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On New Linear Operator Associated with Gaussian Hypergeometric Functions

Research Article

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Abstract. In the present paper, we introduce new classes $\sum_n^*(\alpha,\beta,k,\rho)$ and $\sum_n(\alpha,\beta,k,\rho)$ of meromorphic functions defined by means of the linear operator $L^*(\alpha,\beta)f(z)$ for function in $\mathbb{U}^*=\{z:0<|z|<1\}$ and investigate a number of inclusion relationships of these classes. We also derive some interesting properties of these classes.

Keywords. Meromorphic functions; Hadamard product; Linear operator; Functions with positive real part; Integral operator

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1. Introduction

Let $\sum$ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the punctured unit disk

$$\mathbb{U}^*=\{z:z\in\mathbb{C}\text{ and }0<|z|<1\}=\mathbb{U}\setminus\{0\},$$

$\mathbb{C}$ being (as usual) the set of complex numbers. We denote by $\sum S^*(\beta)$ and $\sum K(\beta)$ $(\beta \geqq 0)$ the subclasses of $\sum$ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in $\mathbb{U}^*$ (see also the recent works [17] and [18]).

For functions $f_j(z)$ $(j = 1,2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \quad (j = 1,2), \tag{1.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{1.3}$$

Let us consider the function $\widetilde{\phi}(\alpha,\beta;z)$ defined by

$$\widetilde{\phi}(\alpha,\beta;z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} a_n z^n \quad \left(\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \alpha \in \mathbb{C}\right), \tag{1.4}$$

where

$$\mathbb{Z}_0^- = \{0,-1,-2,\cdots\} = \mathbb{Z}^- \cup \{0\}.$$

Here, and in the remainder of this paper, $(\lambda)_\kappa$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa = n \in \mathbb{N};\ \lambda \in \mathbb{C}) \\ 1 & (\kappa = 0;\ \lambda \in \mathbb{C}\setminus\{0\}), \end{cases} \tag{1.5}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [19, p. 21 *et seq.*]), $\mathbb{N}$ being the set of positive integers.

It is easy to see that, in the case when $a_k = 1$ $(k = 0,1,2,\cdots)$, the following relationship holds true between the function $\widetilde{\phi}(\alpha,\beta;z)$ and the Gaussian hypergeometric function [15]:

$$\widetilde{\phi}(\alpha,\beta;z) = \frac{1}{z}\, {}_2F_1(1,\alpha;\beta;z). \tag{1.6}$$

where

$${}_2F_1\left(b,\alpha,\beta;z\right) = \sum_{n=0}^{\infty} \frac{(b)_n\,(\alpha)_n}{(\beta)_n} \frac{z^n}{n!}$$

is the well-known Gaussian hypergeometric function. Corresponding to the function $\tilde{\phi}(\alpha,\beta;z)$, using the Hadamard product for $f(z) \in \sum$, we define a new linear operator $L^*(\alpha,\beta)$ on $\sum$ by

$$L^*\left(\alpha,\beta\right) f(z) = \tilde{\phi}\left(\alpha,\beta;z\right) * f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left|\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right| a_n z^n. \tag{1.7}$$

The meromorphic functions related to the generalized and Gaussian hypergeometric functions were considered recently by Cho and Kim [1], Dziok and Srivastava [2], Liu [7], Liu and Srivastava [8], see also [17] and [18]).

For a function $f \in L^*(\alpha,\beta) f(z)$ we define

$$\mathscr{D}^0 \left(L^*(\alpha,\beta) f(z)\right) = L^*(\alpha,\beta) f(z),$$

and for $k = 1,2,3,\dots$,

$$\mathscr{D}^k \left(L^*(\alpha,\beta) f(z)\right) = z\left(\mathscr{D}^{k-1} L^*(\alpha,\beta) f(z)\right)' + \frac{2}{z}$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} n^k \left|\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right| a_n z^n. \tag{1.8}$$

We note that $\mathscr{D}^k$ in (1.6) was studied by Ghanim and Darus [3], [4], [5] and [6].

