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Graphic Requirements for Multiple Attractive Cycles in Boolean Dynamical Systems

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Abstract. E. Remy, P. Ruet and D. Thieffry have proved a Boolean version of Thomas' conjecture: if a map F from $\{0,1\}^n$ to itself has several fixed points, then there exists a positive circuit in the corresponding interaction graph. In this paper, we prove that the presence of a positive circuit in a local interaction graph is also a necessary condition for the presence of several attractive cycles in the Boolean synchronous dynamics.

1. **Introduction**

This paper is related to the problem of providing new sufficient conditions in order that there may exist a positive circuit (i.e. a circuit contains an even number of negative edges) in the corresponding interaction graph.

Let us begin with the celebrated Thomas' conjecture in genetic regulatory systems. At the beginning of the 1980s, the biologist Ren Thomas proposed, in the course of his work on the analysis of gene networks, a conjecture in which *the presence of at least one positive circuit in the corresponding interaction graph* (*i.e. the sign of a circuit being defined as the product of the signs of its edges*) *is a necessary* (*but not sufficient*) *condition for the presence of multiple stable stationary states* (*i.e. the existence of several stable fixed points in the dynamics*) [1, 2, 5, 11, 12]. This is the so-called Thomas' conjecture. This conjecture has already been proved in several different types of formal mathematical models of gene networks (see [3, 5, 9, 10]): the differential, differential with decay, piecewise-linear and multivalued discrete models. Recently, Remy *et al.* proved that Thomas' conjecture is also true for Boolean models [2], by using a recent proof by M.-H. Shih and J.-L. Dong of the Boolean version of the Jacobian conjecture [8].

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A possible generalization of Thomas' conjecture in the Boolean framework was raised by Remy *et al.* [2] in gene networks.

Conjecture 1.1. *Positive circuits are necessary for the coexistence of alternative attractors.*

Motivated by the Conjecture 1.1, let us formulate five conjectures relating circuits to the qualitative behavior of a Boolean dynamical system as follows:

Conjecture 1.2. *If $F : \{0,1\}^n \to \{0,1\}^n$ has one fixed point and one attractive cycle, then there is an $x \in \{0,1\}^n$ such that the interaction graph $G(F'(x))$ has a positive circuit.*

Conjecture 1.3. *If $F : \{0,1\}^n \to \{0,1\}^n$ has at least two attractive cycles, then there is an $x \in \{0,1\}^n$ such that the interaction graph $G(F'(x))$ has a positive circuit.*

Conjecture 1.4. *If $F : \{0,1\}^n \to \{0,1\}^n$ has two disjoint non-attractive cycles, then there is an $x \in \{0,1\}^n$ such that the interaction graph $G(F'(x))$ has a positive circuit.*

Conjecture 1.5. *If $F : \{0,1\}^n \to \{0,1\}^n$ has one attractive cycle and one non-attractive cycle, then there is an $x \in \{0,1\}^n$ such that the interaction graph $G(F'(x))$ has a positive circuit.*

Conjecture 1.6. *If $F : \{0,1\}^n \to \{0,1\}^n$ has one fixed point and one non-attractive cycle, then there is an $x \in \{0,1\}^n$ such that the interaction graph $G(F'(x))$ has a positive circuit.*

Our aim is to study the five conjectures above (i.e. Conjectures 1.2, 1.3, 1.4, 1.5 and 1.6). These notions concerning the fixed points, the attractive cycles, the non-attractive cycles and the attractors in a Boolean dynamical system will be explained in Section 3.1.

The content of this paper is organized as follows. In the next section let us recall some elementary definitions about Boolean Jacobian matrices and interaction graphs, and explain the notion concerning the stability of the interaction graphs under projection, needed to prove the two theorems (i.e. Theorems 3.2 and 3.3) of this paper. Following the above conjecture proposed by Remy and coworkers, we establish new sufficient conditions for the presence of a positive circuit in the corresponding interaction graph. Furthermore, in Section 3 we state and prove the two new results, which are the generalizations of Thomas' conjecture in the Boolean case. Consequently Conjectures 1.2 and 1.3 are true for all $n \geq 2$. In Section 4 we show that for each $n \geq 2$ there is a Boolean map, which provides a counterexample to the Conjecture 1.4. And further, we also show that there are examples which give a negative answer to the Conjectures 1.5 and 1.6 for all $n \geq 3$. These counterexamples hold in any dimension $n \geq 3$. Finally, in Section 5 our results and some remarks are illustrated with several examples.

2. Preliminaries

In the following, we state some definitions, notations and results needed to formulate and prove our results. The material can be found in [1, 2, 4, 8], and also in the books by Robert [6, 7].

The order "$\leq$" on $\{0,1\}$ is given by $0 \leq 0 \leq 1 \leq 1$. Let $\{0,1\}$ be with operations $+, \cdot$ defined as follows:

$$0+0 = 0\cdot 1 = 1\cdot 0 = 0\cdot 0 = 0, \quad 1+0 = 0+1 = 1+1 = 1\cdot 1 = 1,$$
$$\overline{1} = 0, \quad \text{and} \quad \overline{0} = 1.$$

For each positive integer n, let $\{0,1\}^n$ denote the collection of all ordered n-tuples $x = (x_1, \ldots, x_n)$ with components $x_i \in \{0,1\}$ for $i = 1, \ldots, n$. The notation ^{t}, which stands for the transpose of a vector, is used in this paper. We also write $x = (x_1, \ldots, x_n)^t$ interchangeably. For $x \in \{0,1\}^n$ and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, let us define $\tilde{x}^{i_1,\ldots,i_k} = y$ by

$$y_j = \begin{cases} x_j & \text{if the cardinality of } \{l \in \{1,\ldots,k\} \mid i_l = j\} \text{ is even (or zero),} \\ \bar{x}_j & \text{otherwise.} \end{cases}$$

The notation $(\tilde{x}^{i_1,\ldots,i_k})_j$ denotes the j-th component of $\tilde{x}^{i_1,\ldots,i_k}$. When $k = 1$, $\tilde{x}^{i_1} = (x_1, \ldots, \bar{x}_{i_1}, \ldots, x_n)$ and $(\tilde{x}^{i_1})_j$ stands for the j-th component of $\tilde{x}^{i_1}$, that is, $(\tilde{x}^{i_1})_j = \bar{x}_j$ if $j = i_1$, $(\tilde{x}^{i_1})_j = x_j$ if $j \neq i_1$.

We are interested in the evolution of the biological network involving n interacting genes, which are denoted by the integers $1, \ldots, n$. The possible expression levels of each gene $i \in \{1, \ldots, n\}$ are assumed to be either 1 (when the gene i is active) or 0 (when the gene i is inactive). A *state* of the network is an element $x = (x_1, \ldots, x_n) \in \{0,1\}^n$, where x_i is the expression level of gene i. Let $F : \{0,1\}^n \to \{0,1\}^n$, and $F(x) = (f_1(x), \ldots, f_n(x))$. For each $x \in \{0,1\}^n$ and $i \in \{1, \ldots, n\}$, $f_i(x)$ denotes the value to which x_i tends when the network is in state x. In general, we say that the pair $(\{0,1\}^n, F)$ is a *Boolean network* and that it is viewed as a model for the dynamics of a network of n genes. Thus the Boolean synchronous dynamics of this network is described by the map F, in the sense that all the variables x_i are simultaneously updated to $f_i(x)$ in one step.

2.1. *Boolean Jacobian Matrices and Interaction Graphs*

Given a map $F : \{0,1\}^n \to \{0,1\}^n$, we call *Boolean Jacobian matrix* of F evaluated at state $x \in \{0,1\}^n$, and we denote by $F'(x) = (f_{ij}(x))$, the $n \times n$ matrix over $\{0,1\}$ with (i,j)-entry defined by

$$f_{ij}(x) = \begin{cases} 1 & \text{if } f_i(x) \neq f_i(\tilde{x}^j), \\ 0 & \text{otherwise.} \end{cases}$$

For each $x \in \{0,1\}^n$, we define the (*local*) *connectivity graph* of the Boolean Jacobian matrix $F'(x)$, denoted by $\Gamma(F'(x))$, to be the directed graph with vertex set $\{1, \ldots, n\}$ such that there is an edge from j to i when $f_{ij}(x) = 1$. Thus, the adjacency matrix of $\Gamma(F'(x))$ is the transpose of $F'(x)$.

