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Ricci Solitons in Kenmotsu Manifold

Research Article

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Abstract. In this paper we give a characterisation of Ricci solitons in Ricci recurrent and ϕ-recurrent Kenmotsu manifolds based on the 1-form.

Keywords. Ricci solitons; Kenmotsu; ϕ-recurrent; Concircular; Pseudo-projective; Ricci recurrent; Shrinking; Expanding; Steady

MSC. 53D10; 53D15

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1. Introduction

Ricci soliton is a special solution to the Ricci flow introduced by Hamilton [4] in the year 1982. In [8], Ramesh Sharma initiated the study of Ricci solitons in contact Riemannian geometry. Later, Mukut Mani Tripathi [9], Nagaraja et al. [6] and others extensively studied Ricci solitons in contact metric manifolds. Ricci soliton in a Riemannian manifold (M,g) is a natural generalization of an Einstein metric and is defined as a triple (g,V,λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$(\mathscr{L}_{\mathrm{V}}g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0, \tag{1.1}$$

where S is the Ricci tensor of M and $\mathscr{L}_{\mathrm{V}}$ denote the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively.

In 1972, Kenmotsu [5] studied a class of contact Riemannian manifolds satisfying some special conditions and these manifolds are known as Kenmotsu manifolds. The authors in [6] have studied Ricci solitons in Kenmotsu manifolds under semi-symmetry conditions. In this paper, we study the conditions which characterise Ricci solitons in Kenmotsu manifolds. Section 2 contains a brief review of Kenmotsu manifolds and Ricci solitons. In sections 3 – 6, we prove the characterizing conditions for Ricci solitons in ϕ-recurrent, pseudo-projective ϕ-recurrent, concircular ϕ-recurrent and Ricci recurrent Kenmotsu manifolds.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ,ξ,η,g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ,ξ,η) satisfying

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ g(X,\xi) = \eta(X),\ \eta(\xi) = 1,\ \eta\circ\phi = 0, \tag{2.1}$$

and

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

An almost contact metric manifold is said to be a Kenmotsu manifold [5] if

$$(\nabla_X\phi)Y = -g(X,\phi Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

$$\nabla_X\xi = X - \eta(X)\xi, \tag{2.4}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold the following relations hold [1].

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.5}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.7}$$

where R is the Riemannian curvature tensor.

$$S(X,\xi) = -2n\eta(X), \tag{2.8}$$

$$S(\phi X,\phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{2.9}$$

$$(\nabla_X\eta)Y = g(X,Y) - \eta(X)\eta(Y). \tag{2.10}$$

Let (g,V,λ) be a Ricci soliton in a Kenmotsu manifold M.

Taking $\mathrm{V} = \xi$ then from (2.4) and (1.1), we have

$$S(X,Y) = -(\lambda+1)g(X,Y) + \eta(X)\eta(Y). \tag{2.11}$$

The above equation yields

$$QX = -(\lambda+1)X + \eta(X)\xi, \tag{2.12}$$

$$S(X,\xi) = -\lambda\eta(X), \tag{2.13}$$

$$r = -\lambda(2n+1) - 2n. \tag{2.14}$$

Also by definition of covariant derivative, we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{2.15}$$

We will use the following result later.

Lemma 2.1 ([3]). *In a ϕ-recurrent Kenmotsu manifold (M^{2n+1},g), the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by*

$$A(W) = \eta(\rho)\eta(W). \tag{2.16}$$

Replacing W by ξ in (2.16)*, it follows that*

$$A(\xi) = \eta(\rho). \tag{2.17}$$

3. Ricci-recurrent Kenmotsu Manifold

Definition 3.1. A Kenmotsu manifold is said to be Ricci-recurrent manifold if there exists a non-zero 1-form A such that

$$(\nabla_W S)(Y,Z) = A(W)S(Y,Z). \tag{3.1}$$

Replacing Z by ξ in (3.1) and using (2.8), we have

$$(\nabla_W S)(Y,\xi) = -2nA(W)\eta(Y). \tag{3.2}$$

Using (2.8) and (2.4) in (2.15), we obtain

$$(\nabla_W S)(Y,\xi) = -[S(Y,W) + 2ng(Y,W)]. \tag{3.3}$$

In view of (3.2) and (3.3), we have

$$S(Y,W) = -2ng(Y,W) + 2nA(W)\eta(Y). \tag{3.4}$$

Taking $Y = \xi$ in (3.4), we get

$$S(\xi,W) = -2n\eta(W) + 2nA(W). \tag{3.5}$$

Applying Lemma 2.1, (3.5) reduces to

$$S(\xi,W) = -2n\eta(W)[1-\eta(\rho)]. \tag{3.6}$$

Using (2.13) and (2.17) in (3.6), we obtain

$$\lambda = 2n[1 - A(\xi)]. \tag{3.7}$$

Theorem 3.1. *Ricci soliton in Ricci-recurrent Kenmotsu manifold (M,g) with the 1-form A is*

- *expanding if $A(\xi) < 1$,*
- *steady if $A(\xi) = 1$,*
- *shrinking if $A(\xi) > 1$.*

4. ϕ-recurrent Kenmotsu Manifolds

Definition 4.1. A Kenmotsu manifold is said to be ϕ-recurrent manifold [3] if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z, \tag{4.1}$$

for arbitrary vector fields X,Y,Z,W.