It follows from (1.7) that

$$z\left(L^*(\alpha,\beta)f(z)\right)' = \alpha L^*(\alpha+1,\beta)f(z) - (\alpha+1)L^*(\alpha,\beta)f(z). \tag{1.9}$$

Also, from (1.9) we get

$$z\left(\mathscr{D}^k L^*(\alpha,\beta)f(z)\right)' = \alpha \mathscr{D}^k L^*(\alpha+1,\beta)f(z) - (\alpha+1)\, \mathscr{D}^k L^*(\alpha,\beta)f(z). \tag{1.10}$$

Let Ω be the class of all analytic, convex and univalent functions $h(z)$ in the open unit disk satisfying $h(0) = 1$ and

$$\Re\{h(z)\} > 0, \quad |z| < 1. \tag{1.11}$$

For two functions $f, g \in \Omega$, we say that f is *subordinate* to g or g is *superordinate* to f in $\mathbb{U}$ and write $f \prec g$, $z \in \mathbb{U}$, if there exist a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|w(z)| \le 1$ when $z \in \mathbb{U}$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U), \quad (z \in \mathbb{U}).$$

Let $P_n(\rho)$ be the class of functions $p(z)$ analytic in $\mathbb{U}^*$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left|\frac{\Re\left(p(z)\right) - \rho}{1-\rho}\right| d\theta \le n\pi, \tag{1.12}$$

where $z = re^{i\theta}$, $n \ge 2$ and $0 \le \rho < 1$. This class has been introduced in [11]. We note that $P_n(0) = P_n$ [13] and $P_2(\rho) = P(\rho)$ [10], the class of analytic functions with positive real part greater than ρ and $P_2(0) = P$, the class of functions with positive real part. From (1.12) we can write $p \in P_n(\rho)$ as

$$p(z) = \left(\frac{n}{4} + \frac{1}{2}\right) p_1(z) + \left(\frac{1}{2} - \frac{n}{4}\right) p_2(z), \tag{1.13}$$

where $p_i(z) \in P(\rho)$, $i = 1,2$ and $z \in \mathbb{U}^*$.

Making use of the operator $\mathscr{D}^k L^*(\alpha,\beta) f(z)$, we introduce some new classes of meromorphic functions in the punctured unit disk $\mathbb{U}^*$.

Definition 1.1. Let $f \in \sum$. Then $f \in \sum_n^*(\alpha,\beta,k,\rho)$, if and only if

$$-(1-\alpha)z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' - \alpha z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' \in P_n(\rho)$$

where $\alpha > 0, n \ge 2, 0 \le \rho < 1$ and $z \in \mathbb{U}^*$.

Definition 1.2. Let $f \in \sum$. Then $f \in \sum_n(\alpha,\beta,k,\rho)$, if and only if

$$(1-\alpha)z\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' - \alpha z\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' \in P_n(\rho)$$

where $\alpha > 0$, $n \ge 2$, $0 \le \rho < 1$ and $z \in \mathbb{U}^*$.

2. Preliminary Results

Lemma 2.1 ([14]). *Let $p(z)$ be analytic in $\mathbb{U}^*$ with $p(0) = 1$, α be a complex number with $\Re(\alpha) \ge 0$ ($\alpha \ne 0$) be such that*

$$\Re\left\{p(z) + \alpha z p'(z)\right\} > \beta \quad (0 \le \beta < 1).$$

Then

$$\Re\big(p(z)\big) > \beta + (1-\beta)(2\gamma - 1),$$

where γ is given by

$$\gamma = \gamma(\alpha) = \int_0^1 \left(1 + t^{\Re(\alpha)}\right)^{-1} dt,$$

which is an increasing function of $\Re(\alpha)$ and $\frac{1}{2} \le \gamma < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.2 ([16]). *If $p(z)$ is analytic in $\mathbb{U}^*$, $p(0) = 1$ and $\Re\big(p(z)\big) > \frac{1}{2}$, $z \in \mathbb{U}^*$, then for any function F analytic in $\mathbb{U}^*$, the function $p * F$ takes values in the convex hull of the image of $\mathbb{U}^*$ under F.*

Lemma 2.3 ([12]). *Let $p(z) = 1 + b_1 z + b_2 z_2 + \cdots \in P(\rho)$. Then*

$$\Re\big(p(z)\big) \ge 2\rho - 1 + \frac{2(1-\rho)}{1+|z|}.$$

3. Main Results

Theorem 3.1. *Let $f \in \sum_n^*(\alpha,\beta,k,\rho)$. Then*

$$-z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' \in P_n\left(\rho_1\right)$$

where ρ_1 is given by

$$\rho_1 = \rho + (1-\rho)(2\gamma - 1), \tag{3.1}$$

and

$$\gamma = \int_0^1 \left(1 + t^{\Re(\alpha)}\right)^{-1} dt.$$

Proof. Let

$$-z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' = p(z) = \left(\frac{n}{4}+\frac{1}{2}\right)p_1(z)+\left(\frac{1}{2}-\frac{n}{4}\right)p_2(z). \tag{3.2}$$