A directed graph with a sign, $+1$ or -1, attached to each edge, is called a *signed directed graph*. A *circuit of length* k $(1 \le k \le n)$ in a signed directed graph with vertex set $\{1,\dots,n\}$ is a sequence $(i_1,\dots,i_k)$ of k distinct vertices such that there is an edge from i_j to i_{j+1}, $1 \le j \le k-1$, and from i_k to i_1. When $k = 1$, a circuit of length 1 (i.e. an edge of the form (i,i)) is called a *loop*. The *sign of a circuit* is the product of the signs of its edges. In other words, a circuit C is negative if the number of negative edges of C is odd, and positive otherwise. If G is a signed directed graph with vertex set $\{1,\dots,n\}$ and $I \subseteq \{1,\dots,n\}$, the *restriction of G to I* is defined as the signed directed graph obtained from G by removing any vertex not in I and any edge whose source or target is not in I [1].

Definition 2.1. Let $F : \{0,1\}^n \to \{0,1\}^n$ and $x \in \{0,1\}^n$. We define the (local) interaction graph evaluated at state $x \in \{0,1\}^n$, denoted by $G(F'(x))$, to be the signed directed graph with $\{1,\dots,n\}$ as set of vertices and such that there is an edge from j to i if

$$f_{ij}(x) = 1,$$

with positive sign when

$$x_j = f_i(x),$$

and negative sign otherwise.

In this Boolean context, Thomas' conjecture can be reformulated as follow:

Theorem 2.2. *If $F : \{0,1\}^n \to \{0,1\}^n$ has at least two fixed points, then there is an $x \in \{0,1\}^n$ such that $G(F'(x))$ has a positive circuit.*

The above result has been proved in [2].

2.2. *Stability Under Projection*

In order to prove the following two theorems we shall employ the notion concerning the stability of the interaction graph under projection.

Let us begin to explain that the local interaction graphs defined in Section 2.1 are stable under projection in the following sense (see [1] in more detail).

Let $I = \{i_1,\dots,i_m\} \subseteq \{1,\dots,n\}$ with $i_1 < i_2 < \cdots < i_m$. We define the map $\pi_I : \{0,1\}^n \to \{0,1\}^m$ by $\pi_I(x) = z$, where $x = (x_1,\dots,x_{i_1},\dots,x_{i_j},\dots,x_{i_m},\dots,x_n) \in \{0,1\}^n$, $z = (z_1,\dots,z_j,\ \dots,z_m) \in \{0,1\}^m$ and $z_j = x_{i_j}$ for all $j = 1,\dots,m$, (i.e. π_I is the *projection* map on $\{0,1\}^m$), and the map $s : \{0,1\}^m \to \{0,1\}^n$ defined as $s(\pi_I(x)) = y \in x[\{1,\dots,n\} \setminus I \mid I] \subseteq \{0,1\}^n$, where $y = (y_1,\dots,y_{i_1},\dots,y_{i_j},\dots,y_{i_m},\dots,y_n) \in \{0,1\}^n$, $y_{i_j} = x_{i_j}$ for all $j = 1,\dots,m$ and $y_t = (s(\pi_I(x)))_t$ for $t \notin \{i_1,\dots,i_m\}$, is said to be a *section* of π_I. Here we use the notation $x[\{1,\dots,n\} \setminus I \mid I] = \{y \in \{0,1\}^n \mid y_j = x_j \text{ for all } j \in I\}$ to denote a $(n-m)$-subcube generated by $x \in \{0,1\}^n$: see [2, 8] for a thorough survey. Clearly, the composite function of π_I and s, $\pi_I \circ s$ from $\{0,1\}^m$ to itself is the identity map.

Given such a subset I of m genes, there are a lot of ways to define possible dynamics on I : if $F : \{0,1\}^n \to \{0,1\}^n$, let

$$F_{I,s} = \pi_I \circ F \circ s : \{0,1\}^m \to \{0,1\}^m.$$

We are particularly interested in the very specific section map s, which is said to be *regular* if $\pi_k \circ s : \{0,1\}^m \to \{0,1\}$ is constant for each $k \notin I$. In this article, we mainly focus on this kind of regular sections. As an illustration, if $i \in \{1,\dots,n\}$, let us consider $I = \{k_1,\dots,k_{n-1}\} \subseteq \{1,\dots,n\}$ with $k_1 = 1 < \cdots < k_{i-1} = i-1 < k_i = i+1 < \cdots < k_{n-1} = n$ and $\xi \in \{0,1\}^n$ be given. Then the projection map $\pi_I : \{0,1\}^n \to \{0,1\}^{n-1}$ is such that $\pi_I(x) = z$, where $x = (x_{k_1},\dots,x_{k_{i-1}},x_i,x_{k_i},\dots,x_{k_{n-1}}) \in \{0,1\}^n$, $z = (z_1,\dots,z_j,\dots,z_{n-1}) \in \{0,1\}^{n-1}$ and $z_j = x_{k_j}$ for all $j = 1,\dots,n-1$, (i.e. $\pi_I(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$), and we define the specific regular section $s : \{0,1\}^{n-1} \to \{0,1\}^n$ by

$$s(\pi_I(x)) = (x_1,\dots,x_{i-1},\xi_i,x_{i+1},\dots,x_n) \tag{2.1}$$

lying in $\xi[\{1,\dots,i-1,i+1,\dots,n\} \mid i]$. Let us recall that the notation $\xi[\{1,\dots,i-1, i+1,\dots,n\} \mid i] = \{y \in \{0,1\}^n \mid y_i = \xi_i\}$ denotes a $(n-1)$-subcube generated by $\xi \in \{0,1\}^n$. Thus $F_{I,s} : \{0,1\}^{n-1} \to \{0,1\}^{n-1}$ is such that

$$\begin{aligned} F_{I,s}(\pi_I(x)) &= (\pi_I \circ F \circ s)(\pi_I(x)) \\ &= \pi_I(F(s(\pi_I(x)))) \\ &= \pi_I(F(x_1,\dots,x_{i-1},\xi_i,x_{i+1},\dots,x_n)) \\ &= (f_1(s(\pi_I(x))),\dots,f_{i-1}(s(\pi_I(x))),f_{i+1}(s(\pi_I(x))),\dots,f_n(s(\pi_I(x)))). \end{aligned}$$

That is,

$$F_{I,s}(\pi_I(x)) = \begin{pmatrix} f_1(x_1,\dots,x_{i-1},\xi_i,x_{i+1},\dots,x_n) \\ \vdots \\ f_{i-1}(x_1,\dots,x_{i-1},\xi_i,x_{i+1},\dots,x_n) \\ f_{i+1}(x_1,\dots,x_{i-1},\xi_i,x_{i+1},\dots,x_n) \\ \vdots \\ f_n(x_1,\dots,x_{i-1},\xi_i,x_{i+1},\dots,x_n) \end{pmatrix}^t \in \{0,1\}^{n-1}, \quad (x \in \{0,1\}^n). \tag{2.2}$$

The following result has been proved in [1] and plays a crucial role in the proof of our first theorem.

Lemma 2.3. *Let $F : \{0,1\}^n \to \{0,1\}^n$, $I = \{i_1,\dots,i_m\} \subseteq \{1,\dots,n\}$ with $i_1 < i_2 < \cdots < i_m$ and $z \in \{0,1\}^m$. If s is a regular section of π_I, then $G(F'_{I,s}(z))$ coincides with the restriction of $G(F'(s(z)))$ to I.*

3. **Positive Circuits**

We now establish a characterization for a circuit to be positive in the corresponding interaction graph.

Lemma 3.1. *Let* $F : \{0,1\}^n \to \{0,1\}^n$ *and* $x \in \{0,1\}^n$. *Assume that* $C = (i_1, i_2, \ldots, i_l)$ *is a circuit* $(1 \le l \le n)$ *of the local interaction graph* $G(F'(x))$. *Then* C *is positive if, and only if, the cardinality of the set* $\{j \in \{i_1, \ldots, i_l\} \mid f_j(x) \ne x_j\}$ *is even (or zero).*

Proof. Define the map $\sigma : \{0,1\} \to \{0,1\}$ by $\sigma(0) = 1$, $\sigma(1) = 0$.

