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New Generalized Measures of Fuzzy Entropy and Their Properties

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Abstract. Fuzziness is one of the universal attributes of human thinking and objective things whereas fuzzy set theory is one of the efficient means of researching and processing fuzzy phenomena in real world. Taking this fact into consideration, we have introduced and investigated two new generalized fuzzy measures of entropy and studied their essential and desirable properties.

1. Introduction

In real life situation, uncertainty arises in decision-making problem either due to lack of knowledge or due to inherent vagueness. Such types of problems can be solved using probability theory and fuzzy set theory respectively. Fuzziness, a feature of imperfect information, results from the lack of crisp distinction between the elements belonging and not belonging to a set. A measure of fuzziness which is often used and cited in the literature of fuzzy information is an entropy first mentioned by Zadeh [11]. However, the two functions measure fundamentally different types of uncertainty. Basically, the Shannon's [9] entropy measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment. De Luca and Termini [1] introduced some requirements which capture our intuitive comprehension of the degree of fuzziness and consequently developed a measure of fuzzy entropy which corresponds to Shannon's [9] probabilistic entropy, given by,

$$H(A) = -\sum_{i=1}^{n} \left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right] \tag{1.1}$$

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Kapur [5] took the following measure of fuzzy entropy corresponding to Havrada and Charvat's [3] probabilistic entropy:

$$(1.2) \qquad H^{\alpha}(A) = (1-\alpha)^{-1}\sum_{i=1}^{n}\left[\{\mu_A{}^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\} - 1\right];\ \alpha \neq 1, \alpha > 0$$

Parkash [8] introduced a new generalized measure of fuzzy entropy involving two real parameters, given by

$$(1.3) \quad H_{\alpha}^{\beta}(A) = [(1-\alpha)\beta]^{-1}\sum_{i=1}^{n}\{\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\}^{\beta} - 1, \quad \alpha > 0, \alpha \neq 1, \beta \neq 1$$

and called it (α,β) fuzzy entropy which includes some well known fuzzy entropies.

Some other interesting findings related with theoretical measures of fuzzy entropy and their applications have been investigated by Zadeh [11], Kapur [5], Gurdial, Petry and Beaubouef [2], Pal and Bezdek [7], Hu and Yu [4] etc. In Section 2, we have introduced and investigated two new generalized non-probabilistic information measures.

2. Two new generalized parametric measures of fuzzy entropy

In this section, we have proposed two new generalized information measures for a fuzzy distribution $\{\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n);\ 0 < \mu_A(x_i) < 1\}$ and studied their essential and desirable properties. The proposed measures are

2.1. *Generalized fuzzy entropy involving three parameters α, β and γ*

We propose the generalized fuzzy entropy depending upon three real parameters α, β and γ as given by the following mathematical expression:

$$(2.1) \qquad H_{\alpha,\beta,\gamma}(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\frac{\mu_A^{\alpha+\beta+\gamma-1}(x_i) + (1-\mu_A(x_i))^{\alpha+\beta+\gamma-1}}{\mu_A^{\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}},$$
$$\alpha \neq 1,\ \beta \geq 0,\ \gamma \geq 0,\ \alpha+\beta+\gamma-1 > 0,\ \beta+\gamma-1 \geq 0.$$

If $\alpha \to 1$, the measure (2.1) becomes

$$(2.2) \qquad H_{\beta,\gamma}(A) = -\sum_{i=1}^{n}\frac{\mu_A^{\beta+\gamma}(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}\log\{1-\mu_A(x_i)\}}{\mu_A^{\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}},$$

If $\beta = 0$ and $\gamma = 1$, then the above equation (2.2) reduces to

$$H(A) = -\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]$$

which is fuzzy entropy introduced by De Luca and Termini [1].

Thus, we see that the measure introduced in equation (2.1) is a generalized measure of fuzzy entropy.