Then $p(z)$ is analytic in $\mathbb{U}^*$ with $p(0)=1$. Using the identity (1.10) in (3.2) and differentiating the resulting equation with respect to z, we have

$$-(1-\alpha)z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' - \alpha z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' = \left[p(z)+\alpha z p'(z)\right].$$

Since $f\in\sum_n^*(\alpha,\beta,k,\rho)$, then $\left\{p(z)+\alpha z p'(z)\right\}\in P_n(\rho)$ for $z\in\mathbb{U}^*$. This implies that

$$\Re\left\{p_i(z)+\alpha z p_i'(z)\right\} > \rho, \quad i=1,2.$$

Applying Lemma 2.1, we see that $\Re\left(p_i(z)\right) > \rho_1$, where ρ_1 is given by (3.1). Hence, $p\in P_n\left(\rho_1\right)$ for $z\in\mathbb{U}^*$ and the proof is thus complete. □

Theorem 3.2. *Let $f\in\sum_n^*(0,\beta,k,\rho)$ for $z\in\mathbb{U}^*$. Then $f\in\sum_n^*(\alpha,\beta,k,\rho)$ for $|z|<\Re(\alpha)$, where*

$$\Re(\alpha) = \frac{1}{|\alpha|+\sqrt{1+|\alpha|^2}}. \tag{3.3}$$

Proof. Set

$$-z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' = \left(1-\rho\right)h(z)+\rho \quad (h(z)\in P_n).$$

Using the same technique as in Theorem 3.1, we have

$$\begin{aligned} -(1-\alpha)z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' - \alpha z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)f(z)\right)' - \rho \quad &(3.4)\\ = \left(1-\rho\right)\left\{h(z)+\alpha z h'(z)\right\}&\\ = \left(1-\rho\right)\left[\left(\frac{n}{4}+\frac{1}{2}\right)\left\{h_1(z)+\alpha z h_1'(z)\right\}+\left(\frac{1}{2}-\frac{n}{4}\right)\left\{h_2(z)+\alpha z h_2'(z)\right\}\right], \quad &(3.5)\end{aligned}$$

where we have used (1.12) and $h_1(z), h_2(z)\in P, z\in\mathbb{U}^*$. Using the following well known estimate [9]:

$$\left|z h_i'(z)\right| \le \frac{2r}{1-r^2}\Re\left(h_i(z)\right) \quad (|z|=r<1,\ i=1,2),$$

we find that

$$\Re\left\{h_i(z)+\alpha z h_i'(z)\right\} \ge \Re\left\{h_i(z)+\left|\alpha z h_i'(z)\right|\right\} \ge \Re\left(h_i(z)\right)\left\{1-\frac{2|\alpha|r}{1-r^2}\right\}.$$

The right hand side of this inequality is positive if $r<\Re(\alpha)$, where $\Re(\alpha)$ is given by (3.3). Consequently it follows from (3.5) that $f\in\sum_n^*(\alpha,\beta,k,\rho)$ for $|z|<\Re(\alpha)$. Sharpness of this result follows by taking $h_i(z)=\frac{1+z}{1-z}$ in (3.5), $i=1,2$. □

Theorem 3.3. *Let* $f \in \sum_n^*(0,\beta,k,\rho)$ *and let*

$$F_\delta(f(z)) = \frac{\delta}{z^{\delta+1}} \int_0^z t^\delta f(t)dt \quad \left(\delta > 0, z \in \mathbb{U}^*\right). \tag{3.6}$$

Then

$$-z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) F_\delta(f(z))\right)' \in P_n\left(\rho_2\right)$$

where ρ_2 *is given by*

$$\rho_2 = \rho + \left(1-\rho\right)\left(2\gamma_1 - 1\right) \tag{3.7}$$

and

$$\gamma_1 = \int_0^1 \left(1 + t^{\Re\left(\frac{1}{\delta}\right)}\right)^{-1} dt.$$

Proof. Setting

$$-z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) F_\delta(f(z))\right)' = p(z) = \left(\frac{n}{4}+\frac{1}{2}\right)p_1(z) + \left(\frac{1}{2}-\frac{n}{4}\right)p_2(z). \tag{3.8}$$

Then $p(z)$ is analytic in $\mathbb{U}^*$ with $p(0) = 1$. Using the following operator identity:

$$z\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) F_\delta(f(z))\right)' = \delta\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) F_\delta(f(z))\right) - (\delta+1)\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) F_\delta(f(z))\right) \tag{3.9}$$

in (3.8), and differentiating the resulting equation with respect to z, we find that

$$-z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) F_\delta(f(z))\right)' = \left\{p(z) + \frac{1}{\delta} z p'(z)\right\} \in P_n(\rho).$$

Using Lemma 2.1, we see that $-z^2\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) F_\delta(f(z))\right)' \in P_n\left(\rho_2\right)$ for $z \in \mathbb{U}^*$, where ρ_2 is given by (3.7).