We use the notation $\mathbf{I}$ to denote the identity map. Let $m \in \{1, \ldots, l\}$. Define the map $\sigma_m : \{0,1\} \to \{0,1\}$ by

$$\sigma_m = \begin{cases} \sigma & \text{if the edge } (i_m, i_{m+1}) \text{ from } i_m \text{ in } C \text{ is negative,} \\ \mathbf{I} & \text{if the edge } (i_m, i_{m+1}) \text{ is positive.} \end{cases}$$

Define the map $\delta_m : \{0,1\} \to \{0,1\}$ by

$$\delta_m = \begin{cases} \sigma & \text{if } x_{i_m} \ne f_{i_m}(x), \\ \mathbf{I} & \text{if } x_{i_m} = f_{i_m}(x). \end{cases}$$

Since

$$\begin{aligned}
\delta_2(x_{i_2}) = f_{i_2}(x) = \sigma_1(x_{i_1}) &\Rightarrow x_{i_2} = \delta_2(\sigma_1(x_{i_1})) = (\delta_2 \circ \sigma_1)(x_{i_1}), \\
\delta_3(x_{i_3}) = f_{i_3}(x) = \sigma_2(x_{i_2}) &\Rightarrow x_{i_3} = \delta_3(\sigma_2(x_{i_2})) = (\delta_3 \circ \sigma_2)(x_{i_2}), \\
&\vdots \\
\delta_{l-1}(x_{i_{l-1}}) = f_{i_{l-1}}(x) = \sigma_{l-2}(x_{i_{l-2}}) &\Rightarrow x_{i_{l-1}} = \delta_{l-1}(\sigma_{l-2}(x_{i_{l-2}})) = (\delta_{l-1} \circ \sigma_{l-2})(x_{i_{l-2}}), \\
\delta_l(x_{i_l}) = f_{i_l}(x) = \sigma_{l-1}(x_{i_{l-1}}) &\Rightarrow x_{i_l} = \delta_l(\sigma_{l-1}(x_{i_{l-1}})) = (\delta_l \circ \sigma_{l-1})(x_{i_{l-1}}), \\
\delta_1(x_{i_1}) = f_{i_1}(x) = \sigma_l(x_{i_l}) &\Rightarrow x_{i_1} = \delta_1(\sigma_l(x_{i_l})) = (\delta_1 \circ \sigma_l)(x_{i_l}),
\end{aligned}$$

we have

$$\begin{aligned}
x_{i_1} &= (\delta_1 \circ \sigma_l)(x_{i_l}) \\
&= (\delta_1 \circ \sigma_l)((\delta_l \circ \sigma_{l-1})(x_{i_{l-1}})) \\
&= (\delta_1 \circ \sigma_l \circ \delta_l \circ \sigma_{l-1})(x_{i_{l-1}}) \\
&= \cdots \\
&= (\delta_1 \circ \sigma_l \circ \delta_l \circ \sigma_{l-1} \circ \delta_{l-1} \circ \sigma_{l-2} \circ \cdots \circ \delta_2 \circ \sigma_1)(x_{i_1}) \\
&= \sigma^{p+q}(x_{i_1}),
\end{aligned}$$

where p is the cardinality of the set $\{j \in \{i_1, \ldots, i_l\} \mid f_j(x) \ne x_j\}$ and, q is the number of negative edges in C.

This implies that $p + q$ is even. Therefore, q is even (or zero) if and only if p is even (or zero).

Hence, we prove that C is positive if and only if p is even (or zero), and we complete the proof of Lemma 3.1. □

3.1. *Attractive Cycles and Fixed Points*

Let F always be a map of $\{0,1\}^n$ into itself. First of all let us recall that a *fixed point* of F is a state x such that $F(x) = x$. We define a *synchronous cycle of length* $r \geq 2$ for F to be a sequence $(x^1, \ldots, x^r)$ of r distinct states in $\{0,1\}^n$ such that $F(x^i) = x^{i+1}$ for $1 \leq i \leq r-1$ and $F(x^r) = x^1$. Such a cycle is said to be an *attractive cycle* $C = (x^1, \ldots, x^r)$ *with strategy* $\varphi : \{1, \ldots, r\} \to \{1, \ldots, n\}$ if, for all $i = 1, \ldots, r$, $F(x^i) = \widetilde{x^i}^{\varphi(i)}$. If a synchronous cycle is not attractive, then it is said to be *non-attractive*. Observe that if $C = (x^1, \ldots, x^r)$ is an attractive cycle of F with strategy $\varphi : \{1, \ldots, r\} \to \{1, \ldots, n\}$, then for any $i = 1, \ldots, n$, the cardinality of $\{p \in \{1, \ldots, r\} \mid \varphi(p) = i\}$ is even (or zero). As a consequence, r is then even.

We now make the definition of the attractor precise. A *trap domain* of the Boolean synchronous dynamics for F is a nonempty subset A of $\{0,1\}^n$ such that, for all $x \in A$, $F(x) \in A$. In other words, a trap domain is a set of states that we cannot leave in the Boolean synchronous dynamics. A trap domain A is said to be an *attractor*, or a *smallest trap domain*, if there is no trap domain strictly included in A. We notice that if $\xi \in \{0,1\}^n$ is a fixed point of F and if $C = (x^1, \ldots, x^r)$ is a synchronous cycle, then $\{\xi\}$ and $\{x^1, \ldots, x^r\}$ are attractors. Remark also that an attractor is not necessarily an attractive cycle. As an illustration, we consider the map $F : \{0,1\}^3 \to \{0,1\}^3$ defined by

$$F(x) = \begin{pmatrix} \bar{x}_1\bar{x}_2x_3 + x_1\bar{x}_2 + x_1x_2x_3 \\ \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1x_2x_3 \\ \bar{x}_1x_2 + x_1\bar{x}_3 \end{pmatrix}, \quad (x \in \{0,1\}^3).$$

Let us compute the map F at each state $x \in \{0,1\}^3$. Then, F is given by the table:

x	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
$F(x)$	(0,0,0)	(1,1,0)	(0,1,1)	(0,0,1)	(1,0,1)	(1,0,0)	(0,0,1)	(1,1,0)

The Boolean synchronous dynamics for F contains 27 trap domains. It is easy to see that the fixed point is $(0,0,0)$, the attractive cycle is $((1,0,0),(1,0,1))$ with strategy $\varphi : \{1,2\} \to \{1,2,3\}$ defined by $\varphi(1) = \varphi(2) = 3$, and the non-attractive cycle is $((0,0,1), (1,1,0))$. Here, $\{(0,0,1), (1,1,0)\} \cup \{(1,0,0), (1,0,1)\}$, $\{(0,0,1), (1,1,0)\} \cup \{(0,0,0)\}$, $\{(0,0,0), (1,0,0), (1,0,1)\}$, and $\{(0,0,1), (1,1,0)\} \cup \{(0,1,1)\}$ are other examples of trap domains.

Our first result of this paper follows.