Next, we study the essential and desirable properties of the generalized measure of fuzzy entropy (2.1):

2.2. *Essential properties*

(i) $H_\alpha, \beta, \gamma(A) \geq 0$

(ii) $\dfrac{\partial^2 H_{\alpha,\beta,\gamma}(A)}{\partial {\mu_A}^2(x_i)} < 0$. Thus $H_{\alpha,\beta,\gamma}(A)$ is a concave function of $\mu_A(x_i)\ \forall\ i$

(iii) $H_{\alpha,\beta,\gamma}(A)$ does not change when $\mu_A(x_i)$ is replaced by $1 - \mu_A(x_i)$

(iv) $H_{\alpha,\beta,\gamma}(A)$ is an increasing function of $0 \leq \mu_A(x_i) \leq \frac{1}{2}$

$\{H_{\alpha,\beta,\gamma}(A)/\mu_A(x_i) = 0\} = 0$ and $\{H_{\alpha,\beta,\gamma}(A)/\mu_A(x_i) = \frac{1}{2}\} = n \log 2 > 0$

(v) $H_{\alpha,\beta,\gamma}(A)$ is decreasing function of $\mu_A(x_i)$ for $\frac{1}{2} \leq \mu_A(x_i) \leq 1$

$\{H_{\alpha,\beta,\gamma}(A)/\mu_A(x_i) = \frac{1}{2}\} = n \log 2$

$\{H_{\alpha,\beta,\gamma}(A)/\mu_A(x_i) = 1\} = 0$

(vi) $[H_{\alpha,\beta,\gamma}(A) = 0]$ for $\mu_A(x_i) = 0$ or 1.

Under these conditions, the measure $[H_{\alpha,\beta,\gamma}(A)]$ is a valid measure of fuzzy entropy.

Next with the help of data, we have presented the measure (2.1) graphically as shown in Figure 2.1. For this purpose, we have fixed $\beta = 2$ and $\gamma = 3$. Then, for different α, we have computed different values of $[H_{\alpha,\beta,\gamma}(A)]$ as shown in Table 2.1.

Table 2.1

$\mu_A(x_i)$	α	$H_{\alpha,\beta,\gamma}(A)$	α	$H_{\alpha,\beta,\gamma}(A)$	α	$H_{\alpha,\beta,\gamma}(A)$
0.0		0.0000		0.0000		0.0000
0.1		0.1054		0.1053		0.1053
0.2		0.2239		0.2239		0.2232
0.3		0.3648		0.3610		0.3576
0.4		0.5767		0.5393		0.5190
0.5	2	0.6931	4	0.6931	16	0.6931
0.6		0.5767		0.5393		0.5190
0.7		0.3648		0.3610		0.3576
0.8		0.2239		0.2236		0.2232
0.9		0.1054		0.1053		0.1053
1.0		0.0000		0.0000		0.0000

2.2.1. *Desirable properties.*

(i) Maximum value

Differentiating (2.1) with respect to $\mu_A(x_i)$ and then putting $\frac{\partial H_{\alpha,\beta,\gamma}(A)}{\partial \mu_A(x_i)} = 0$, we get

$$\mu_A(x_i) = \frac{1}{2}.$$

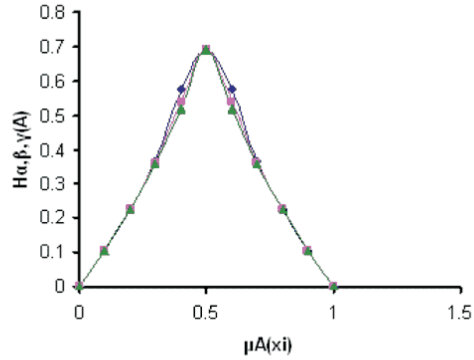

Figure 2.1. Concavity of $H_{\alpha,\beta,\gamma}(A)$

Again, at $\mu_A(x_i) = \frac{1}{2}$, we have

$$\frac{\partial^2 H_{\alpha,\beta,\gamma}(A)}{\partial \mu_A^2(x_i)} = -\{\alpha + 2(\beta + \gamma - 1)\} < 0\,.$$

Thus, we see that the maximum value of the fuzzy entropy exists at

$$\mu_A(x_i) = \frac{1}{2}\,.$$

If, we denote the maximum value by $f(n)$, then $f(n) = n \log 2$.

Further, $f'(n) = \log 2 > 0$.

This shows that the maximum value of the generalized fuzzy entropy is an increasing function of which is most desirable result.

