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The Pell and Pell-Lucas Numbers via Square Roots of Matrices

Research Article

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Abstract. In this paper, the Pell and Pell-Lucas numbers with specialized rational subscripts are derived from general expressions by square roots of the matrices M^n and N^n. Besides, we reveal that the identities involving these numbers are produced by square roots matrices $M^{n/2}$ and $N^{n/2}$. Further we show that the matrices M^n and N^n are generalized to rational powers by using the Abel's functional equation.

Keywords. The Pell and Pell-Lucas numbers; Square roots of matrices

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1. Introduction

Two members in the vast array of integer sequences are the Pell $\{P_n\}_{n=0}^{\infty}$ and Pell-Lucas $\{Q_n\}_{n=0}^{\infty}$ sequences, which are defined by the recurrence relations

$$P_{n+2} = 2P_{n+1} + P_n, \ \ Q_{n+2} = 2Q_{n+1} + Q_n; \quad n \ge 0,$$

where $P_0 = 0$, $P_1 = 1$, $Q_0 = 2$ and $Q_1 = 2$ [1,5]. The Pell and Pell-Lucas numbers are generated by the Binet's formula and matrices as well as the recurrence relations. For the characteristic equation $\phi^2 - 2\phi - 1 = 0$, since the roots of this equation are $\phi_{1,2} = 1 \pm \sqrt{2}$, it is known that P_n and Q_n numbers can be expressed with the Binet's formula in the form [1]:

$$P_n = \frac{\phi_1^n - \phi_2^n}{2\sqrt{2}}, \quad Q_n = \phi_1^n + \phi_2^{-n},$$

where $\phi_1 = 1+\sqrt{2}$ is the silver ratio. Also, P_n and Q_n numbers can be derived by taking successive integer powers of 2×2 matrices and multiplying of their different powers [1,2]:

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}, \quad N^n = \begin{bmatrix} \frac{1}{2}Q_n & 2P_n \\ P_n & \frac{1}{2}Q_n \end{bmatrix}, \quad H^n = \begin{bmatrix} \frac{-1}{2}Q_{n-1} & 2P_n \\ -P_n & \frac{1}{2}Q_{n+1} \end{bmatrix}.$$

As known, the integer sequences are not widely identified for any rational or real numbers that are not integers. However, in literature authors have defined a lot of methods that may be used to characterize the Pell and Pell-Lucas numbers with rational or real subscripts [6–8]. The characterisation of the Pell and Pell-Lucas numbers is given in [6,7] studies of Horadam who has observed geometrical connection between Pell-type numbers and circles. These studies show how the Pell and Pell-Lucas numbers are associated with sets of coaxal circles. Thus, author revealed the point with Euclidean plane (x, y), where x and y are given by

$$x = \begin{cases} \left(\phi^{2n} - \cos(n-1)\pi\right)/2\sqrt{2}\phi^n, & y=0, \text{ for } \{P_n\}, \\ \left(\phi^{2n} + \cos(n-1)\pi\right)/\phi^n, & y=0, \text{ for } \{Q_n\},\ \phi = 1+\sqrt{2}. \end{cases}$$

Each of these points makes one function which gives classical the Pell P_n and Pell-Lucas Q_n numbers when n is any integer, and also yields the real Pell P_x and Pell-Lucas Q_x numbers for any real number $n = x$ in the following:

$$P_x = \frac{\phi^{2x} - \cos(\pi(x-1))}{2\sqrt{2}\phi^x}, \quad Q_x = \frac{\phi^{2x} + \cos(\pi(x-1))}{\phi^x}, \quad (\phi = 1+\sqrt{2}).$$

