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Some Special V_4-magic Graphs

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Abstract. For any abelian group A, a graph $G = (V, E)$ is said to be A-magic if there exists a labeling $l : E(G) \to A - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to A$ defined by $l^+ := \sum\{l(uv)/uv \in E(G)\}$ is a constant map. In this paper, we consider the Klein-four group $V_4 = Z_2 \oplus Z_2$ and investigate graphs that are V_4-magic

1. Introduction

For any abelian group A, written additively, any mapping $l : E(G) \to A - \{0\}$ is called a labeling. Given a labeling on the edge set of G, one can introduce a vertex set labeling $l^+ : V(G) \to A$ as follows: $l^+(v) = \Sigma\{l(uv)/uv \in E(G)\}$. A graph G is said to be A-magic if there is a labeling $l : E(G) \to A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$.

The original concept of A-magic graph is due to Sedlacek [1, 2], who defined it to be a graph with a real-valued edge labeling such that

(1) distinct edges have distinct nonnegative labels; and
(2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Observation 1.1. Any regular graph is fully magic.

Observation 1.2. If G is A-magic, then so is $G \times K_2$, hence so is $G \times Q_n$.

Observation 1.3. For any $n \geq 3$, the path of order n is non-magic.

Observation 1.4. C_4, the cycle of order four, with a pendant edge is non-magic. In fact, any even cycle C_{2n} with a pendant edge is non-magic.

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2. Existing results

Theorem 2.1 ([3]). *A tree T is V_4-magic if and only if all its vertices have odd degrees.*

Theorem 2.2 ([3]). *For $m, n \geq 2$, the complete bipartite graph $K(m, n)$ is V_4-magic.*

Theorem 2.3 ([3]). *For any $n > 3$, K_n - e, the complete graph with one edge removed, is V_4-magic.*

Theorem 2.4 ([3]). (a) *Any even cycle with k pendant edges is V_4-magic if and only if k is even.*
(b) *Any odd cycle with k pendant edges is V_4-magic if and only if k is odd.*

Theorem 2.5 ([3]). *The wheel W_n is V_4-magic ($n \geq 3$).*

3. Main results

Definition 3.1. A *shell* $S_{n,n-3}$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n on n vertices. The vertex at which all the chords are concurrent is called *apex*. The two vertices adjacent to the apex have degree 2, apex has degree $n-1$ and all the other vertices have degree 3.

Theorem 3.2. *Shell graphs $S_{n,n-3}$ are V_4-magic.*

Proof. ***Case* 1.** n is even.

Let $n = 2r+2$. Then the number of edges is $4r+1$. Number of chords are $n-3$. Let the vertices and edges be as follows:

$$V(S_{n,n-3}) = \{a_0, b_0, a_1, b_1, \ldots, a_r, b_r\},$$
$$E(S_{n,n-3}) = \{a_i b_i, a_i b_{i-1}/0 \leq i \leq r\} \cup \{b_r v/v \neq a_0, a_r\}.$$

Label the edges as $l(b_r v) = c$, $v \neq a_0$, $a_r, l(b_r a_r) = l(a_r b_{r-1}) = a$. Then label all the edges $b, a, b, a, \ldots,$ up to $b_0 a_0$. Then $l(a_0 b_r) = b$. So $l^+(b_r) = a + b + c + (n-2)c$, $l^+(a_0) = 2b = 0$. $l^+(a_r) = 2a = 0$. $l^+(b_i) = a + b + c = 0$, $i = 0, 1, 2, \ldots, r-1$. $l^+(a_i) = a + b + c = 0$, $i = 1, 2, \ldots, r-1$.

Thus $l^+(v) = 0$ for all vertices.

***Case* 2.** n is odd.

Let $n = 2r+3$. Number of chords is $n-3$. $V(S_{n,n-3}) = \{a_0, b_0, \ldots, a_r, b_r\} \cup \{a_{r+1}\}$, $E(S_{n,n-3}) = \{a_i b_i, a_i b_{i-1}/1 \leq i \leq r\} \cup \{a_{r+1} v/v \neq a_0, b_r\} \cup \{a_0 b_0, a_0 a_{r+1}\}$.