Let us consider a ϕ-recurrent Kenmotsu manifold. By virtue of (2.1) and (4.1), we have

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z. \tag{4.2}$$

Contracting (4.2) with U, we obtain

$$-g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z,U). \tag{4.3}$$

Let e_i $(i = 1,2,\dots,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Taking $X = U = e_i$ in (4.3) and taking summation over i, $1 \le i \le 2n+1$, we get

$$-(\nabla_W S)(Y,Z) = A(W)S(Y,Z). \tag{4.4}$$

Replacing Z by ξ in (4.4) and using (2.8), we have

$$-(\nabla_W S)(Y,\xi) = -2nA(W)\eta(Y). \tag{4.5}$$

Using (2.8) and (2.4) in (2.15), we obtain

$$(\nabla_W S)(Y,\xi) = -[S(Y,W) + 2ng(Y,W)]. \tag{4.6}$$

In view of (4.5) and (4.6), we have

$$S(Y,W) = -2ng(Y,W) - 2nA(W)\eta(Y). \tag{4.7}$$

Taking $Y = \xi$ in (4.7), we get

$$S(\xi,W) = -2n\eta(W) - 2nA(W). \tag{4.8}$$

Applying Lemma 2.1, (4.8) reduces to

$$S(\xi,W) = -2n\eta(W)[1-\eta(\rho)]. \tag{4.9}$$

Using (2.13) and (2.17) in (4.9), we obtain

$$\lambda = 2n[1 - A(\xi)]. \tag{4.10}$$

Theorem 4.1. *Ricci soliton in ϕ-recurrent Kenmotsu manifold (M,g) with the 1-form A is*

- *expanding if $A(\xi) < 1$,*
- *steady if $A(\xi) = 1$,*
- *shrinking if $A(\xi) > 1$.*

5. Pseudo-projective ϕ-recurrent Kenmotsu Manifold

In a Kenmotsu manifold M, the pseudo-projective curvature tensor $\overline{P}$ is given by [7]

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)[g(Y,Z)X - g(X,Z)Y]$$

where a and b are constants such that $a,b \neq 0$.

Definition 5.1. A Kenmotsu manifold is said to be pseudo-projective ϕ-recurrent manifold if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_W \overline{P})(X,Y)Z) = A(W)\overline{P}(X,Y)Z, \tag{5.1}$$

for arbitrary vector fields X,Y,Z,W.

Let us consider a pseudo-projective ϕ-recurrent Kenmotsu manifold. By virtue of (2.1) and (5.1), we have

$$-(\nabla_W \overline{P})(X,Y)Z + \eta((\nabla_W \overline{P})(X,Y)Z)\xi = A(W)\overline{P}(X,Y)Z. \tag{5.2}$$

Contracting (5.2) with U, we obtain

$$-g((\nabla_W \overline{P})(X,Y)Z,U) + \eta((\nabla_W \overline{P})(X,Y)Z)\eta(U) = A(W)g(\overline{P}(X,Y)Z,U). \tag{5.3}$$

Let e_i $(i = 1,2,\dots,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (5.3) and taking summation over i, $1 \le i \le 2n+1$, we get

$$(\nabla_W S)(Y,Z) = A(W)\left\{S(Y,Z) - \frac{r}{(2n+1)}g(Y,Z)\right\}. \tag{5.4}$$

Replacing Z by ξ in (5.4) and using (2.1) and (2.8), we have

$$(\nabla_W S)(Y,\xi) = A(W)\left\{2n + \frac{r}{(2n+1)}\right\}\eta(Y). \tag{5.5}$$

Using (2.8) and (2.4) in (2.15), we obtain

$$(\nabla_W S)(Y,\xi) = -[S(Y,W) + 2ng(Y,W)]. \tag{5.6}$$

In view of (5.5) and (5.6), we have

$$S(Y,W) = -2ng(Y,W) - \left\{2n + \frac{r}{(2n+1)}\right\}A(W)\eta(Y). \tag{5.7}$$

Taking $Y = \xi$ in (5.7), we get

$$S(\xi,W) = -2n\eta(W) - \left\{2n + \frac{r}{(2n+1)}\right\}A(W). \tag{5.8}$$

Applying Lemma 2.1, (5.8) reduces to

$$S(\xi,W) = -2n\eta(W) - \left\{2n + \frac{r}{(2n+1)}\right\}\eta(\rho)\eta(W). \tag{5.9}$$