Although the identities $P_{x-1} + P_{x+1} = Q_x$ and $2P_x + Q_x = 2P_{x+1}$ are valid for the real Pell P_x and Pell-Lucas Q_x numbers, the identity $P_{2x} = P_x Q_x$ is destroyed. Thus, Horadam [8] defined the following Pell and Pell-Lucas curves with complex notation:

$$P_x = \frac{\phi^x - e^{x\pi i}\phi^{-x}}{2\sqrt{2}}, \quad Q_x = \phi^x + e^{x\pi i}\phi^{-x}. \tag{1}$$

The P_x and Q_x can be called as the Binet's formula for the Pell and Pell-Lucas numbers with real subscripts, respectively. The P_x and Q_x defined in (1) hold for analogous identities of the classical Pell and Pell-Lucas numbers [5,8]:

$$P_{x-1} + P_{x+1} = Q_x, \quad 2P_x + Q_x = 2P_{x+1}, \quad P_{2x} = P_x Q_x.$$

Finally, for all $x \geq 0$ real quantity, in [3] authors described the exponential representations for P_x and Q_x by given

$$P_x = \left[\phi^x - (-1)^{\lambda(x)}\phi^{-x}\right]/\sqrt{8},$$
$$Q_x = \phi^x + (-1)^{\lambda(x)}\phi^{-x}, \qquad \phi = 1+\sqrt{2},$$

and in the same study, authors defined the polynomial exponential representations for P_x, $(x \geq 0)$ and Q_x, $(x > 0)$ by given

$$P_x = \sum_{j=0}^{\lambda[(x-1)/2]} \binom{x-1-j}{j} 2^{x-1-2j}, \quad Q_x = \sum_{j=0}^{\lambda[x/2]} \frac{x}{x+j}\binom{x-j}{j} 2^{x-2j}.$$

These representations coincide with P_n and Q_n numbers when n is a positive integer. Besides, certain properties of these numbers such as

$$P_x Q_x = \begin{cases} P_{2x}, & \text{if } \phi(x) < 1/2 \\ Q_{2x}/\sqrt{8}, & \text{if } \phi(x) \geq 1/2 \end{cases}, \quad \begin{matrix} P_{-x} = (-1)^{\lambda(x)} Q_x/\sqrt{8} \\ Q_{-x} = (-1)^{\lambda(x)+1}\sqrt{8}P_x \end{matrix}$$

are established, where $\lambda(x)$ is the greatest integer, not exceeding x and $\phi(x) = x - \lambda(x) > 0$ is the fractional part of x.

The purpose of this paper is to reveal the Pell and Pell-Lucas numbers given with the Binet's formula in (1), by using the square root of the matrices M^n and N^n [1,2,9]. However, the main goal of this study is not to calculate the square roots of a 2×2 matrix. On the other hand, in literature, certain methods have been studied [4,10,11] for computing the square roots of arbitrary 2×2 matrices.

2. Main Results

The main results of the paper are the following.

Theorem 1. *Let $M_i^{n/2}$ $(i = 1,2,3,4)$ denotes a square roots of the Pell matrix M^n. Then,*

$$M_{1,2}^{n/2} = \pm\begin{bmatrix} P_{(n+2)/2} & P_{n/2} \\ P_{n/2} & P_{(n-2)/2} \end{bmatrix}, \quad M_{3,4}^{n/2} = \frac{\pm 1}{2\sqrt{2}}\begin{bmatrix} Q_{(n+2)/2} & Q_{n/2} \\ Q_{n/2} & Q_{(n-2)/2} \end{bmatrix}.$$

Proof. We apply the Cayley-Hamilton method for computing square roots of the Pell matrix M^n [11]. The matrix M^n has got the square roots matrices in forms:

$$\sqrt{M^n} = \frac{\pm 1}{\sqrt{T \pm 2\sqrt{\det(M^n)}}}\left[M^n \pm \sqrt{\det(M^n)}I\right], \tag{2}$$