The labeling of edges is as follows:

$l(a_{r+1} b_r) = l(a_r b_r) = a$. Then consecutively we label the edges by $b, a, b, a, \ldots$ up to the edge $b_0 a_0$. Then $l(a_0 a_{r+1}) = a$, $l(a_{r+1} v) = c$, $v \neq a_0, b_r$. Then $l^+(a_r) = 2rc + 2a = 0$. $l^+(a_i) = a + b + c = 0$, for $i = 1, 2, \ldots, r$, $l^+(b_i) = a + b + c = 0$, for $i = 0, 1, 2, \ldots, r-1$, $l^+(a_0) = 2a = 0$, $l^+(b_r) = 2a = 0$. Then $l^+(v) = 0$ for all vertices. Hence, shell graphs are V_4-Magic. □

Definition 3.3. For positive integers n, k, $1 \le k \le n-3$, the family $C(n,k)$ is the family of graphs obtained by taking k concurrent chords in a cycle C_n on n vertices. In general $C(n,k)$ consists of many graphs. The shell graph $S_{n,n-3}$ is the unique member of $C(n, n-3)$. If we take maximum number of alternate concurrent chords, then for $n = 2s$ there is unique such graph. It belongs to $C(2s, s-1)$. For $n = 2s+1$, we cannot take maximum number of alternate chords without taking some consecutive ones. Here we are interested in alternate chords symmetrically placed on two sides of the apex. If $n \equiv 1 \mod 4$, we have to take two consecutive chords exactly in the middle. This graph is denoted by $S_{4t+1,2t}$. If $n \equiv 3 \mod 4$,we have to take four consecutive chords in the middle. This graph is denoted by $S_{4t+3,2t+2}$.

Theorem 3.4. *The graph $C(2s, s-1)$, with alternative concurrent chords is V_4-magic.*

Proof. ***Case 1.*** $s = 2t$.

Let the graph be denoted by $S_{4t,2t-1}$. This graph has $4t$ vertices and $6t-1$ edges. Let the vertex set be $\{a_0, b_0, \ldots, a_{2t-1}, b_{2t-1}\}$, with the cycle $C = (a_0, b_0, \ldots, a_{2t-1}, b_{2t-1})$. Let a_0 be the apex vertex with the chords $a_0a_1, a_0a_2, \ldots, a_0a_{2t-1}$. Label the edges as $l(a_0a_i) = c$, for $i = 1, 2, \ldots, 2t-1$. And label the edges of C as $a, a, b, b, \ldots$, starting with $a_0b_0, b_0a_1, a_1b_1, b_1a_2, \ldots$. Then $l^+(a_0) = a + b + c + (2t-2)c = 0$, $l^+(a_i) = a + b + c = 0$, $i = 1, 2, \ldots, 2t-1$, $l^+(b_i) = a + a = 0$, $i = 0, 2, 4, \ldots, 2t-2$, $l^+(b_i) = b + b = 0$, $i = 1, 3, 5, \ldots, 2t-1$. Thus $l^+(v) = 0$ for all $v \in V$.

Case 2. $s = 2t+1$.

Let the graph be denoted by $S_{4t+2,2t}$. This graph has $4t+2$ vertices and $6t+2$ edges. Let the vertex set be $\{a_0, b_0, \ldots, a_{2t}, b_{2t}\}$, with the cycle $C = (a_0, b_0, \ldots, a_{2t}, b_{2t})$ and the chords are $a_0a_1, a_0a_2, \ldots, a_0a_{2t}$.