(ii) Monotonicity

Differentiating equation (2.1) with respect to α, we get

$$\begin{aligned}&\{1-\alpha\}^2 \frac{dH_{\alpha,\beta,\gamma}(A)}{d\alpha}\\ &\quad = \sum_{i=1}^{n} \frac{\left\{\mu_A^{\alpha+\beta+\gamma-1}(x_i)\log \mu_A^{1-\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha+\beta+\gamma-1}\log(1-\mu_A(x_i))^{1-\alpha}\right\}}{\left\{\mu_A^{\alpha+\beta+\gamma-1}(x_i) + (1-\mu_A(x_i))^{\alpha+\beta+\gamma-1}\right\}}\\ &\qquad + \sum_{i=1}^{n} \log\left[\frac{\mu_A^{\alpha+\beta+\gamma-1}(x_i) + (1-\mu_A(x_i))^{\alpha+\beta+\gamma-1}}{\mu_A^{\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}}\right]\\ &\quad = f(\alpha) \quad \text{(say)}.\end{aligned}$$

Now, we discuss the following two cases:

Case I. When $\alpha > 1$, we have

$$\left[\frac{dH_{\alpha,\beta,\gamma}(A)}{d\alpha} \leq 0\right]$$

which shows that $[H_{\alpha,\beta,\gamma}(A)]$ is monotonically decreasing function of $\alpha > 1$.

Case II. When $\alpha < 1$.

We have

$$f(0) = \sum_{i=1}^{n} \frac{\left\{\mu_A^{\beta+\gamma}(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}\log(1-\mu_A(x_i))\right\}}{\left\{\mu_A^{\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}\right\}} < 0$$

since $\beta+\gamma \geq 0$ and $0 < \mu_A(x_i) < 1$.

Also

$$f(1) = \sum_{i=1}^{n} \frac{\left\{\mu_A^{\beta+\gamma}(x_i)\log 1 + (1-\mu_A(x_i))^{\beta+\gamma}\log 1\right\}}{\left\{\mu_A^{\beta+\gamma}(x_i) + (1-\mu_A(x_i))^{\beta+\gamma}\right\}} = 0.$$

Thus, $f(0) < 0$ and $f(1) = 0$ which shows that $f(\alpha)$ is negative for $0 < \alpha < 1$. Hence $\frac{dH_{\alpha,\beta,\gamma}(A)}{d\alpha} \leq 0$ which shows that $H_{\alpha,\beta,\gamma}(A)$ is monotonically decreasing function of α.

Next with the help of data, we have computed different values of $H_{\alpha,\beta,\gamma}(A)$ for different values of α for fixed $\beta = 2$ and $\gamma = 3$ and presented $H_{\alpha,\beta,\gamma}(A)$ graphically as shown in following Figure 2.2 which shows that the fuzzy entropy $H_{\alpha,\beta,\gamma}(A)$ introduced in (2.1) is monotonically decreasing function of α for $0 < \alpha < 1$.

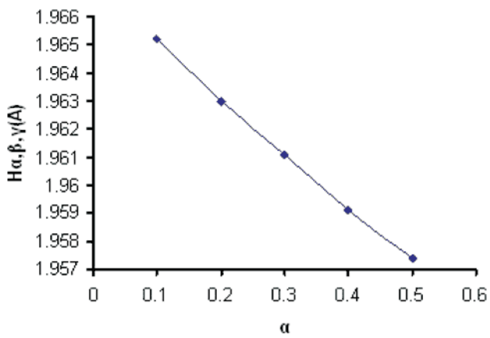

Figure 2.2. Monotonicity of $H_{\alpha,\beta,\gamma}(A)$ for $0 < \alpha < 1$

2.3. *Another generalized fuzzy entropy involving two parameters α and β*

Now, we introduce another parametric measure of fuzzy entropy depending upon two real parameters α and β. This measure is given by

$$(2.3) \qquad H^{\alpha,\beta}(A) = \frac{\sum_{i=1}^{n}\left\{\mu_A^{\alpha+\beta}(x_i) + (1-\mu_A(x_i))^{\alpha+\beta} - 1\right\}}{2^{1-\alpha-\beta} - 1}, \quad \alpha+\beta \neq 1, \alpha+\beta > 0.$$

If $\alpha = 0$ and $\beta = 1$, then the above measure (2.3) becomes

$$H(A) = -\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]$$

which is fuzzy entropy introduced by De Luca and Termini [1].