where I is the identity matrix, and $T = Q_n$ is the trace of the matrix M^n. As $\det(M^n) = e^{n\pi i}$, we have $\sqrt{\det(M^n)} = e^{n\pi i/2}$. Thus, it is obtained that $\sqrt{T \pm 2\sqrt{\det(M^n)}} = \sqrt{\phi^n} \pm \sqrt{e^{n\pi i}\phi^{-n}}$. It is seen that the matrix M^n has four non-integral square roots. Firstly, we choose that

$$\sqrt{M^n} = \frac{\pm 1}{\phi^{n/2} + e^{n\pi i/2}\phi^{-n/2}}\begin{bmatrix} P_{n+1} + e^{n\pi i/2} & P_n \\ P_n & P_{n-1} + e^{n\pi i/2} \end{bmatrix}.$$

Using the Binet's formula and algebraic manipulation, we can write the values of elements $(1,1)$ and $(1,2)$ for the right side matrix as

$$\frac{\pm\left(\phi^{n/2} - e^{n\pi i/2}\phi^{-n/2}\right)\left(P_{n+1} + e^{n\pi i/2}\right)}{\phi^n - e^{n\pi i}\phi^{-n}} = \pm P_{(n+2)/2}$$

and

$$\frac{\pm\left(\phi^{n/2} - e^{n\pi i/2}\phi^{-n/2}\right)P_n}{\phi^n - e^{n\pi i}\phi^{-n}} = \frac{\pm\left(\phi^{n/2} - e^{n\pi i/2}\phi^{-n/2}\right)P_n}{2\sqrt{2}P_n} = \pm P_{n/2}.$$

The elements $(2,1)$ and $(2,2)$ can be found in the similar way. In this case, we find the two square roots matrices,

$$M_1^{n/2} = \begin{bmatrix} P_{(n+2)/2} & P_{n/2} \\ P_{n/2} & P_{(n-2)/2} \end{bmatrix}, \quad M_2^{n/2} = -\begin{bmatrix} P_{(n+2)/2} & P_{n/2} \\ P_{n/2} & P_{(n-2)/2} \end{bmatrix}. \tag{3}$$

When we pick the other circumstance of the matrix equation (2), we have

$$\sqrt{M^n} = \frac{\pm\left(\phi^{n/2} + e^{n\pi i/2}\phi^{-n/2}\right)}{\phi^n - e^{n\pi i}\phi^{-n}}\begin{bmatrix} P_{n+1} - e^{n\pi i/2} & P_n \\ P_n & P_{n-1} - e^{n\pi i/2} \end{bmatrix}.$$

The value of the element $(1,1)$ is written by algebraic manipulation for the right side matrix as

$$\frac{\pm\left(\phi^{n/2} + e^{n\pi i/2}\phi^{-n/2}\right)\left(P_{n+1} - e^{n\pi i/2}\right)}{\phi^n - e^{n\pi i}\phi^{-n}} = \frac{\pm Q_{(n+2)/2}}{2\sqrt{2}}.$$

The other elements are given in the same way. Thus, the other two square roots matrices are acquired as follow:

$$M_3^{n/2}=\frac{1}{2\sqrt{2}}\begin{bmatrix}Q_{(n+2)/2} & Q_{n/2}\\ Q_{n/2} & Q_{(n-2)/2}\end{bmatrix},\quad M_4^{n/2}=\frac{-1}{2\sqrt{2}}\begin{bmatrix}Q_{(n+2)/2} & Q_{n/2}\\ Q_{n/2} & Q_{(n-2)/2}\end{bmatrix}. \tag{4}$$

□

Moreover, it is known that the square root of the 2×2 matrix M^n is another 2×2 matrix $M_i^{n/2}$ $(i=1,2,3,4)$ such that $M^n=M_i^{n/2}M_i^{n/2}$. If we take the matrix $M_i^{n/2}$ $(i=1,2)$, these identities

$$P^2_{(n+2)/2}+P^2_{n/2}=P_{n+1},\quad P_{(n+2)/2}P_{n/2}+P_{n/2}P_{(n-2)/2}=P_n$$

are obtained by equating of corresponding elements for equal matrices. In addition, as $M_i^{n/2}$ $(i=3,4)$, considering the elements for the square roots matrices leads to

$$Q^2_{(n+2)/2}+Q^2_{n/2}=8P_{n+1},\quad Q_{(n+2)/2}Q_{n/2}+Q_{n/2}Q_{(n-2)/2}=8P_n.$$