Hence, the proof is complete. □

Theorem 3.4. *Let* $\phi(z) \in \sum$ *satisfy the inequality:*

$$\Re\left(z\phi(z)\right) > \frac{1}{2}, \quad z \in \mathbb{U}^*. \tag{3.10}$$

If $f \in \sum_n(\alpha,\beta,k,\rho)$. *Then* $\phi * f \in \sum_n(\alpha,\beta,k,\rho)$.

Proof. Let $G = \phi * f$. Then

$$\begin{aligned}
&(1-\alpha)z\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) G(z)\right) + \alpha z\left(\mathscr{D}^k L^*\left(\alpha,\beta\right) G(z)\right) \\
&\quad = (1-\alpha)z\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)\left(\phi * f\right)(z)\right) + \alpha z\left(\mathscr{D}^k L^*\left(\alpha,\beta\right)\left(\phi * f\right)(z)\right) \\
&\quad = z\phi(z) * h(z) \qquad \left(h(z) \in P_n(\rho)\right)
\end{aligned}$$

where

$$z\phi(z)*h(z)=\left(\frac{n}{4}+\frac{1}{2}\right)\left[(1-\rho)\left\{z\phi(z)*h_1(z)+\rho\right\}\right]$$
$$+\left(\frac{1}{2}-\frac{n}{4}\right)\left[(1-\rho)\left\{z\phi(z)*h_2(z)+\rho\right\}\right] \qquad (h_1(z),h_2(z)\in P).$$

Since $\Re\left(z\phi(z)\right)>\frac{1}{2}$ and by using Lemma 2.2, we can conclude that $G=\phi*f\in\sum_n(\alpha,\beta,k,\rho)$. □

Theorem 3.5. *Let $\phi(z)\in\sum$ satisfy the inequality* (3.10) *and $f\in\sum_n^*(0,\beta,k,\rho)$. Then $\phi*f\in\sum_n^*(0,\beta,k,\rho)$.*

Proof. We have

$$-z^2\left(\mathscr{D}^kL^*(\alpha,\beta)(\phi*f)(z)\right)'=-z^2\left(\mathscr{D}^kL^*(\alpha,\beta)f(z)\right)'*z\phi(z), \qquad z\in\mathbb{U}^*.$$

The rest of the proof of Theorem 3.5 follows by using the same techniques as in the one of Theorem 3.4. □

Theorem 3.6. *Let $f\in\sum_n(\alpha,\beta,k,\rho_3)$ and $g\in\sum_n(\alpha,\beta,k,\rho_4)$. Let also $F=f*g$. Then $F\in\sum_n(\alpha,\beta,k,\rho_5)$, where*

$$\rho_5=1-4(1-\rho_3)(1-\rho_4)\left[1-\frac{1}{\alpha}\int_0^1\frac{u^{\left(\frac{1}{(1-\alpha)}\right)-1}}{1+u}du\right]. \tag{3.11}$$

This result is sharp.

Proof. Since $f\in\sum_n(\alpha,\beta,k,\rho_3)$ and $g\in\sum_n(\alpha,\beta,k,\rho_4)$, it follows that

$$S(z)=(1-\alpha)z\left(\mathscr{D}^kL^*(\alpha,\beta)f(z)\right)+\alpha z\left(\mathscr{D}^kL^*(\alpha,\beta)f(z)\right)\in P_n(\rho_3)$$

and

$$S^*(z)=(1-\alpha)z\left(\mathscr{D}^kL^*(\alpha,\beta)g(z)\right)+\alpha z\left(\mathscr{D}^kL^*(\alpha,\beta)g(z)\right)\in P_n(\rho_4).$$