Further, if $\alpha = 1$ and $\beta = 0$, then the above equation (2.3) again reduces to De Luca and Termini's [1] fuzzy entropy.

Moreover, if $\alpha + \beta \to 1$, then the equation (2.3) reduces to

$$H^{\alpha,\beta}(A) = -\sum_{i=1}^{n} [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i))]$$

which is again a fuzzy entropy introduced by De Luca and Termini [1].

Thus, we see that the measure of entropy introduced in equation (2.3) is a generalized measure of fuzzy entropy.

Next, we study some essential and desirable properties of the generalized measure of fuzzy entropy (2.3):

2.3.1. *Essential properties.* (i) $H^{\alpha,\beta}(A) \geq 0$.

Here, two cases arise:

Case I. If $\alpha + \beta > 1$, then $\frac{\partial^2 H^{\alpha,\beta}(A)}{\partial {\mu_A}^2(x_i)} < 0$.

Case II. If $\alpha + \beta < 1$, $\frac{\partial^2 H^{\alpha,\beta}(A)}{\partial {\mu_A}^2(x_i)} < 0$

Thus $H^{\alpha,\beta}(A)$ is a concave function of $\mu_A(x_i)\ \forall\ i$.

(iii) $H^{\alpha,\beta}(A)$ does not change when $\mu_A(x_i)$ is replaced by $1 - \mu_A(x_i)$

(iv) $H^{\alpha,\beta}(A)$ is an increasing function of $\mu_A(x_i)$ for $0 \leq \mu_A(x_i) \leq 1/2$

(v) $[H^{\alpha,\beta}(A)$ is decreasing function of $\mu_A(x_i)$ for $1/2 \leq \mu_A(x_i) \leq 1$

(vi) $H^{\alpha,\beta}(A) = 0$ for $\mu_A(x_i) = 0$ or 1

Under these conditions, $H^{\alpha,\beta}(A)$ is a valid measure of fuzzy entropy.

Next with the help of the data, we have presented the measure (2.3) graphically. For this purpose, we have fixed $\beta = 2$. Then, for different values of α, we have computed different values of $H^{\alpha,\beta}(A)$ as shown in Table 2.2.

Table 2.2

$\mu_A(x_i)$	α	$H^{\alpha,\beta}(A)$	α	$H^{\alpha,\beta}(A)$	α	$H^{\alpha,\beta}(A)$	α	$H^{\alpha,\beta}(A)$
0.0		0.0000		0.0000		0.0000		0.0000
0.1		0.3929		0.4836		0.6500		0.8400
0.2		0.7620		0.7616		0.8900		0.9820
0.3		0.8592		0.9100		0.9736		0.9984
0.4		0.9654		0.9799		0.9958		0.9999
0.5	2	1.0000	4	1.0000	8	1.0000	16	1.0000
0.6		0.9654		0.9799		0.9958		0.9999
0.7		0.8592		0.9100		0.9736		0.9984
0.8		0.7620		0.7616		0.8900		0.9820
0.9		0.3929		0.4836		0.6500		0.8400
1.0		0.0000		0.0000		0.0000		0.0000

Next, we have presented the values of $H^{\alpha,\beta}(A)$ graphically for $\alpha = 4$ and obtained the following Figure 2.3 which shows that the measure introduced in equation (2.3) is a concave function. Similarly, for other values of α, we get different concave curves.

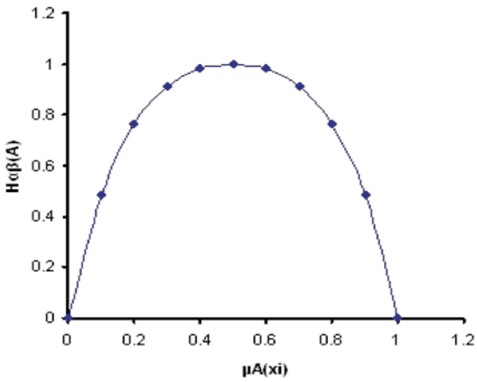

Figure 2.3. Concavity of $H^{\alpha,\beta}(A)$ for $\alpha = 4$, $\beta = 2$

2.3.2. *Desirable properties.*

(i) Maximum value

Differentiating (2.3) with respect to $\mu_A(x_i)$ and taking $\frac{\partial H^{\alpha,\beta}(A)}{\partial \mu_A(x_i)} = 0$, we get

$$\mu_A(x_i) = \frac{1}{2} \quad \forall\ i.$$

Again, at $\mu_A(x_i) = \frac{1}{2}$, we have

$$\frac{\partial^2 H^{\alpha,\beta}(A)}{\partial \mu_A^2(x_i)} = -8\left[\frac{(\alpha+\beta)^2 - (\alpha+\beta)}{2^{\alpha+\beta} - 2}\right] < 0\,.$$

