,n}$ with central vertex u and cyclic vertices v_i, $1 \le i \le n$. So $\diamond(W_{1,n}) = n$ and $s^\diamond(W_{1,n}) = (n_3)$. If any technology unit fails the fault signal indicates a maximal clique of order 3 has failed. Choosing uniformly at random any 3 vertices say $v_i, v_j v_k$ and finding that $\langle v_i, v_j, v_k\rangle = K_3$ has probability $p_3(W_{1,n}) = \frac{n}{\binom{n+1}{3}} = \frac{6}{n^2-1}$.

Case (i): If the failure is at u the probability of detecting the fault on the first fault search by uniformly at randon, choosing any 3 vertices is exactly $\frac{6}{n^2-1}$ because u is common to all m-cliques.

Case (ii): However, if the failure is at vertex v_i the probability of detecting the fault on the first fault search by uniformly at random, choosing any 3 vertices is $\frac{2}{n} \cdot \frac{6}{n^2-1} = \frac{12}{n(n^2-1)}$, because v_i is common to two K_3 amongst the $\diamond(W_{1,n}) = n$, m-cliques (all K_3).

Remark 4.1. Search and detection by m-clique density is not optimal. By using an efficient clique detection algorithm, Case (i) can be resolved at a probability of 1. Hence, find any K_3 and the fault will be detected at u. Case (ii) can be resolved at a probability of $\frac{2}{n}$.

5. Conclusion and Scope for Research

The paper reported on a variety of introductory results of a new graph invariant called the *m-clique load* of a graph G. The concept of the *m-clique sequence* or $\diamond$-sequence of a graph G with $\epsilon(G) \geq 1$ was also introduced. Determining the $\diamond$-sequence of different classes of graphs and of different graph products will certainly be interesting to unravel. Because this is a new sequence defined for graphs the determination of the curling number of the corresponding $\diamond$-sequences will be worthy research. To research the latter sensibly, the order subscript(s) will will have to be lobbed off. That is, an entry $a_{i_{c_i}}$ must be considered to be a_i only. Finding Nordhaus-Gaddum-type inequalities seems intuitively possible and remains open at this stage. Undoubtedly graph coloring and labeling will lead to results in terms of the clique load of a graph G. It means that the notion of clique load has at least an auxiliary application to coloring and labeling of graphs.

Conjecture 2.2.1 remains open. Also the brief introduction to maximal m-clique density in Section 4 calls for further research.

Open problem. Up to isimorphism, find the number of distinct $G_t^{(p)}$ graphs corresponding to a finite sequence $(a_{i,c_i} : 1 \leq i \leq t)$, t the number of entries (elements) in the sequence and $a_i, c_i \in \mathbb{N}$, $c_i > c_{i+1}$.

Formalising Observation 2.2 remains open as well.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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