Theorem 3.2. *If* $F : \{0,1\}^n \to \{0,1\}^n$ *has an attractive cycle* $C = (x^1, \ldots, x^r)$ *with strategy* $\varphi : \{1, \ldots, r\} \to \{1, \ldots, n\}$ *and one fixed point, then there is an* $x \in \{0,1\}^n$ *such that* $G(F'(x))$ *has a positive circuit.*

Proof. We prove the theorem by mathematical induction on the dimension n of the n-cube $\{0,1\}^n$. The induction begins with $n = 2$; for in this case $r = 2$, and so we let $C = (x^1, x^2)$ be an attractive cycle of F with strategy $\varphi : \{1,2\} \to \{1,2\}$, and let $x^1 = \xi \in \{0,1\}^2$. We thus obtain $F(\xi) = \tilde{\xi}^{\varphi(1)}$ and $F(\tilde{\xi}^{\varphi(1)}) = \xi$. Without loss of generality, we suppose that $\varphi(1) = 1$. Since there exists one fixed point of F in $\{0,1\}^2$ by the hypothesis, it follows that either $F(\tilde{\xi}^2) = \tilde{\xi}^2$ or $F(\tilde{\xi}^{1,2}) = \tilde{\xi}^{1,2}$. If $F(\tilde{\xi}^2) = \tilde{\xi}^2$, then $f_2(\tilde{\xi}^2) = (\tilde{\xi}^2)_2$ and $f_2(\xi) = (\tilde{\xi}^1)_2$ together imply that $f_{22}(\xi) = 1$, and so the local connectivity graph $\Gamma(F'(\xi))$ has a loop at a vertex 2. Moreover, it follows from the definition of the local interaction graph that the loop at a vertex 2 is also contained in $G(F'(\xi))$. As the cardinality of $\{j \in \{2\} \mid f_j(\xi) \neq \xi_j\}$ is zero, Lemma 3.1 shows that it is positive and consequently, $G(F'(\xi))$ has a positive loop at a vertex 2 in this case. If, on the other hand, $F(\tilde{\xi}^{1,2}) = \tilde{\xi}^{1,2}$, we see that similar arguments establish the existence of a positive loop at a vertex 2 in the local interaction graph $G(F'(\tilde{\xi}^1))$. Hence the case of $n = 2$ is valid.

Now we suppose that the theorem is true for some integer $n - 1 \geq 2$. We will prove that the theorem is true for n.

To see that, we assume that the attractive cycle and the fixed point of the map F from $\{0,1\}^n$ to itself are $C = (x^1, \ldots, x^r)$ with strategy $\varphi : \{1, \ldots, r\} \to \{1, \ldots, n\}$ and η, respectively. We wish to show that there exists a positive circuit in the corresponding interaction graph. Since F has an attractive cycle $C = (x^1, \ldots, x^r)$ with strategy φ, we let $x^1 = \xi \in \{0,1\}^n$ and consequently,

$$
\begin{aligned}
F(\xi) &= \tilde{\xi}^{\varphi(1)}, \\
F(\tilde{\xi}^{\varphi(1)}) &= \tilde{\xi}^{\varphi(1),\varphi(2)}, \\
F(\tilde{\xi}^{\varphi(1),\varphi(2)}) &= \tilde{\xi}^{\varphi(1),\varphi(2),\varphi(3)}, \\
&\vdots \\
F(\tilde{\xi}^{\varphi(1),\ldots,\varphi(r-2)}) &= \tilde{\xi}^{\varphi(1),\ldots,\varphi(r-2),\varphi(r-1)}, \\
F(\tilde{\xi}^{\varphi(1),\ldots,\varphi(r-2),\varphi(r-1)}) &= \tilde{\xi}^{\varphi(1),\ldots,\varphi(r-2),\varphi(r-1),\varphi(r)} = \xi.
\end{aligned}
\tag{3.1}
$$

If the Boolean Jacobian matrix $F'(\eta)$ has no zero columns, as argued in [6, 7], this implies that the local connectivity graph $\Gamma(F'(\eta))$ contains a circuit $(i_1, i_2, \ldots, i_l)$ with $\{i_1, i_2, \ldots, i_l\} \subseteq \{1, \ldots, n\}$. By definition of the local interaction graph, the circuit $(i_1, i_2, \ldots, i_l)$ is also contained in $G(F'(\eta))$. Since $F(\eta) = \eta$, it follows that the cardinality of $\{j \in \{i_1, \ldots, i_l\} \mid f_j(\eta) \neq \eta_j\}$ is zero. Now, with the use of Lemma 3.1, we have that this circuit $(i_1, i_2, \ldots, i_l)$ is positive in the corresponding interaction graph $G(F'(\eta))$. Hence we can conclude that $G(F'(\eta))$ has a positive circuit in this case.

On the other hand, if $F'(\eta)$ has at least one zero column, then there exists an $i \in \{1, \ldots, n\}$ such that $f_j(\tilde{\eta}^i) = f_j(\eta)$ for all $j = 1, \ldots, n$. Let $I = \{1, \ldots, i-1, i+1, \ldots, n\}$ and we choose this regular section s from $\{0,1\}^{n-1}$ to $\{0,1\}^n$ defined as in (2.1); that is, $s(\pi_I(x)) = (x_1, \ldots, x_{i-1}, \xi_i, x_{i+1}, \ldots, x_n)$ for all $x \in \{0,1\}^n$. In order to use the induction hypothesis, we consider the map $F_{I,s}$ from $\{0,1\}^{n-1}$ to itself defined as in (2.2). Since $\{\varphi(1), \ldots, \varphi(r)\} \subseteq \{1, \ldots, n\}$, we split the arguments into the following two cases.

Case I. $i \in \{\varphi(1), \dots, \varphi(r)\}$.

We let $k = \min\{p \in \{1, \dots, r\} \mid \varphi(p) = i\}$. Here, we use the notation $\min\{p \in \{1, \dots, r\} \mid \varphi(p) = i\}$ to denote the minimum element in the set $\{p \in \{1, \dots, r\} \mid \varphi(p) = i\}$. Therefore, $1 \leq k \leq r - 1$. (The reason of $k \neq r$ is explained as follows: for if not, we could have $\varphi(r) = i$ and for all $p = 1, \dots, r-1$, $\varphi(p) \neq i$, and so this contradicts the fact that for any $\alpha = 1, \dots, n$, the cardinality of $\{p \in \{1, \dots, r\} \mid \varphi(p) = \alpha\}$ is even (or zero).)

If $\xi_i = \bar{\eta}_i$, it follows from $F(\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)}) = \tilde{\xi}^{\varphi(1),\dots,\varphi(k-1),\varphi(k)}$ in (3.1) and $F(\tilde{\eta}^i) = \eta$ that there exist two states $\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)}$ and $\tilde{\eta}^i$ in the $(n-1)$-subcube $\xi[\{1, \dots, i-1, i+1, \dots, n\} \mid i]$ such that

$$f_j(\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)}) = (\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)})_j \tag{3.2}$$

and

$$f_j(\tilde{\eta}^i) = (\tilde{\eta}^i)_j \tag{3.3}$$

for all $j = 1, \dots, i-1, i+1, \dots, n$. We notice that $(\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1),\varphi(k)})_j = (\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)})_j$ for all $j = 1, \dots, i-1, i+1, \dots, n$, since $\varphi(k) \notin \{\varphi(1), \dots, \varphi(k-1)\}$, $\varphi(k) = i$ and $j \neq i$. Moreover, $\eta_j = (\tilde{\eta}^i)_j$ for all $j = 1, \dots, i-1, i+1, \dots, n$.

According to (2.2), (3.2) and (3.3), we thus obtain $F_{I,s}(\pi_I(\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)})) = \pi_I(\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)})$ and $F_{I,s}(\pi_I(\tilde{\eta}^i)) = \pi_I(\tilde{\eta}^i)$.

Consequently $F_{I,s} : \{0,1\}^{n-1} \to \{0,1\}^{n-1}$ has two fixed points $\pi_I(\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)})$ and $\pi_I(\tilde{\eta}^i)$ in $\{0,1\}^{n-1}$, and it follows from Theorem 2.2 that there exists an $z \in \{0,1\}^{n-1}$ such that the local interaction graph $G(F'_{I,s}(z))$ contains a positive circuit $(j_1, j_2, \dots, j_m)$ with $\{j_1, j_2, \dots, j_m\} \subseteq \{1, \dots, i-1, i+1, \dots, n\}$. For the same reason we deduce that if $\xi_i = \eta_i$ then $\pi_I(\tilde{\xi}^{\varphi(1),\dots,\varphi(k-1)})$ and $\pi_I(\eta)$ are the two fixed points of $F_{I,s}$ in $\{0,1\}^{n-1}$. Thus, by Theorem 2.2, there exists an $w \in \{0,1\}^{n-1}$ such that the local interaction graph $G(F'_{I,s}(w))$ contains a positive circuit $(h_1, h_2, \dots, h_t)$ with $\{h_1, h_2, \dots, h_t\} \subseteq \{1, \dots, i-1, i+1, \dots, n\}$.