Label the edges as $l(a_0a_i) = c$, for $i = 1, 2, \ldots, 2t$. And edges of C as $a, a, b, b, \ldots$, starting with a_0b_0, b_0a_1, a_1b_1, $b_1a_2, \ldots$. Then $l^+(a_0) = a+a+2tc = 0$, $l^+(a_i) = a + b + c = 0$, for $i = 1, 2, \ldots, 2t$, $l^+(b_i) = a + a = 0$, for $i = 0, 2, 4, \ldots, 2t$. $l^+(b_i) = b + b = 0$, for $i = 1, 3, 5, \ldots, 2t-1$. Thus $l^+(v) = 0$ for all the vertices. Hence the graph is V_4-magic. □

Theorem 3.5. *The graphs $S_{4t+1,2t}$, $S_{4t+3,2t+2}$, $t \ge 1$ are V_4-magic.*

Proof. ***Case 1.*** $n = 4t+1$.

Consider the odd cycle on the vertices $a_0, b_0, \ldots, a_{2t-1}, b_{2t-1}, a_{2t}$ with $2t$ chords $a_ta_i/i \ne t$. Label the edges as follows: $l(a_ta_i) = c$ for $i \ne t$ and $l(a_0a_{2t}) = a$. From the edge a_0a_{2t}, we label as $b, b, a, a, b, b, \ldots$, on both sides up to the vertex a_t. Then $l^+(a_t) = 2tc + 2b = 0$ if t is odd, $2tc + 2a = 0$ if t is even. $l^+(a_i) = a + b + c = 0$, $i = 0, 1, 2, \ldots, t-1$, $t+1$, $\ldots$, $2t$. $l^+(b_i) = 2b = 0$ or $l^+(b_i) = 2a = 0$, $i = 0, 1, 2, \ldots, 2t-1$. Thus $l^+(v) = 0$ for all the vertices.

***Case* 2.** $n = 4t + 3$.

Let $V = \{a_0, b_0, a_1, b_1, \ldots, a_{2t}, b_{2t}, 4a_{2t+1}\}$. The edge set is given by $E = \{a_i b_i, b_i a_{i+1}, 0 \le i \le 2t\} \cup \{a_{2t+1} a_0\} \cup \{a_0 a_i, a_0 b_{t-1+i}, 1 \le i \le t+1\}$. There are $2t+2$ chords. Label all the chords as c. $l(a_0 a_i) = l(a_0 b_{t-1+i}) = c, 1 \le i \le t+1$, and $l(b_r a_{r+1}) = a$. On both sides of $b_r a_{r+1}$ label the edges as $b, a, a, b, b, a, a, b, b, \ldots$, on both sides up to a_0.

Then

$$l^+(a_0) = \begin{cases} (2t+2)c + 2a, & \text{if } t \text{ is odd,} \\ (2t+2)c + 2b, & \text{if } t \text{ is even.} \end{cases}$$

$$l^+(a_i) = \begin{cases} a + b + c, & \text{if } i = 1, 2, \ldots, t+1. \\ 2a \text{ or } 2b, & \text{if } i = t+2, \ldots, 2t+1. \end{cases}$$

$$l^+(b_i) = 2b \text{ or } 2a, \quad \text{if } i = 0, 1, 2, \ldots, t-1.$$

$$l^+(b_i) = a + b + c, \quad \text{if } i = t, t+1, \ldots, 2t.$$

Then $l^+(v) = 0$ for all the vertices. □

Theorem 3.6. *The graph $S_{2s+1,s}$ is V_4-magic.*

Proof. Let the vertex set be $\{a_0, b_0, a_1, b_1, \ldots, a_{s-1}, b_{s-1}, a_s\}$ with cycle $C = \{a_0, b_0, a_1, b_1, \ldots, b_{s-1}, a_s\}$. The chords are $\{a_0 a_i, 1 \le i \le s-1\}$ and $a_0 b_{s-1}$. Label the chords as $l(a_0 a_i) = l(a_0 b_{s-1}) = c$, $i = 1, 2, \ldots, s-1$ and $l(a_{s-1} b_{s-1}) = a$. Both sides of $a_{s-1} b_{s-1}$, we label the edges $b, b, a, a, \ldots$, up to a_0. Then $l^+(a_0) = sc + 2b$ if s is even, $(s-1)c + a + b + c$ if s is odd. $l^+(a_i) = a + b + c$, $i = 1, 2, \ldots, s-1$, $l^+(a_s) = 2b$, $l^+(b_{s-1}) = a + b + c$. $l^+(b_i) = 2a$ or $2b$, $i = 0, 1, 2, \ldots, s-2$. Then $l^+(v) = 0$ for all the vertices. □