Thus, we see that the maximum value of the fuzzy entropy exists at $\mu_A(x_i) = \frac{1}{2}$.

Further, if we denote the maximum value by $f(n)$, then $f(n) = n$. which implies that $f'(n) = 1 > 0$

This shows that the maximum value of the generalized fuzzy entropy is an increasing function of n which is most desirable result.

(ii) Monotonicity

Differentiating equation (2.3) with respect to α, we get

$$\{2^{1-\alpha-\beta} - 1\}^2 \frac{dH^{\alpha,\beta}(A)}{d\alpha} \tag{2.4}$$

$$= \{2^{1-\alpha-\beta} - 1\} \sum_{i=1}^{n} \left[\mu_A^{\alpha+\beta}(x_i) \log \mu_A(x_i) + (1-\mu_A(x_i))^{\alpha+\beta} \log(1-\mu_A(x_i))\right]$$

$$+\, 2^{1-\alpha-\beta} \sum_{i=1}^{n} \left[\mu_A^{\alpha+\beta}(x_i) + (1-\mu_A(x_i))^{\alpha+\beta} - 1\right]$$

Next, we discuss the following two cases:

Case I. When $\alpha + \beta > 1$.

From equation (2.4), we have

$$\{2^{1-\alpha-\beta} - 1\}^2 \frac{dH^{\alpha,\beta}(A)}{d\alpha} \geq 0$$

which shows that $H^{\alpha,\beta}(A)$ is monotonically increasing function of α.

Next, with the help of the data, we have computed different values of $H^{\alpha,\beta}(A)$ for different values of α for fixed $\beta = 2$ and presented $H^{\alpha,\beta}(A)$ graphically as shown in Figure 2.4 which shows that the measure of generalized fuzzy entropy $H^{\alpha,\beta}(A)$ introduced in (2.3) is monotonically increasing function of α when $\alpha + \beta > 1$.

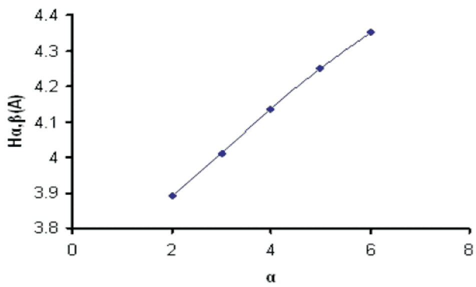

Figure 2.4. Concavity of $H^{\alpha,\beta}(A)$ for $\alpha + \beta > 1$

Case II. When $\alpha + \beta < 1$. In this case, again, we have

$$\frac{dH^{\alpha,\beta}(A)}{d\alpha} \leq 0$$

which shows that $H^{\alpha,\beta}(A)$ is monotonically decreasing function of α.

Next, with the help of the data, we have computed different values of $H^{\alpha,\beta}(A)$ for different values of α for fixed $\beta = 0.2$, presented $H^{\alpha,\beta}(A)$ graphically and obtained the Figure 2.5 which shows that the measure of generalized fuzzy entropy $H^{\alpha,\beta}(A)$ is monotonically decreasing function of α when $\alpha + \beta < 1$.

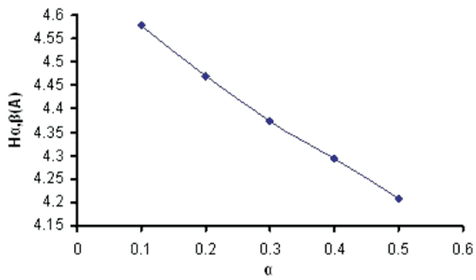

Figure 2.5. Monotonicity of $H^{\alpha,\beta}(A)$ for $0 < \alpha + \beta < 1$

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