By using Lemma 2.3, we show that if $\xi_i = \bar{\eta}_i$ then the positive circuit $(j_1, j_2, \dots, j_m)$ is also contained in the local interaction graph $G(F'(s(z)))$ with vertex set $\{1, \dots, n\}$; on the other hand, if $\xi_i = \eta_i$ then the positive circuit $(h_1, h_2, \dots, h_t)$ is also contained in the local interaction graph $G(F'(s(w)))$ with vertex set $\{1, \dots, n\}$.

Since $\xi_i = \bar{\eta}_i$ or $\xi_i = \eta_i$, we can conclude that there exists an $x = s(z)$ or $x = s(w)$ in $\{0,1\}^n$ such that $G(F'(x))$ has a positive circuit.

Case II. $i \notin \{\varphi(1), \dots, \varphi(r)\}$.

If $\xi_i = \bar{\eta}_i$, it follows from (3.1) and $F(\tilde{\eta}^i) = \eta$ that there exist $r+1$ states $\xi, \tilde{\xi}^{\varphi(1)}, \tilde{\xi}^{\varphi(1),\varphi(2)}, \dots, \tilde{\xi}^{\varphi(1),\dots,\varphi(r-1)}$ and $\tilde{\eta}^i$ in the $(n-1)$-subcube $\xi[\{1, \dots, i-1, i+1, \dots, n\} \mid i]$ such that

$$
\begin{aligned}
f_j(\xi) &= (\tilde{\xi}^{\varphi(1)})_j,\\
f_j(\tilde{\xi}^{\varphi(1)}) &= (\tilde{\xi}^{\varphi(1),\varphi(2)})_j,\\
f_j(\tilde{\xi}^{\varphi(1),\varphi(2)}) &= (\tilde{\xi}^{\varphi(1),\varphi(2),\varphi(3)})_j,\\
&\vdots\\
f_j(\tilde{\xi}^{\varphi(1),\dots,\varphi(r-1)}) &= (\tilde{\xi}^{\varphi(1),\dots,\varphi(r-1),\varphi(r)})_j,
\end{aligned}
\tag{3.4}
$$

and

$$f_j(\tilde{\eta}^i) = (\tilde{\eta}^i)_j \tag{3.5}$$

for all $j = 1, \dots, i-1, i+1, \dots, n$.

By the hypothesis, we have that the strategy of this attractive cycle $C = (x^1, \dots, x^r)$ is the map $\varphi : \{1, \dots, r\} \to \{1, \dots, n\}$. Now we define a new map $\varphi' : \{1, \dots, r\} \to \{1, \dots, n-1\}$ by setting

$$\varphi'(p) = \begin{cases} \varphi(p) & \text{if } \varphi(p) \le i-1, \\ \varphi(p) - 1 & \text{if } \varphi(p) \ge i+1. \end{cases}$$

Thus, for any $\beta = 1, \dots, n-1$, the cardinality of $\{p \in \{1, \dots, r\} \mid \varphi'(p) = \beta\}$ is also even (or zero).

According to (2.2), (3.4), (3.5) and the new map φ', we thus obtain $F_{I,s}(\pi_I(\xi)) = \widetilde{\pi_I(\xi)}^{\varphi'(1)}$, $F_{I,s}(\widetilde{\pi_I(\xi)}^{\varphi'(1)}) = \widetilde{\pi_I(\xi)}^{\varphi'(1),\varphi'(2)}$, $F_{I,s}(\widetilde{\pi_I(\xi)}^{\varphi'(1),\varphi'(2)}) = \widetilde{\pi_I(\xi)}^{\varphi'(1),\varphi'(2),\varphi'(3)}, \dots$, $F_{I,s}(\widetilde{\pi_I(\xi)}^{\varphi'(1),\dots,\varphi'(r-1)}) = \widetilde{\pi_I(\xi)}^{\varphi'(1),\dots,\varphi'(r-1),\varphi'(r)} = \pi_I(\xi)$, and $F_{I,s}(\pi_I(\tilde{\eta}^i)) = \pi_I(\tilde{\eta}^i)$.

Consequently $F_{I,s} : \{0,1\}^{n-1} \to \{0,1\}^{n-1}$ has an attractive cycle $C' = (a^1, \dots, a^r)$ where $a^1 = \pi_I(\xi) \in \{0,1\}^{n-1}$ with strategy $\varphi' : \{1, \dots, r\} \to \{1, \dots, n-1\}$ and one fixed point $\pi_I(\tilde{\eta}^i)$ in $\{0,1\}^{n-1}$, so we may apply the induction hypothesis to $F_{I,s}$ and obtain that there exists an $v \in \{0,1\}^{n-1}$ such that the local interaction graph $G(F'_{I,s}(v))$ contains a positive circuit. For the same reason we deduce that if $\xi_i = \eta_i$ then the attractive cycle and the fixed point of $F_{I,s}$ in $\{0,1\}^{n-1}$ are $C' = (\pi_I(\xi), \widetilde{\pi_I(\xi)}^{\varphi'(1)}, \dots, \widetilde{\pi_I(\xi)}^{\varphi'(1),\dots,\varphi'(r-1)})$ with strategy $\varphi' : \{1, \dots, r\} \to \{1, \dots, n-1\}$ and $\pi_I(\eta)$, respectively. Also, by the induction hypothesis, there exists an $u \in \{0,1\}^{n-1}$ such that the local interaction graph $G(F'_{I,s}(u))$ contains a positive circuit.

By using Lemma 2.3, the desired result follows as above.

Hence, in either case, the result is established.

This completes the inductive proof of Theorem 3.2, and thus proves the theorem. □

Therefore Conjecture 1.2 is now a theorem. Our next result extends Theorem 3.2 to the presence of two disjoint attractive cycles and thus shows that the existence of a positive circuit in the corresponding interaction graph is also required for the existence of multiple attractive cycles in Boolean dynamical systems.

3.2. *The Principal Theorem*

The aim of this paper is to prove the following theorem giving a new sufficient condition for the presence of a positive circuit in the corresponding interaction graph. From Theorems 1 and 2 we get the main result, which is a generalization of Thomas' conjecture for Boolean dynamical systems.

Theorem 3.3. *If $F:\{0,1\}^n \to \{0,1\}^n$ has at least two attractive cycles, then there is an $x \in \{0,1\}^n$ such that $G(F'(x))$ has a positive circuit.*

Proof. By the hypothesis, we can assume that $C_1 = (x^1,\dots,x^{r_1})$ and $C_2 = (y^1,\dots,y^{r_2})$ are the two disjoint attractive cycles of the map F from $\{0,1\}^n$ into itself with strategies $\varphi:\{1,\dots,r_1\}\to\{1,\dots,n\}$ and $\psi:\{1,\dots,r_2\}\to\{1,\dots,n\}$, respectively. For simplicity of notation, we let $x^1=\xi$ and $y^1=\eta$. It follows that

$$\begin{aligned} F(\xi) &= \tilde{\xi}^{\varphi(1)},\\ F(\tilde{\xi}^{\varphi(1)}) &= \tilde{\xi}^{\varphi(1),\varphi(2)},\\ F(\tilde{\xi}^{\varphi(1),\varphi(2)}) &= \tilde{\xi}^{\varphi(1),\varphi(2),\varphi(3)},\\ &\vdots\\ F(\tilde{\xi}^{\varphi(1),\dots,\varphi(r_1-2)}) &= \tilde{\xi}^{\varphi(1),\dots,\varphi(r_1-2),\varphi(r_1-1)},\\ F(\tilde{\xi}^{\varphi(1),\dots,\varphi(r_1-2),\varphi(r_1-1)}) &= \tilde{\xi}^{\varphi(1),\dots,\varphi(r_1-2),\varphi(r_1-1),\varphi(r_1)} = \xi, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} F(\eta) &= \tilde{\eta}^{\psi(1)},\\ F(\tilde{\eta}^{\psi(1)}) &= \tilde{\eta}^{\psi(1),\psi(2)},\\ F(\tilde{\eta}^{\psi(1),\psi(2)}) &= \tilde{\eta}^{\psi(1),\psi(2),\psi(3)},\\ &\vdots\\ F(\tilde{\eta}^{\psi(1),\dots,\psi(r_2-2)}) &= \tilde{\eta}^{\psi(1),\dots,\psi(r_2-2),\psi(r_2-1)},\\ F(\tilde{\eta}^{\psi(1),\dots,\psi(r_2-2),\psi(r_2-1)}) &= \tilde{\eta}^{\psi(1),\dots,\psi(r_2-2),\psi(r_2-1),\psi(r_2)} = \eta. \end{aligned} \tag{3.7}$$

Since $\{\varphi(1),\dots,\varphi(r_1)\}\subseteq\{1,\dots,n\}$ and $\{\psi(1),\dots,\psi(r_2)\}\subseteq\{1,\dots,n\}$, we split the proof into two cases.

