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Research Article

A Study on Neutrosophic Anti-Fuzzy Subsemiring of a Semiring

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Abstract. In this research, we endeavor to explore the algebraic properties of *neutrosophic anti-fuzzy subsemirings* (NAFSSR) within a *semiring* (SR), and we present several theorems related to NAFSSR in the context of a *semiring* (SR).

Keywords. Fuzzy set, Neutrosophic fuzzy set, Anti-fuzzy subsemiring, Neutrosophic anti-fuzzy subsemiring, Neutrosophic anti-fuzzy normal subsemiring, Homomorphism, Anti-homomorphism, Isomorphism, Anti-isomorphism

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1. Introduction

Various notions in universal algebra expand upon the structure of an associative ring $(R;+,\cdot)$. Among these, nearrings and different forms of semirings have proven to be highly valuable. An algebra $(R;+,\cdot)$ is defined as a *semiring* (SR) when $(R;+)$ and $(R;\cdot)$ form semigroups and adhere to the distributive laws $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(y+z)\cdot x=y\cdot x+z\cdot x$ universally for x, y, and z in R. A SRR is deemed additively commutative if $x+y=y+x$ is true universally for x, y in R. Furthermore, a semiring R might include an identity element 1, where $1\cdot x=x=x\cdot 1$, and a zero element 0, where $0+x=x=x+0$ and $x\cdot 0=0=0\cdot x$ hold universally for x in R. After L. A. Zadeh introduced fuzzy sets in his work [13], scholars began exploring extensions of this idea. Smarandache, in publications from 2002 and 2006 [10, 11], put forth Neutrosophy as an innovative philosophical perspective. Later, Abou-Zaid [1] introduced the notions of fuzzy subnearrings and ideals. In this paper, we propose several theorems concerning *neutrosophic algebraic fuzzy subsemirings* (NAFSSR) within the framework of a *semiring* (SR).

2. Preliminaries

Definition 2.1 ([13])**.** Imagine X as a non-empty collection. A *fuzzy subset* (FS) D of X is characterized as a mapping $D: X \to [0,1]$.

Definition 2.2 ([10])**.** A *neutrosophic fuzzy set* (NFS) D over a universal set X is defined through a truth membership function $T_D(x)$, an indeterminacy function $I_D(x)$, and a falsity membership function $F_D(x)$, represented as $D=\{\langle x, T_D(x), I_D(x), F_D(x)\rangle; x \in X\}$, where $T_D, I_D, F_D: X \to [0,1]$ and the condition $0 \le T_D(x)+I_D(x)+F_D(x) \le 3$ holds.

Definition 2.3. Let R be a SR. An FS D of R is identified as an '*anti-fuzzy subsemiring*' (AFSSR) of R provided it adheres to:

(i) $\mu D(x+y) \le \max\{\mu D(x), \mu D(y)\}$,

(ii) $\mu D(xy) \le \max\{\mu D(x), \mu D(y)\}$, for every x and y in R.

Definition 2.4. Assume R is an SR. An FS D of R is termed a '*neutrosophic anti-fuzzy subsemiring*' (NAFSSR) of R if it satisfies these criteria:

(i) (a) $T_D(x+y) \le \max\{T_D(x), T_D(y)\}$,
  (b) $I_D(x+y) \le \max\{I_D(x), I_D(y)\}$,
  (c) $F_D(x+y) \ge \min\{F_D(x), F_D(y)\}$,

(ii) (a) $T_D(xy) \le \max\{T_D(x), T_D(y)\}$,
  (b) $I_D(xy) \le \max\{I_D(x), I_D(y)\}$,
  (c) $F_D(xy) \ge \min\{F_D(x), F_D(y)\}$, universally for x and y in R.