And also, taking the determinant of the matrices $M^{n/2}$ yields

$$P_{(n+2)/2}P_{(n-2)/2}-P^2_{n/2}=e^{n\pi i/2},\quad Q_{(n+2)/2}Q_{(n-2)/2}-Q^2_{n/2}=-8e^{n\pi i/2}$$

which are a general case of the Cassini-like formula for the Pell and Pell-Lucas numbers with specialized rational subscripts, respectively.

Hence the matrices $M_i^{n/2}$ $(i=1,2,3,4)$ are nonsingular, the notation $M_i^{-n/2}$ denotes the inverse of matrices $M_i^{n/2}$ in (3) and (4), which are given as:

$$M_{1,2}^{-n/2}=\frac{\pm 1}{e^{n\pi i/2}}\begin{bmatrix}P_{(n-2)/2} & -P_{n/2}\\ -P_{n/2} & P_{(n+2)/2}\end{bmatrix},\quad M_{3,4}^{-n/2}=\frac{\pm 1}{2\sqrt{2}e^{n\pi i/2}}\begin{bmatrix}-Q_{(n-2)/2} & Q_{n/2}\\ Q_{n/2} & -Q_{(n+2)/2}\end{bmatrix}.$$

A lot of elementary formula for these numbers can be found by equating of corresponding elements for the equal matrices such as $M^{k/2}M^{(n+1)/2}=M^{(k+n+1)/2}$, $M^nM^{1/2}=M^{(2n+1)/2}$ and $M^{n/2}M^{(n+1)/2}=M^{(2n+1)/2}$.

Theorem 2. *For all integers k and n, the following equalities are valid:*

(i) $P_{(k+n+1)/2}=P_{k/2}P_{(n+3)/2}+P_{(k-2)/2}P_{(n+1)/2}$,

(ii) $8P_{(k+n+1)/2}=Q_{(k+2)/2}Q_{(n+1)/2}+Q_{k/2}Q_{(n-1)/2}$,

(iii) $P_{(2n+1)/2}=P_nP_{3/2}+P_{n-1}P_{1/2}$,

(iv) $Q_{(2n+1)/2}=P_{n+1}Q_{1/2}+P_nQ_{-1/2}$,

(v) $P_{(2n+1)/2}=P_{(n+2)/2}P_{(n+1)/2}+P_{n/2}P_{(n-1)/2}$,

(vi) $8P_{(2n+1)/2}=Q_{n/2}Q_{(n+3)/2}+Q_{(n-2)/2}Q_{(n+1)/2}$,

(vii) $e^{n\pi i/2}P_{(k-n)/2}=P_{k/2}P_{(n+2)/2}-P_{(k+2)/2}P_{n/2}$,

(viii) $8e^{n\pi i/2}P_{(k-n)/2}=Q_{(k+2)/2}Q_{n/2}-Q_{k/2}Q_{(n+2)/2}$.

Proof. The equalities (i)-(ii) can be found by the matrix equation $M_i^{k/2}M_i^{(n+1)/2}=M_1^{(k+n+1)/2}$ $(i=1,2,3,4)$. For $i=1$ (or $i=2$), $M_1^{(k+n+1)/2}$ can be written as

$$M_1^{(k+n+1)/2}=\begin{bmatrix}P_{(k+n+3)/2} & P_{(k+n+1)/2}\\ P_{(k+n+1)/2} & P_{(k+n-1)/2}\end{bmatrix} \tag{5}$$

and

$$M_1^{k/2}M_1^{(n+1)/2}=\begin{bmatrix} P_{(k+2)/2}P_{(n+3)/2}+P_{k/2}P_{(n+1)/2} & P_{(k+2)/2}P_{(n+1)/2}+P_{k/2}P_{(n-1)/2} \\ P_{k/2}P_{(n+3)/2}+P_{(k-2)/2}P_{(n+1)/2} & P_{k/2}P_{(n+1)/2}+P_{(k-2)/2}P_{(n-1)/2} \end{bmatrix}. \tag{6}$$