Case I. The intersection of these two sets $\{\varphi(1),\dots,\varphi(r_1)\}$ and $\{\psi(1),\dots,\psi(r_2)\}$ is not empty.

Let $i\in\{\varphi(1),\dots,\varphi(r_1)\}\cap\{\psi(1),\dots,\psi(r_2)\}$. We assume that $k=\min\{p\in\{1,\dots,r_1\}\mid\varphi(p)=i\}$ and $l=\min\{q\in\{1,\dots,r_2\}\mid\psi(q)=i\}$. That is, k and l are the minimum elements in the sets $\{p\in\{1,\dots,r_1\}\mid\varphi(p)=i\}$ and

$\{q \in \{1,\ldots,r_2\} \mid \psi(q) = i\}$, respectively. Therefore, $1 \le k \le r_1 - 1$ and $1 \le l \le r_2 - 1$. (The reason of $k \ne r_1$ is explained as follows: for if not, we could have $\varphi(r_1) = i$ and for all $p = 1,\ldots,r_1 - 1$, $\varphi(p) \ne i$, and so this contradicts the fact that for any $\alpha = 1,\ldots,n$, the cardinality of $\{p \in \{1,\ldots,r_1\} \mid \varphi(p) = \alpha\}$ is even (or zero). Similarly, $l \ne r_2$.)

If $\xi_i = \bar{\eta}_i$, it follows from $F(\tilde{\eta}^{\psi(1),\ldots,\psi(l-1)}) = \tilde{\eta}^{\psi(1),\ldots,\psi(l-1),\psi(l)}$ and $F(\tilde{\eta}^{\psi(1),\ldots,\psi(r_2-1)}) = \eta$ in (3.7) that there exists an $m_1 = \min\{q \in \{l+1,\ldots,r_2\} \mid \psi(q) = i\}$ such that

$$F(\tilde{\eta}^{\psi(1),\ldots,\psi(l-1),\psi(l),\ldots,\psi(m_1-1)}) = \tilde{\eta}^{\psi(1),\ldots,\psi(l-1),\psi(l),\psi(l+1),\ldots,\psi(m_1-1),\psi(m_1)}. \tag{3.8}$$

Therefore, the two states $\tilde{\xi}^{\varphi(1),\ldots,\varphi(k-1)}$ and $\tilde{\eta}^{\psi(1),\ldots,\psi(m_1-1)}$ are in the $(n-1)$-subcube $\xi[\{1,\ldots,i-1,i+1,\ldots,n\} \mid i]$ such that $f_j(\tilde{\xi}^{\varphi(1),\ldots,\varphi(k-1)}) = (\tilde{\xi}^{\varphi(1),\ldots,\varphi(k-1)})_j$ and $f_j(\tilde{\eta}^{\psi(1),\ldots,\psi(m_1-1)}) = (\tilde{\eta}^{\psi(1),\ldots,\psi(m_1-1)})_j$ for all $j = 1,\ldots,i-1,i+1,\ldots,n$.

Similarly, it can be shown that if $\xi_i = \eta_i$ then the two states $\tilde{\xi}^{\varphi(1),\ldots,\varphi(k-1)}$ and $\tilde{\eta}^{\psi(1),\ldots,\psi(l-1)}$ are in the $(n-1)$-subcube $\xi[\{1,\ldots,i-1,i+1,\ldots,n\} \mid i]$ such that $f_j(\tilde{\xi}^{\varphi(1),\ldots,\varphi(k-1)}) = (\tilde{\xi}^{\varphi(1),\ldots,\varphi(k-1)})_j$ and $f_j(\tilde{\eta}^{\psi(1),\ldots,\psi(l-1)}) = (\tilde{\eta}^{\psi(1),\ldots,\psi(l-1)})_j$ for all $j = 1,\ldots,i-1,i+1,\ldots,n$.

Similar arguments as in the proof of Case I of Theorem 3.2 establish the result whether $\xi_i = \bar{\eta}_i$ or $\xi_i = \eta_i$.

Case II. The intersection of these two sets $\{\varphi(1),\ldots,\varphi(r_1)\}$ and $\{\psi(1),\ldots,\psi(r_2)\}$ is the empty set.

Let $i \in \{\psi(1),\ldots,\psi(r_2)\}$. Obviously, $i \notin \{\varphi(1),\ldots,\varphi(r_1)\}$. We assume that $l = \min\{q \in \{1,\ldots,r_2\} \mid \psi(q) = i\}$. Therefore, $1 \le l \le r_2 - 1$. (The reason of $l \ne r_2$ is explained as follows: for if not, we could have $\psi(r_2) = i$ and for all $q = 1,\ldots,r_2-1$, $\psi(q) \ne i$, and so this contradicts the fact that for any $\alpha = 1,\ldots,n$, the cardinality of $\{q \in \{1,\ldots,r_2\} \mid \psi(q) = \alpha\}$ is even (or zero).)

If $\xi_i = \bar{\eta}_i$, it follows from (3.7) that there exists an $m_2 = \min\{q \in \{l+1,\ldots,r_2\} \mid \psi(q) = i\}$ such that

$$F(\tilde{\eta}^{\psi(1),\ldots,\psi(l-1),\psi(l),\ldots,\psi(m_2-1)}) = \tilde{\eta}^{\psi(1),\ldots,\psi(l-1),\psi(l),\psi(l+1),\ldots,\psi(m_2-1),\psi(m_2)}. \tag{3.9}$$

Thus, according to (3.6) and (3.9), the $r+1$ states $\xi, \tilde{\xi}^{\varphi(1)}, \tilde{\xi}^{\varphi(1),\varphi(2)},\ldots, \tilde{\xi}^{\varphi(1),\ldots,\varphi(r_1-1)}$ and $\tilde{\eta}^{\psi(1),\ldots,\psi(m_2-1)}$ are in the $(n-1)$-subcube $\xi[\{1,\ldots,i-1, i+1,\ldots,n\} \mid i]$ such that $f_j(\xi) = (\tilde{\xi}^{\varphi(1)})_j$, $f_j(\tilde{\xi}^{\varphi(1)}) = (\tilde{\xi}^{\varphi(1),\varphi(2)})_j$, $f_j(\tilde{\xi}^{\varphi(1),\varphi(2)}) = (\tilde{\xi}^{\varphi(1),\varphi(2),\varphi(3)})_j,\ldots, f_j(\tilde{\xi}^{\varphi(1),\ldots,\varphi(r_1-1)}) = (\tilde{\xi}^{\varphi(1),\ldots,\varphi(r_1-1),\varphi(r_1)})_j$, and $f_j(\tilde{\eta}^{\psi(1),\ldots,\psi(m_2-1)}) = (\tilde{\eta}^{\psi(1),\ldots,\psi(m_2-1)})_j$ for all $j = 1,\ldots,i-1,i+1,\ldots,n$.

Similarly, it can be shown that if $\xi_i = \eta_i$ then the $r+1$ states $\xi, \tilde{\xi}^{\varphi(1)}, \tilde{\xi}^{\varphi(1),\varphi(2)},\ldots, \tilde{\xi}^{\varphi(1),\ldots,\varphi(r_1-1)}$ and $\tilde{\eta}^{\psi(1),\ldots,\psi(l-1)}$ are in the $(n-1)$-subcube $\xi[\{1,\ldots,i-1,i+1,\ldots,n\} \mid i]$ such that $f_j(\xi) = (\tilde{\xi}^{\varphi(1)})_j$, $f_j(\tilde{\xi}^{\varphi(1)}) = (\tilde{\xi}^{\varphi(1),\varphi(2)})_j$, $f_j(\tilde{\xi}^{\varphi(1),\varphi(2)}) = (\tilde{\xi}^{\varphi(1),\varphi(2),\varphi(3)})_j,\ldots, f_j(\tilde{\xi}^{\varphi(1),\ldots,\varphi(r_1-1)}) = (\tilde{\xi}^{\varphi(1),\ldots,\varphi(r_1-1),\varphi(r_1)})_j$, and $f_j(\tilde{\eta}^{\psi(1),\ldots,\psi(l-1)}) = (\tilde{\eta}^{\psi(1),\ldots,\psi(l-1)})_j$ for all $j = 1,\ldots,i-1,i+1,\ldots,n$.