Using identity (1.10) in the above equations, we get

$$\mathscr{D}^kL^*(\alpha,\beta)f(z)=\frac{1}{\alpha}z^{-1-\frac{1}{\alpha}}\int_0^z t^{\frac{1}{\alpha}-1}S(t)dt \tag{3.12}$$

and

$$\mathscr{D}^kL^*(\alpha,\beta)g(z)=\frac{1}{\alpha}z^{-1-\frac{1}{\alpha}}\int_0^z t^{\frac{1}{\alpha}-1}S^*(t)dt. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\mathscr{D}^kL^*(\alpha,\beta)F(z)=\frac{1}{\alpha}z^{-1-\frac{1}{\alpha}}\int_0^z t^{\frac{1}{\alpha}-1}Q(t)dt, \tag{3.14}$$

where

$$Q(z) = \left(\frac{n}{4}+\frac{1}{2}\right) q_1(z) + \left(\frac{1}{2}-\frac{n}{4}\right) q_2(z)$$
$$= \frac{1}{\alpha} z^{-1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-1} \left(S(t) * S^*(t)\right) dt. \tag{3.15}$$

Now

$$S(z) = \left(\frac{n}{4}+\frac{1}{2}\right) s_1(z) + \left(\frac{1}{2}-\frac{n}{4}\right) s_2(z), \tag{3.16}$$

$$S^*(z) = \left(\frac{n}{4}+\frac{1}{2}\right) s_1^*(z) + \left(\frac{1}{2}-\frac{n}{4}\right) s_2^*(z) \tag{3.17}$$

where $s_i \in P\left(\rho_3\right)$ and $s_i^* \in P\left(\rho_4\right)$, $i = 1,2$. Since

$$P_i^*(z) = \frac{s_i^*(z)-\rho_4}{2\left(1-\rho_4\right)} + \frac{1}{2} \in P\left(\frac{1}{2}\right), \quad i = 1,2$$

we obtain $\left(s_i * s_i^*\right)(z) \in P\left(\rho_3\right)$ using the Herglotz formula. Thus

$$\left(s_i * s_i^*\right)(z) \in P\left(\rho_5\right)$$

with

$$\rho_5 = 1 - 2\left(1-\rho_3\right)\left(1-\rho_4\right). \tag{3.18}$$

Using (3.14), (3.15), (3.17), (3.18) and Lemma 2.3, we have

$$\Re\left(q_i(z)\right) = \frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1} \Re\left\{\left(s_i * s_i^*\right)(uz)\right\} du$$
$$\geq \frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}\left(2\rho_5 - 1 + \frac{2\left(1-\rho_5\right)}{1+u|z|}\right) du$$
$$\geq \frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}\left(2\rho_5 - 1 + \frac{2\left(1-\rho_5\right)}{1+u}\right) du$$
$$= 1 - 4\left(1-\rho_3\right)\left(1-\rho_4\right)\left[1 - \frac{1}{\alpha}\int_0^1 \frac{u^{\left(\frac{1}{(1-\alpha)}\right)-1}}{1+u} du\right].$$

From this, we conclude that $F \in \sum_n(\alpha,\beta,k,\rho_5)$ where ρ_5 is given by (3.11).

Now, let us consider

$$S(z) = \left(\frac{n}{4}+\frac{1}{2}\right)\frac{1+\left(1-\rho_3\right)z}{1-z} + \left(\frac{1}{2}-\frac{n}{4}\right)\frac{1-\left(1-\rho_3\right)z}{1+z}$$

and

$$S^*(z) = \left(\frac{n}{4}+\frac{1}{2}\right)\frac{1+\left(1-\rho_4\right)z}{1-z} + \left(\frac{1}{2}-\frac{n}{4}\right)\frac{1-\left(1-\rho_4\right)z}{1+z}.$$

Since

$$\left(\frac{1+(1-\rho_3)z}{1-z}\right) * \left(\frac{1+(1-\rho_4)z}{1-z}\right) = 1-4(1-\rho_3)(1-\rho_4) + \frac{4(1-\rho_3)(1-\rho_4)}{1-z},$$

it follows from (3.15) that

$$q_i(z) = \frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}\left\{1-4(1-\rho_3)(1-\rho_4) + \frac{4(1-\rho_3)(1-\rho_4)}{1-z}\right\}du$$

$$\to 1-4(1-\rho_3)(1-\rho_4)\left\{1-\frac{1}{\alpha}\int_0^1 \frac{u^{\left(\frac{1}{(1-\alpha)}\right)-1}}{1+u}du\right\}$$

as $z \to -1$. This completes the proof of the sharpness of the result. □

4. Concluding Remarks and Observations

In our present investigation, we have successfully applied a linear operator which is associated with the gaussian hypergeometric function. By means of this general linear operator, we have introduced and investigated various interesting properties of some new subclasses of meromorphically univalent functions in the punctured unit disk $\mathbb{U}^*$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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