We obtained the equality (i) from the right side equations in (5) and (6). If we pick $M_1^{(k+n+1)/2}=M_3^{k/2}M_3^{(n+1)/2}$ for $i=3$ (or $i=4$), the equality (ii) is established. Considering the equations $M^nM_i^{1/2}=M_1^{(2n+1)/2}$ $(i=1,3)$ and $M_i^{n/2}M_i^{(n+1)/2}=M_1^{(2n+1)/2}$ $(i=1,2,3,4)$, the equalities (iii)-(iv) and the equalities (v)-(vi) are found, respectively. Computing the equation $M_i^{(k-n)/2}=M_i^{k/2}M_i^{-n/2}$ $(i=1,2,3,4)$, we have the equalities (vii)-(viii). □

The next goal is to find different relations between the Pell and Pell-Lucas numbers with specialized rational subscripts by using the square roots of matrix N^n [2].

Theorem 3. *Let $N_i^{n/2}$ $(i=1,2,3,4)$ denotes a square root of the matrix N^n. Then it has a type of following form*

$$N_{1,2}^{n/2}=\frac{\pm1}{2}\begin{bmatrix} Q_{n/2} & 4P_{n/2} \\ 2P_{n/2} & Q_{n/2} \end{bmatrix},\quad N_{3,4}^{n/2}=\pm\sqrt{2}\begin{bmatrix} P_{n/2} & Q_{n/2}/2 \\ Q_{n/2}/4 & P_{n/2} \end{bmatrix}.$$

Proof. By applying of the Cayley-Hamilton method for computing the square roots of the matrix N^n, we have got square roots matrices in the form

$$\sqrt{N^n}=\frac{\pm1}{\sqrt{T\pm2\sqrt{\det(N^n)}}}\left[N^n\pm\sqrt{\det(N^n)}I\right], \tag{7}$$

where I is the identity matrix, and $T=Q_n$ is the trace of the matrix N^n. As $\det(N^n)=e^{n\pi i}$, we have $\sqrt{\det(N^n)}=e^{n\pi i/2}$. Therefore, it is obtained that $\sqrt{T\pm2\sqrt{\det(N^n)}}=\sqrt{\phi^n}\pm\sqrt{e^{n\pi i}\phi^{-n}}$. It is seen that the matrix N^n has four non-integral square roots matrices. From this point of view, we first select as

$$N^{n/2}=\frac{\pm1}{\left(\phi^{n/2}+e^{n\pi i/2}\phi^{-n/2}\right)}\begin{bmatrix} \frac{Q_n}{2}+e^{n\pi i/2} & 2P_n \\ P_n & \frac{Q_n}{2}+e^{n\pi i/2} \end{bmatrix}.$$

Using the Binet's formula and algebraic manipulation, we write down the values of elements $(1,1)$ and $(1,2)$ for the right side matrix

$$\frac{\pm\left(\phi^{n/2}-e^{n\pi i/2}\phi^{-n/2}\right)\left(Q_n+2e^{n\pi i/2}\right)}{2\left(\phi^n-e^{n\pi i}\phi^{-n}\right)}=\frac{\pm\left(\phi^n-e^{n\pi i}\phi^{-n}\right)\left(\phi^{n/2}+e^{n\pi i/2}\phi^{-n/2}\right)}{4\sqrt{2}P_n}=\frac{\pm Q_{n/2}}{2}$$

and

$$\frac{\pm2P_n\left(\phi^{n/2}-e^{n\pi i/2}\phi^{-n/2}\right)}{\phi^n-e^{n\pi i}\phi^{-n}}=\frac{\pm2P_n\left(\phi^{n/2}-e^{n\pi i/2}\phi^{-n/2}\right)}{2\sqrt{2}P_n}=\pm2P_{n/2}.$$