Similar arguments, together with Theorem 3.2, give the desired result whether $\xi_i = \bar{\eta}_i$ or $\xi_i = \eta_i$.

Thus, in either case, the result follows.

Hence we complete the proof of Theorem 3.3. □

Therefore Conjecture 1.3 is now a theorem.

4. **Counterexamples to Several Conjectures**

Recently, a possible generalization of Theorem 2.2 to the coexistence of alternative attractors was proposed by Remy and coworkers in [1, 2]. Following Remy, Ruet and Thieffry (2007), let us bring up five different kinds of questions which are explicitly stated in Section 1. Some (i.e. Conjectures 1.2 and 1.3) are shown in the previous section that the answer is affirmative for any dimension ≥ 2, and the others (i.e. Conjectures 1.4, 1.5 and 1.6) relating the behavior of a Boolean dynamical system to the topology of its interaction graph are studied in this section. Finally, we show that there always exists a Boolean map, which provides a counterexample to the Conjectures 1.4, 1.5, and 1.6 for all dimensions ≥ 3.

Theorem 4.1. *Let $n \geq 2$ and $F : \{0,1\}^n \to \{0,1\}^n$ be defined by*

$$F(x_1, x_2, x_3, \ldots, x_n) = (\bar{x}_1, \bar{x}_2, 0, \ldots, 0)^t, \quad (x \in \{0,1\}^n).$$

Then F is a counterexample to the Conjecture 1.4. More precisely, each of the local interaction graphs $G(F'(x))$, $x \in \{0,1\}^n$, is the signed directed graph whose vertex set is $\{1, \ldots, n\}$ and whose edge set consists of exactly two negative loops at vertices 1 and 2.

Proof. One easily verifies that the Boolean synchronous dynamics for F has neither attractive cycles nor fixed points, but has only two non-attractive cycles $\{(0,0,0,\ldots,0),(1,1,0,\ldots,0)\}$ and $\{(0,1,0,\ldots,0),(1,0,0,\ldots,0)\}$. Moreover, by definition of the map F, for each x in $\{0,1\}^n$, we have

$$F'(x) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For each $x \in \{0,1\}^n$, the local connectivity graph $\Gamma(F'(x))$ thus contains only two edges, two loops at vertices 1 and 2. Moreover, it follows from the definition of the local interaction graph that for each $x \in \{0,1\}^n$, $G(F'(x))$ contains only two loops at vertices 1 and 2. As $x_1 \neq f_1(x)$ and $x_2 \neq f_2(x)$, Lemma 3.1 shows that the two loops at their respective vertices 1 and 2 are negative.

We conclude that for each x in $\{0,1\}^n$, the local interaction graph $G(F'(x))$ is the signed directed graph with vertex set $\{1,\ldots,n\}$ and two negative loops at vertices 1 and 2. □

Theorem 4.2. *Let $n \geq 3$ and $F : \{0,1\}^n \to \{0,1\}^n$ be defined by*

$$F(x_1, x_2, x_3, x_4, \ldots, x_n) = (\bar{x}_2, x_3, x_1, 0, \ldots, 0)^t, \quad (x \in \{0,1\}^n).$$

Then F is a counterexample to the Conjecture 1.5. More precisely, each of the local interaction graphs $G(F'(x))$, $x \in \{0,1\}^n$, is the signed directed graph whose vertex set is $\{1,\ldots,n\}$ and whose edge set consists of exactly one negative circuit with vertices 1, 2 and 3.

Proof. One easily verifies that the Boolean synchronous dynamics for F has no fixed points, but has only two alternative attractors: the attractive cycle

$$((0,0,0,0,\ldots,0),(1,0,0,0,\ldots,0),(1,0,1,0,\ldots,0),(1,1,1,0,\ldots,0),$$
$$(0,1,1,0,\ldots,0),(0,1,0,0,\ldots,0))$$

with strategy $\varphi : \{1,2,3,4,5,6\} \to \{1,2,3\}$ defined by $\varphi(1) = \varphi(4) = 1$, $\varphi(3) = \varphi(6) = 2$, $\varphi(2) = \varphi(5) = 3$, and the non-attractive cycle $\{(0,0,1,0,\ldots,0),$ $(1,1,0,0,\ldots,0)\}$. Moreover, by definition of the map F, for each x in $\{0,1\}^n$, we have

$$F'(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For each $x \in \{0,1\}^n$, the local connectivity graph $\Gamma(F'(x))$ thus contains only three edges, one from vertex 1 to vertex 3, another from vertex 3 to vertex 2 and the other from vertex 2 to vertex 1. By definition of the local interaction graph, we obtain that for each $x \in \{0,1\}^n$, $G(F'(x))$ contains only one circuit $(1,3,2)$ of length 3. Since $x_2 \neq f_1(x)$, $x_1 = f_3(x)$ and $x_3 = f_2(x)$, it follows that the edge from vertex 2 to vertex 1 is negative, but the others are positive. Consequently, this circuit $(1,3,2)$ is negative.

We conclude that for each x in $\{0,1\}^n$, the local interaction graph $G(F'(x))$ is the signed directed graph with vertex set $\{1,\ldots,n\}$ and one negative circuit with vertices 1, 2 and 3. □

To close this section we also give a counterexample to the Conjecture 1.6 for all $n \geq 3$.

Theorem 4.3. *Let $n \geq 3$ and $F : \{0,1\}^n \to \{0,1\}^n$ be defined by*

$$F(x_1, x_2, x_3, x_4, \ldots, x_n) = (\bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2, \bar{x}_1\bar{x}_2x_3, x_1x_2\bar{x}_3, 0, \ldots, 0)^t, \quad (x \in \{0,1\}^n).$$

Then F is a counterexample to the Conjecture 1.6. *More precisely, none of the local interaction graphs $G(F'(x))$, $x \in \{0,1\}^n$, contains a positive circuit.*

Proof. One easily verifies that the Boolean synchronous dynamics for F has no attractive cycles, but has only two alternative attractors: the fixed point $(0,0,0,0,\dots,0)$ and the non-attractive cycle $\{(0,0,1,0,\dots,0),(1,1,0,0,\dots,0)\}$. Moreover, by definition of the map F, for each x in $\{0,1\}^n$, we have

$$F'(x) = \begin{pmatrix} \bar{x}_2x_3 + x_2 & \bar{x}_1\bar{x}_3 & \bar{x}_1\bar{x}_2 & 0 & \cdots & 0 \\ \bar{x}_2x_3 & \bar{x}_1x_3 & \bar{x}_1\bar{x}_2 & 0 & \cdots & 0 \\ x_2\bar{x}_3 & x_1\bar{x}_3 & x_1x_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

A computation for the Boolean Jacobian matrix $F'(x)$ of F evaluated at each state $x \in \{0,1\}^n$ now shows that the local interaction graph associated to any state contains no positive circuits. □

5. **Concluding Remarks**

In this section, a generalization of Theorem 2.2 to the existence of multiple attractive cycles in Boolean dynamical systems (i.e. Theorem 3.3) and some remarks about Theorems 3.2 and 3.3 are illustrated with several examples and counterexamples for which we give detailed computations.

5.1. *Illustrations of Theorem 3.3*

As an illustration of Theorem 3.3, the next two examples exhibit two disjoint attractive cycles in a Boolean dynamical system, therefore they must have a positive circuit in the local interaction graph.