Using (2.13), (2.14) and (2.17) in (5.9), we obtain

$$\lambda = \frac{2n(2n[1+A(\xi)]+1)}{(2n+1)[1+A(\xi)]}. \tag{5.10}$$

Theorem 5.1. *Ricci soliton in a pseudo-projective ϕ-recurrent Kenmotsu manifold (M,g) with 1-form A is expanding, provided $A(\xi)$ is non-negative.*

6. Concircular ϕ-recurrent Kenmotsu Manifold

The Concircular curvature tensor of (M,g) is given by [10]

$$\overline{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y].$$

Definition 6.1. A Kenmotsu manifold is said to be concircular ϕ-recurrent manifold if there exist a non-zero 1-form A such that

$$\phi^2((\nabla_W \overline{C})(X,Y)Z) = A(W)\overline{C}(X,Y)Z. \tag{6.1}$$

for arbitrary vector fields X,Y,Z,W.

Let us consider a concircular ϕ-recurrent Kenmotsu manifold. By virtue of (2.1) and (6.1), we have

$$-(\nabla_W \overline{C})(X,Y)Z + \eta((\nabla_W \overline{C})(X,Y)Z)\xi = A(W)\overline{C}(X,Y)Z. \tag{6.2}$$

Contracting (6.2) with U, we obtain

$$-g((\nabla_W \overline{C})(X,Y)Z,U) + \eta((\nabla_W \overline{C})(X,Y)Z)\eta(U) = A(W)g(\overline{C}(X,Y)Z,U). \tag{6.3}$$

Let e_i $(i = 1,2,\dots,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (6.3) and taking summation over i, $1 \le i \le 2n+1$, we get

$$(\nabla_W S)(Y,Z) = \frac{dr(W)}{2n+1}g(Y,Z) - A(W)\left\{S(Y,Z) - \frac{r}{2n+1}g(Y,Z)\right\}. \tag{6.4}$$

Replacing Z by ξ in (6.4) and using (2.1) and (2.8), we have

$$(\nabla_W S)(Y,\xi) = \frac{dr(W)}{2n+1}\eta(Y) + A(W)\left\{2n\eta(Y) + \frac{r}{2n+1}\eta(Y)\right\}. \tag{6.5}$$

For a constant r (6.5) reduces to

$$(\nabla_W S)(Y,\xi) = A(W)\eta(Y)\left\{2n + \frac{r}{2n+1}\right\}. \tag{6.6}$$

Using (2.8) and (2.4) in (2.15), we obtain

$$(\nabla_W S)(Y,\xi) = -[S(Y,W) + 2ng(Y,W)]. \tag{6.7}$$

In view of (6.6) and (6.7), we have

$$S(Y,W) = -\left\{2n + \frac{r}{2n+1}\right\}A(W)\eta(Y) - 2ng(Y,W). \tag{6.8}$$

Taking $Y = \xi$, a characteristic vector field in (6.8), we get

$$S(\xi,W) = -2n\eta(W) - \left\{2n + \frac{r}{(2n+1)}\right\}A(W). \tag{6.9}$$

Applying Lemma 2.1, (6.9) reduces to

$$S(\xi,W) = -2n\eta(W) - \left\{2n + \frac{r}{(2n+1)}\right\}\eta(\rho)\eta(W). \tag{6.10}$$

Using (2.13), (2.14) and (2.17) in (6.10), we obtain

$$\lambda = \frac{2n(2n[1+A(\xi)]+1)}{(2n+1)[1+A(\xi)]}. \tag{6.11}$$

Theorem 6.1. *Ricci soliton in a Concircular ϕ-recurrent Kenmotsu manifold M with 1-form A and constant scalar curvature r is expanding for non-negative $A(\xi)$.*

Summary of the results proved can be put in the following table:

S. No.	Curvature tensor	Condition	λ
1	Ricci curvature tensor	$(\nabla_W S)(Y,Z) = A(W)S(Y,Z)$	$2n[1-A(\xi)]$
2	Riemann curvature tensor	$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z$	$2n[1-A(\xi)]$
3	Pseudo-projective curvature tensor	$\phi^2((\nabla_W \overline{P})(X,Y)Z) = A(W)\overline{P}(X,Y)Z$	$\frac{2n(2n(1+A(\xi))+1)}{(2n+1)(1+A(\xi))}$
4	Concircular curvature tensor	$\phi^2((\nabla_W \overline{C})(X,Y)Z) = A(W)\overline{C}(X,Y)Z$	$\frac{2n(2n[1+A(\xi)]+1)}{(2n+1)(1+A(\xi))}$

7. Conclusion

Ricci solitons in Ricci recurrent, ϕ-recurrent, pseudo-projective ϕ-recurrent and concircular ϕ-recurrent. Kenmotsu manifolds have been classified into expanding, shrinking and steady based on the nature of one form associated with the curvature conditions. This study may be extended to η-Ricci solitons in real hypersurfaces of complex space forms.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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