The other elements are given in the same way. In the present case, we determine the two square roots matrices

$$N_1^{n/2}=\begin{bmatrix} \frac{Q_{n/2}}{2} & 2P_{n/2} \\ P_{n/2} & \frac{Q_{n/2}}{2} \end{bmatrix},\ N_2^{n/2}=-\begin{bmatrix} \frac{Q_{n/2}}{2} & 2P_{n/2} \\ P_{n/2} & \frac{Q_{n/2}}{2} \end{bmatrix}.$$

When we prefer the other situation in the matrix equation (7), the other two square roots matrices are procured by computing of values for all elements in the right side matrix

$$N_3^{n/2} = \begin{bmatrix} \sqrt{2}P_{n/2} & Q_{n/2}/\sqrt{2} \\ Q_{n/2}/2\sqrt{2} & \sqrt{2}P_{n/2} \end{bmatrix}, \ N_4^{n/2} = -\begin{bmatrix} \sqrt{2}P_{n/2} & Q_{n/2}/\sqrt{2} \\ Q_{n/2}/2\sqrt{2} & \sqrt{2}P_{n/2} \end{bmatrix}. \qquad \square$$

We suppose that $N_i^{n/2}$ $(i = 1,2,3,4)$ is one of the square roots of the matrix N^n. By equating corresponding elements of the matrix equation $N_i^{n/2}N_i^{n/2} = N^n$: the following identities are given

$$Q_{n/2}^2 + 8P_{n/2}^2 = 2Q_n, \ P_{n/2}Q_{n/2} = P_n,$$

and so taking the determinant of the matrices $N_i^{n/2}$ $(i = 1,2,3,4)$ yields

$$Q_{n/2}^2 - 8P_{n/2}^2 = 4e^{n\pi i/2}.$$

In addition to equalities in the Theorem 2, different equalities for the Pell and Pell-Lucas numbers with specialized rational subscripts can be found by equating of corresponding elements for the matrix equations $N^{k/2}N^{(n+1)/2} = N^{(k+n+1)/2}$, $N^nN^{1/2} = N^{(2n+1)/2}$ and $N^{n/2}N^{(n+1)/2} = N^{(2n+1)/2}$. Hence the matrices $N_i^{n/2}$ $(i = 1,2,3,4)$ are nonsingular, the inverse of matrices $N_i^{n/2}$ are shown with the matrix notation $N_i^{-n/2}$, which are given by

$$N_{1,2}^{-n/2} = \frac{\pm 1}{2e^{n\pi i/2}} \begin{bmatrix} Q_{n/2} & -4P_{n/2} \\ -2P_{n/2} & Q_{n/2} \end{bmatrix}, \ N_{3,4}^{-n/2} = \frac{\pm\sqrt{2}}{e^{n\pi i/2}} \begin{bmatrix} -P_{n/2} & Q_{n/2}/2 \\ Q_{n/2}/4 & -P_{n/2} \end{bmatrix}.$$