Example 5.1. Let $F : \{0,1\}^3 \to \{0,1\}^3$ be defined by

$$F(x) = \begin{pmatrix} \bar{x}_1\bar{x}_2 + x_1x_3 \\ x_1x_3 + x_2\bar{x}_3 \\ \bar{x}_1x_2 + \bar{x}_2x_3 \end{pmatrix}, \quad (x \in \{0,1\}^3).$$

Let us compute the map F at each state $x \in \{0,1\}^3$ and the Boolean Jacobian matrix $F'(0,0,1)$. Then, F is given by the table:

x	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
$F(x)$	(1,0,0)	(1,0,1)	(0,1,1)	(0,0,1)	(0,0,0)	(1,1,1)	(0,1,0)	(1,1,0)

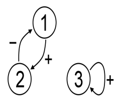
and

$$F'(0,0,1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently, the Boolean synchronous dynamics has exactly two attractive cycles $C_1 = ((0,0,0),(1,0,0))$ and $C_2 = ((0,1,0),(0,1,1),(0,0,1),(1,0,1),(1,1,1),(1,1,0))$ with strategies $\varphi : \{1,2\} \to \{1,2,3\}$ and $\psi : \{1,2,3,4,5,6\} \to \{1,2,3\}$ defined by $\varphi(1) = \varphi(2) = 1$, and $\psi(3) = \psi(6) = 1$, $\psi(2) = \psi(4) = 2$, $\psi(1) = \psi(5) = 3$, respectively. And further, the intersection of these two sets $\{\varphi(1),\varphi(2)\}$ and $\{\psi(1),\ldots,\psi(6)\}$ is not empty. Hence Theorem 3.3 tells us that there needs to exist a positive circuit in the corresponding interaction graph. We now demonstrate it as follows.

The above computations show that the local interaction graph $G(F'(0,0,1))$ already contains three edges, one from vertex 2 to vertex 1, another from vertex 1 to vertex 2 and the other is a loop at a vertex 3. As $x = (0,0,1)$, $x_3 = f_3(x)$, $x_2 = f_2(x)$ and $x_1 \neq f_1(x)$, Lemma 3.1 shows that the loop at a vertex 3 is positive and the circuit $(1,2)$ is negative.

Therefore, the local interaction graph $G(F'(0,0,1))$ is



Example 5.2. Let $F : \{0,1\}^3 \to \{0,1\}^3$ be defined by

$$F(x) = \begin{pmatrix} \bar{x}_1\bar{x}_2 + x_1x_3 + x_1x_2\bar{x}_3 \\ x_1x_3 + x_2\bar{x}_3 \\ \bar{x}_1x_2 + \bar{x}_2x_3 + x_1x_2\bar{x}_3 \end{pmatrix}, \quad (x \in \{0,1\}^3).$$

Let us compute the map F at each state $x \in \{0,1\}^3$ and the Boolean Jacobian matrix $F'(0,0,0)$. Then, F is given by the table:

x	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
$F(x)$	(1,0,0)	(1,0,1)	(0,1,1)	(0,0,1)	(0,0,0)	(1,1,1)	(1,1,1)	(1,1,0)

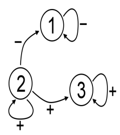
and

$$F'(0,0,0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Consequently, the Boolean synchronous dynamics has only two attractive cycles $C_1 = ((0,0,0),(1,0,0))$ and $C_2 = ((1,1,1),(1,1,0))$ with strategies $\varphi : \{1,2\} \to \{1,2,3\}$ and $\psi : \{1,2\} \to \{1,2,3\}$ defined by $\varphi(1)=\varphi(2)=1$ and $\psi(1)=\psi(2)=3$, respectively. And further, the intersection of these two sets $\{\varphi(1),\varphi(2)\}$ and $\{\psi(1),\psi(2)\}$ is empty. We can conclude from Theorem 3.3 that there needs to exist a positive circuit in the corresponding interaction graph. We now demonstrate it as follows.

The above computations show that the local interaction graph $G(F'(0,0,0))$ thus contains five edges, one from vertex 2 to vertex 1, another from vertex 2 to vertex 3 and the others are three loops at vertices 1, 2 and 3. As $x = (0,0,0)$, $x_3 = f_3(x)$, $x_2 = f_2(x)$ and $x_1 \neq f_1(x)$, Lemma 3.1 shows that the two loops at vertices 2 and 3 are positive and the loop at a vertex 1 is negative. Since $x_2 = f_3(x)$ and $x_2 \neq f_1(x)$, it follows that the local interaction graph associated to state $(0,0,0)$ contains one positive edge from vertex 2 to vertex 3 and one negative edge from vertex 2 to vertex 1.

Therefore, the local interaction graph $G(F'(0,0,0))$ is



5.2. *Remarks about Theorems* 3.2 *and* 3.3

We make the following remark. The conclusion of Theorems 3.2 and 3.3 is not a sufficient condition for the prerequisite in either theorem. So to conclude this paper let us give two examples to illustrate these. For instance, the Boolean synchronous dynamics given in the following Example 5.3 for $n = 2$ has exactly one non-attractive cycle, whereas the local interaction graph associated to state $(0,0,0)$ contains a positive loop at a vertex 1. Likewise, the Boolean synchronous dynamics given in the following Example 5.4 for $n = 2$ has no attractive cycles, whereas the local interaction graph associated to state $(0,0,0)$ is a positive circuit of length 2.

Example 5.3. Let $F : \{0,1\}^2 \to \{0,1\}^2$ be defined by

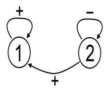
$$F(x) = \begin{pmatrix} \bar{x}_1 x_2 + x_1 \bar{x}_2 \\ \bar{x}_2 \end{pmatrix}, \quad (x \in \{0,1\}^2).$$

Let us compute the map F at each state $x \in \{0,1\}^2$ and the Boolean Jacobian matrix $F'(0,0)$. Then, we obtain $F(0,0) = (0,1)$, $F(0,1) = (1,0)$, $F(1,0) = (1,1)$, $F(1,1) = (0,0)$, and so the Boolean synchronous dynamics for F has neither fixed points nor attractive cycles, but has exactly one non-attractive cycle $\{(0,0),(0,1),(1,0),(1,1)\}$. Moreover,

$$F'(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The local connectivity graph $\Gamma(F'(0,0))$ thus contains three edges, one from vertex 2 to vertex 1, and the others are two loops at vertices 1 and 2. By definition of the local interaction graph, we obtain that $G(F'(0,0))$ contains an edge from vertex 2 to vertex 1, and two loops at vertices 1 and 2. As $x = (0,0)$, $x_1 = f_1(x)$ and $x_2 \neq f_2(x)$, Lemma 3.1 shows that the loop at a vertex 1 is positive and the loop at a vertex 2 is negative. Since $x_2 = f_1(x)$, it follows that the edge from vertex 2 to vertex 1 is positive.

Therefore, the local interaction graph $G(F'(0,0))$ is



It contains a positive loop at a vertex 1.

Example 5.4. Let $F : \{0,1\}^2 \to \{0,1\}^2$ be defined by

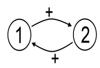
$$F(x) = \begin{pmatrix} \bar{x}_1 x_2 \\ x_1 \bar{x}_2 \end{pmatrix}, \quad (x \in \{0,1\}^2).$$

Let us compute the map F at each state $x \in \{0,1\}^2$ and the Boolean Jacobian matrix $F'(0,0)$. Then, we obtain $F(0,0) = (0,0)$, $F(0,1) = (1,0)$, $F(1,0) = (0,1)$, $F(1,1) = (0,0)$, and so the Boolean synchronous dynamics for F has no attractive cycles, but has a unique fixed point $(0,0)$ and a single non-attractive cycle $\{(0,1),(1,0)\}$. Moreover,

$$F'(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The local connectivity graph $\Gamma(F'(0,0))$ thus contains two edges, one from vertex 1 to vertex 2 and the other from vertex 2 to vertex 1. By definition of the local interaction graph, we obtain that $G(F'(0,0))$ contains exactly one circuit $(1,2)$ of length 2. As $x = (0,0)$, $x_1 = f_1(x)$ and $x_2 = f_2(x)$, Lemma 3.1 shows that this circuit $(1,2)$ is positive.

Therefore, the local interaction graph $G(F'(0,0))$ is



It corresponds to a positive circuit of length 2.

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