Definition 3.7. A *snake graph* is formed by taking n copies of a cycle C_m and identifying exactly one edge of each copy to a distinct edge of the path P_{n+1}, which we will call the backbone of the snake. We will use $T_n^{(m)}$ to denote this snake graph.

Theorem 3.8. *All snake graphs $T_n^{(m)}$ are V_4-magic.*

Proof. Label all the edges as a or b.

Then $l^+(v) = 0$ for all the vertices; Otherwise label all the edges of first cycle C_m as a and the edges of second cycle C_m as b. By labeling the edges of cycles as a and b alternatively, every vertex of degree four have $l^+(v) = 2a + 2b = 0$. Other vertices of degree two has $l^+(v) = 2a$ or $2b = 0$. □

Definition 3.9. Let $\{(G_i, x_i, y_i)\}$ be a finite collection of graphs, each with a fixed edge which is oriented, Then the edge amalgamation Edgeamal $\{(G_i, x_i, y_i)\}$ is formed by taking the union of all the G and identifying their fixed edges, all with the same orientation.

When we consider the edge amalgamation of cycles, we have a generalization of the book graph $S_n \times P_2$. When $\{G_i\}$ is a collection of cycles, we call Edgeamal

$\{(G_i, x_i, y_i)\}$ a generalized book. The spine xy of the generalized book is the edge we obtain from the identification of the edges x_i, y_i and each cycle G_i containing this edge is called a page. For each page $G_i = xyv_1v_2\dots v_{ki}x$ of length k_i+2,we say that $v_1, v_2, \dots, v_{ki}$ is the non spine path of a cycle.

Note that for edge amalgamation of collections of cycles, the choice of edges and orientations is irrelevant. For this reason, we simply use Edgeamal $\{G_i\}$ to denote the edge amalgamation of a collection of cycles.

Theorem 3.10. *All generalized books are V_4-magic.*

Proof. Let G be a generalized book. We consider the following cases.

If the number of pages is odd, then all the vertices of generalized books will be even. We label all the edges by a. Then $l^+(v) = 0$.

If the number of pages is even, we can label the common edge by a, the other edges of first page by b, and all other pages by c. Then $l^+(v) = 0$. □

Corollary. *Let G be a generalized book with P_m as a spine, where $m \geq 2$. Then G is V_4 magic.*

Theorem 3.11. *$C_n^{(t)}$, one point union of t cycles each of length n is V_4-magic whenever t is odd or even.*

Proof. If t is odd, label the edges of first cycle by a, second cycle by b, and the remaining cycles by c. Then $l^+(v) = 0$.

If t is even, label the edges of all cycles by a. Then $l^+(v) = 0$. □

Theorem 3.12. *Ladders L_{n+2} with n steps are V_4-magic.*

Proof. Let $u_0, u_1, \dots, u_{n+1}$ and $v_0, v_1, \dots, v_{n+1}$ be the vertices of a ladder G such that $E(G) = \{u_iu_{i+1}/i = 0,1,2,\dots,n\} \cup \{v_jv_{j+1}/j = 0,1,2,\dots,n\} \cup \{u_iv_i/i = 1,2,\dots,n\}$. Label all the edges by a. Then $l^+(v) = a$. □

Theorem 3.13. *Ladders $P_2 \times P_n$ is V_4-magic.*

Proof. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, v_3, \dots, v_n$ be the vertices of a ladder L_n such that $E(G) = \{u_iu_{i+1}/i = 1,2,\dots,n\} \cup \{v_jv_{j+1}/j = 1,2,\dots,n\} \cup \{u_iv_i/i = 1,2,\dots,n\}$.