Theorem 4. *For all integer n and k, the following equalities are valid:*

(i) $2Q_{(n+k+1)/2} = Q_{(n+1)/2}Q_{k/2} + 8P_{(n+1)/2}P_{k/2}$,

(ii) $2P_{(n+k+1)/2} = P_{(n+1)/2}Q_{k/2} + Q_{(n+1)/2}P_{k/2}$,

(iii) $2P_{(2n+1)/2} = P_nQ_{1/2} + Q_nP_{1/2}$,

(iv) $2Q_{(2n+1)/2} = Q_nQ_{1/2} + 8P_nP_{1/2}$,

(v) $2Q_{(2n+1)/2} = Q_{n/2}Q_{(n+1)/2} + 8P_{n/2}P_{(n+1)/2}$,

(vi) $2P_{(2n+1)/2} = P_{n/2}Q_{(n+1)/2} + Q_{n/2}P_{(n+1)/2}$,

(vii) $2e^{n\pi i/2}Q_{(k-n)/2} = Q_{k/2}Q_{n/2} - 8P_{k/2}P_{n/2}$,

(viii) $2e^{n\pi i/2}P_{(k-n)/2} = P_{k/2}Q_{n/2} - Q_{k/2}P_{n/2}$.

Proof. The equalities (i)-(ii) can be found by equating of corresponding elements for the matrix equation $N_i^{(n+1)/2}N_i^{k/2} = N_1^{(n+k+1)/2}$ $(i = 1,2,3,4)$. The equations of matrices $N^nN_i^{1/2} = N_1^{(2n+1)/2}$ $(i = 1,3)$ and $N_i^{n/2}N_i^{(n+1)/2} = N_1^{(2n+1)/2}$ $(i = 1,2,3,4)$ give to equalities (iii)-(iv) and (v)-(vi), respectively. The equalities (vii)-(viii) are derived from the equations $N_i^{k/2}N_i^{-n/2} = N_1^{(k-n)/2}$ $(i = 1,2,3,4)$. $\square$

Clearly, the computing square roots of different matrix generators of the Pell and Pell-Lucas numbers [2, 9] can be carried out in the above mentioned fashion.

Now, we write the extended matrices $M^{r/q}$ and $N^{r/q}$ for $q \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. So we can obtain the Pell and Pell-Lucas number with rational subscripts. To do this, we use the connection between the equation $p(x) = a_{21}x^2 + (a_{22} - a_{11})x - a_{12}$, where a_{ij} are elements of the 2×2

matrix M^r (or N^r), and the roots of these matrices by the Abel's functional equation [10]. The polynomial $p(x) = P_r(x^2 - 2x - 1)$ is related to the matrices M^r. Let $A(x) = \int \frac{dx}{p(x)}$ be, then

$$A(x) = \int \frac{dx}{P_r\left(x-\phi\right)\left(x+\phi^{-1}\right)} = \frac{1}{2\sqrt{2}P_r}\ln\left(\frac{x-\phi}{x+\phi^{-1}}\right).$$

We define $\Phi_{M^r}(x) = \frac{P_{r+1}x+P_r}{P_r x+P_{r-1}}$ for the matrix M^r. The Abel's functional equation $A(\Phi_{M^r}(x)) = A(x) + k$ is satisfied for the certain real constant k. Then,

$$\begin{aligned} A\left(\frac{P_{r+1}x+P_r}{P_r x+P_{r-1}}\right) &= \frac{1}{2\sqrt{2}P_r}\ln\left(\frac{x\left(P_{r+1}-\phi P_r\right)+P_r-\phi P_{r-1}}{x\left(P_{r+1}+\phi^{-1}P_r\right)+P_r+\phi^{-1}P_{r-1}}\right) \\ &= A(x) + \frac{1}{2\sqrt{2}P_r}\ln\left(\frac{e^{r\pi i}\phi^{-r}}{\phi^r}\right), \qquad \left(\phi = 1+\sqrt{2}\,\right). \end{aligned}$$