***Case* 1.** n is odd.
Label the edges as follows:
$l(u_1v_1) = l(u_iu_{i+1}) = l(v_iv_{i+1}) = a$, for $i = 1,3,\dots,n-2$, $l(u_nv_n) = l(u_iu_{i+1}) = l(v_iv_{i+1}) = b$, for $i = 2,4,\dots,n-1, l(u_iv_i) = c$, for $i = 2,\dots,n-1$.Then $l^+(u_1) = l^+(v_1) = 2a = 0$, $l^+(u_n) = l^+(v_n) = 2b = 0$ and $l^+(v) = a + b + c = 0$ for all other vertices.

***Case* 2.** n is even.
Label the edges as follows:
$l(u_1v_1) = l(u_nv_n) = l(u_iu_{i+1}) = l(v_iv_{i+1}) = a$, for $i = 1,3,5,\dots,n-1$, $l(u_iu_{i+1}) = l(v_iv_{i+1}) = b$, for $i = 2,4,\dots,n-2$ and $l(u_iv_i) = c$, for $i = 2,3,\dots,n-1$. Then

$l^+(u_1) = l^+(v_1) = l^+(v_n) = l^+(u_n) = 2a = 0$ and $l^+(v) = a + b + c = 0$ for remaining vertices. □

Definition 3.14. The graph G with the vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i u_{i+1}, v_i v_{i+1}, v_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\}$ is called *a semi ladder* of length n.

Theorem 3.15. *Semi ladders are V_4-magic.*

Proof. Let G be a semi ladder of length n. Then G has $2n$ vertices and $4n-3$ edges. Label the edges as follows:
$l(u_1 v_1) = l(u_i u_{i+1}) = a$, for $i = 1, 2, \ldots, n-1$, $l(u_n v_n) = l(v_i v_{i+1}) = b$, for $i = 1, 2, \ldots, n-1, l(u_i v_i) = c$, for $i = 2, 3, \ldots, n-1$ and $l(v_i u_{i+1}) = c$, for $i = 1, 2, \ldots, n-1$. Then $l^+(u_1) = 2a = 0$, $l^+(v_n) = 2b = 0$, $l^+(u_i) = 2a + 2c = 0$, for $2 \le i \le n-1, l^+(u_n) = l^+(v_1) = a + b + c = 0$ and $l^+(vi) = 2b + 2c = 0$, for $2 \le i \le n-1$. □

Definition 3.16. The composition of two graphs $G[H]$ has $V(G) \times V(H)$ as vertex set in which (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

Theorem 3.17. *The composition $P_n[K_2^c]$ is V_4-magic.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. Let x, y be the vertices of K_2^c. Denote the vertex (v_i, x) of $P_n[K_2^c]$ by u_i, and (v_i, y) by $u_i', 1 \le i \le n$. The size of P_n $[K_2^c]$ is given by $q = 4n - 4$.

Label the edges as follows:
$l(u_i u_{i+1}) = l(u_i' u_{i'+1}) = a$, for $i = 1, 2, \ldots, n-1$ and $l(u_i u_{i'+1}) = l(u_i' u_{i+1}) = a$, for $i = 1, 2, \ldots, n-1$. Then $l^+(u_i) = l^+(u_i') = 4a = 0$, for $2 \le i \le n-1$ and $l^+(u_1) = l^+(u_n) = l^+(u_1') = l^+(u_n') = 2a = 0$. Hence $P_n[K_2^c]$ is V_4-magic. □

Note. Paths are not V_4-magic. But Cartesian product of paths $P_m \times P_n$ is V_4-magic.

Theorem 3.18. *The planar grid $P_m \times P_n$ is V_4-magic.*

Proof. The planar gird $P_m \times P_n$, $m, n \ge 2$ contains mn vertices and $2mn - (m+n)$ edges. Note that 4 vertices are of degree 2 each, $2(m+n-4)$ vertices are of degree 3 each and $(m-2)(n-2)$ are of degree 4 each. Let $V(P_m \times P_n) = \{u_{ij}, 1 \le i \le n, 1 \le j \le m\}$. $E(P_m \times P_n) = \{u_{ij} u_{i(j+1)} : 1 \le i \le n, 1 \le j \le m-1\} U \{u_{ij} u_{(i+1)j} : 1 \le i \le n-1, 1 \le j \le m\}$.

***Case* 1.** Both m and n are even.

Label the edges as follows:
$l(u_{1j} u_{1j+1}) = l(u_{i1} u_{(i+1)1}) = b$, for $j = 1, 3, 5, \ldots, m-1$, $i = 1, 3, 5, \ldots, n-1$, $l(u_{1j} u_{1(j+1)}) = l(u_{i1} u_{(i+1)1}) = c$, for $j = 2, 4, \ldots, m-2$, $i = 2, 4, 6, \ldots, n-2$, $l(u_{im} u_{(i+1)m}) = l(u_{nj} u_{n(j+1)}) = b$, for $i = 1, 3, \ldots, n-1$, $j = 1, 3, 5, \ldots, m-1$, $l(u_{im} u_{(i+1)m}) = l(u_{nj} u_{n(j+1)}) = c$, for $i = 2, 4, \ldots, n-2$, $j = 2, 4, \ldots, m-2$ and

$l(e) = a$ for all remaining edges. Then $l^+(u_{11}) = l^+(u_{1m}) = l^+(u_{n1}) = l^+(u_{nm}) = 2b = 0$, $l^+(u_{1j}) = a+b+c = 0$, $2 \leq j \leq m-1$, $l^+(u_{nj}) = a+b+c = 0$, $2 \leq j \leq m-1$, $l^+(u_{i1}) = a+b+c = 0$, $2 \leq i \leq n-1$ and $l^+(u_{im}) = a+b+c = 0$, $2 \leq i \leq n-1$. For the remaining vertices $l^+(v) = 2a = 0$.

***Case* 2.** Both m and n are odd.

Label the edges as follows:
$l(u_{1j}u_{1(j+1)}) = l(u_{i1}u_{(i+1)1}) = b$, for $i = 1,3,5,\ldots,n-2$, $j = 1,3,5,\ldots,m-2$, $l(u_{1j}u_{(1j+1)}) = l(u_{i1}u_{(i+1)1}) = c$, for $i = 2,4,\ldots,n-1, j = 2,4,\ldots,m-1$, $l(u_{im}u_{(i+1)m}) = l(u_{nj}u_{n(j+1)}) = c$, for $i = 1,3,5,\ldots,n-2, j = 1,3,5,\ldots,m-2$, $l(u_{im}u_{(i+1)m}) = l(u_{nj}u_{n(j+1)}) = b$, for $i = 2,4,\ldots,n-1, j = 2,4,6,\ldots,m-1$ and $l(e) = a$ for all remaining edges. Then $l^+(u_{11}) = l^+(u_{nm}) = 2b = 0$, $l^+(u_{n1}) = l^+(u_{1m}) = 2c = 0$, $l^+(u_{1j}) = l^+(u_{nj}) = a+b+c = 0$, $2 \leq j \leq m-1$, $l^+(u_{i1}) = l^+(u_{im}) = a+b+c = 0$, $2 \leq i \leq n-1$. For the remaining vertices $l^+(v) = 2a = 0$.