A closed formula for the qth roots of matrix M^r is obtained from the functional equation $\Phi_{M^{r/q}}(x) = A^{-1}\left(A(x) + \frac{k}{q}\right)$, where the inverse function of $A(x)$ is shown with $A^{-1}(x)$ given by

$$A^{-1}(x) = \frac{\phi^{-1}e^{x2\sqrt{2}P_r}+\phi}{1-e^{x2\sqrt{2}P_r}}.$$

It follows that

$$\begin{aligned} \Phi_{M^{r/q}}(x) &= A^{-1}\left(\ln\left(\frac{x-\phi}{x+\phi^{-1}}\right)^{\frac{1}{2\sqrt{2}P_r}} + \frac{1}{2\sqrt{2}P_r}\ln\left(\frac{e^{r\pi i/q}\phi^{-r/q}}{\phi^{r/q}}\right)\right) \\ &= \frac{\phi^{-1}\left(x-\phi\right)e^{r\pi i/q}\phi^{-r/q}+\phi\left(x+\phi^{-1}\right)\phi^{r/q}}{\left(x+\phi^{-1}\right)\phi^{r/q}-\left(x-\phi\right)e^{r\pi i/q}\phi^{-r/q}} \\ &= \frac{x\left(\phi^{r/q+1}-e^{(r+q)\pi i/q}\phi^{-r/q-1}\right)+\phi^{r/q}-e^{r\pi i/q}\phi^{-r/q}}{x\left(\phi^{r/q}-e^{r\pi i/q}\phi^{-r/q}\right)+\left(\phi^{r/q-1}-e^{(r-q)\pi i/q}\phi^{-r/q+1}\right)} = \frac{xP_{r/q+1}+P_{r/q}}{xP_{r/q}+P_{r/q-1}}. \end{aligned}$$

Hence the matrix $M^{r/q}$ is related to the function $\Phi_{M^{r/q}}(x)$, we have

$$M^{r/q} = \pm\begin{bmatrix} P_{(r+q)/q} & P_{r/q} \\ P_{r/q} & P_{(r-q)/q} \end{bmatrix}. \tag{8}$$

If we take the matrix $N^{r/q}$, that is, it can be computed by the function $\Phi_{N^{r/q}}(x)$, we have

$$N^{r/q} = \frac{\pm 1}{2}\begin{bmatrix} Q_{r/q} & 4P_{r/q} \\ 2P_{r/q} & Q_{r/q} \end{bmatrix}. \tag{9}$$

Taking the determinant of matrix equations (8) and (9) yields

$$\begin{aligned} P_{(r+q)/q}P_{(r-q)/q} - P_{r/q}^2 &= e^{r\pi i/q}, \\ Q_{r/q}^2 - 8P_{r/q}^2 &= 4e^{r\pi i/q}. \end{aligned}$$

By considering different rational powers of the matrices M and N, the following matrices equations can be used to obtain identities involving terms of the Pell and Pell-Lucas numbers with rational subscripts:

$$\begin{aligned} M^{(rs+qt)/qs} &= M^{r/q}M^{t/s}, \quad (r,t \in \mathbb{Z} \text{ and } q,s \in \mathbb{Z}^+). \\ N^{(rs+qt)/qs} &= N^{r/q}N^{t/s}. \end{aligned}$$

We have

$$P_{(rs+qt)/qs} = P_{r/q+1}P_{t/s} + P_{r/q}P_{t/s-1} = P_{r/q}P_{t/s+1} + P_{r/q-1}P_{t/s},$$
$$2P_{(rs+qt)/qs} = P_{r/q}Q_{t/s} + Q_{r/q}P_{t/s},\ 2Q_{(rs+qt)/qs} = Q_{r/q}Q_{t/s} + 8P_{r/q}P_{t/s}.$$

Thus, we see that analogous identities of the Pell and Pell-Lucas numbers with integral subscripts seems to be hold for the Pell and Pell-Lucas numbers with rational subscripts.

3. Conclusion

By using the generalized Binet's forms of the Pell and Pell-Lucas numbers given in (1), we presented the two method which generated identities for the Pell and Pell-Lucas numbers with rational subscripts. One of them is based on the square roots of the 2×2 matrices M^n and N^n. The second method is based on the Abel's functional equation. By using different matrix generators of the Pell and Pell-Lucas numbers, more general identities could be obtained for the Pell and Pell-Lucas numbers with rational subscripts (or real subscripts).

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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