***Case* 3.** m is odd and n is even.

Label the edges as follows:
$l(u_{1j}u_{1(j+1)}) = l(u_{i1}u_{(i+1)1}) = b$, for $i = 1,3,\ldots,n-1, j = 1,3,\ldots,m-2$, $l(u_{1j}u_{1j+1}) = l(u_{i1}u_{(i+1)1}) = c$, for $i = 2,4,6,\ldots,n-2$, $j = 2,4,\ldots,m-1$, $l(u_{nj}u_{n(j+1)}) = l(u_{im}u_{(i+1)m}) = b$, for $j = 1,3,5,\ldots,m-2$, $i = 2,4,6,\ldots,n-2$, $l(u_{nj}u_{n(j+1)}) = l(u_{im}u_{(i+1)m}) = c$, for $j = 2,4,\ldots,m-1$, $i = 1,3,\ldots,n-1$ and $l(e) = a$ for the remaining edges. Then $l^+(u_{11}) = l^+(u_{n1}) = 2b = 0$, $l^+(u_{1m}) = l^+(u_{nm}) = 2c = 0, l^+(u_{nj}) = l^+(u_{1j}) = a+b+c = 0$, $2 \leq j \leq m-1$ and $l^+(u_{im}) = l^+(u_{i1}) = a+b+c = 0$, for $2 \leq i \leq n-1$. For the remaining vertices $l^+(v) = 2a = 0$.

***Case* 4.** m is even and n is odd.

Label the edges as follows:
$l(u_{1j}u_{1(j+1)}) = l(u_{i1}u_{(i+1)1}) = b$, for $j = 1,3,m-1$, $i = 1,3,\ldots,n-2$, $l(u_{1j}u_{1(j+1)}) = l(u_{i1}u_{(i+1)1}) = c$, for $j = 2,4,\ldots,m-2, i = 2,4,\ldots,n-1$, $l(u_{nj}u_{n(j+1)}) = l(u_{im}u_{(i+1)m}) = c$, for $j = 1,3,\ldots,m-1$, $i = 2,4,\ldots,n-1$, $l(u_{nj}u_{n(j+1)}) = l(u_{im}u_{(i+1)m}) = b$, for $j = 2,4,\ldots,m-2$, $i = 1,3,5,\ldots,n-2$ and $l(e) = a$ for the remaining edges. Then $l^+(u_{11}) = l^+(u_{1m}) = 2b = 0$, $l^+(u_{n1}) = l^+(u_{nm}) = 2c = 0$, $l^+(u_{1j}) = l^+(u_{nj}) = a+b+c = 0$, $2 \leq j \leq m-1$ and $l^+(u_{im}) = l^+(u_{i1}) = a+b+c = 0$, $2 \leq i \leq n-1$. For the remaining vertices $l^+(v) = 2a = 0$. Hence the planar grid is V_4-magic. □

Definition 3.19. The sequential join of graphs $G_1, G_2, \ldots, G_n$ is formed from $G_1 \cup G_2 \cup \ldots \cup G_n$ by adding edges joining each vertex of G_i with each vertex of G_{i+1} for $1 < i < n-1$.

Theorem 3.20. *The sequential join of m copies of $K_2, m > 2$ is V_4-magic.*

Proof. The sequential join of m copies of K_2 contains $2m$ vertices and $5m-4$ edges. $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$. $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : i = 1, 2, \dots, n\} \cup \{v_i u_{i+1}, u_i v_{i+1} : i = 1, 2, \dots, n-1\}$

Label the edges as follows:

$l(u_i v_i) = l(u_i v_{i+1}) = l(u_i u_{i+1}) = l(v_i v_{i+1}) = l(v_i u_{i+1}) = c$.

Then $l^+(u_1) = l^+(v_1) = l^+(u_n) = l^+(v_n) = 3c = c$, $l^+(u_i) = l^+(v_i) = 5c = c$, $2 \leq i \leq n-1$. □

Definition 3.21. A comb is a graph obtained by joining a single edge to each vertex of a path.

Theorem 3.22. *Comb is not V_4-magic.*

Proof. Let $P_n = (u_1, u_2, \dots, u_n)$ and v_i be the pendent vertex attached to u_i, $1 \leq i \leq n$. Suppose comb is V_4-magic, then $l^+(v_i) = l^+(u_i)$, $1 \leq i \leq n$. Hence $l(u_1 v_1) = l(u_1 v_1) + l(u_1 u_2) = 0$. This implies $l(u_1 u_2) = 0$. which is a contradiction. Hence comb is not V_4-magic. □

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