Definition 2.5. Consider R as an SR. An AFSSR D of R is recognized as a '*neutrosophic anti-fuzzy normal subsemiring*' (NAFNSSR) of R when it meets:

(i) (a) $T_D(x+y) = T_D(y+x)$,
  (b) $I_D(x+y) = I_D(y+x)$,
  (c) $F_D(x+y) = F_D(y+x)$,

(ii) (a) $T_D(xy) = T_D(yx)$,
  (b) $I_D(xy) = I_D(yx)$,
  (c) $F_D(xy) = F_D(yx)$, universally for x and y in R.

Definition 2.6. Let D and B be FSs of sets G and H, respectively. The neutrosophic anti-product, denoted $D \times B$, is expressed as $D \times B = \{\langle (x,y), T_D \times B(x,y), I_D \times B(x,y), F_D \times B(x,y)\rangle /$

universally for x in G and y in $H\}$, with $T_D \times B(x,y) = \max\{T_D(x), T_B(y)\}$, $I_D \times B(x,y) = \max\{I_D(x), I_B(y)\}$, and $F_D \times B(x,y) = \min\{F_D(x), F_B(y)\}$.

Definition 2.7. Assume D is an FS within a set S. The neutrosophic anti-strongest fuzzy relation on S, represented as V, is a fuzzy relation on D given by $TV(x,y) = \max\{T_D(x), T_D(y)\}$, $IV(x,y) = \max\{I_D(x), I_D(y)\}$, and $FV(x,y) = \min\{F_D(x), F_D(y)\}$ universally for x and y in S.

Definition 2.8. Take $(R,+,\cdot)$ and $(R_1,+,\cdot)$ as two SRs. For, $f: R \to R_1$ and an NAFSSR D in R, let V be an NAFSSR in $f(R) = R_1$, defined by $TV(y) = T_D(x)$, $IV(y) = I_D(x)$, $FV(y) = F_D(x)$ universally for x in R and y in R_1. Then D is known as the preimage of V under f, symbolized as $f^{-1}(V)$.

Definition 2.9. View $(R,+,\cdot)$ and $(R_1,+,\cdot)$ as two SRs. For, $f: R \to R_1$ is classified as a *semiring homomorphism* (SR Hom) if $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ hold universally for x and y in R.

Definition 2.10. Consider $(R,+,\cdot)$ and $(R_1,+,\cdot)$ as two SRs. For, $f: R \to R_1$ is labeled a *semiring anti-homomorphism* (SRA Hom) if $f(x+y) = f(y)+f(x)$ and $f(xy) = f(y)f(x)$ are true universally for x and y in R.

Definition 2.11. Let $(R,+,\cdot)$ and $(R_1,+,\cdot)$ be two SRs. For, $f: R \to R_1$ is an SR Hom that is both one-to-one and onto, it is termed a *semiring isomorphism* (SR Iso).

Definition 2.12 ([**?**])**.** Assume $(R,+,\cdot)$ and $(R_1,+,\cdot)$ are two SRs. For, $f: R \to R_1$ is an SRA Hom that is both one-to-one and onto, it is called a *semiring anti-isomorphism* (SRA Iso).

Definition 2.13 ([**?**])**.** Imagine D as an NAFSSR of an SR $(R,+,\cdot)$ with a as an element in R. *The pseudo Neutrosophic anti-fuzzy coset* $(aD)_p$ is outlined by $((aT_D)_p)(x) = p(a)T_D(x), ((aI_D)_p)(x) = p(a)I_D(x), ((aF_D)_p)(x) = p(a)F_D(x)$, for each x in R and some p in P.

3. Properties of Anti-Fuzzy Subsemiring of a Semiring

Theorem 3.1. *Union of any two NAFSSR of a SRR is an NAFSSR of R.*

Proof. Let D and B beany two NAFSSR's of a SRR and x and y in R.
Let $D = \{(x, T_D(x), I_D(x), F_D(x))/x \in R\}$ and $B = \{(x, T_B(x), I_B(x), F_B(x))/x \in R\}$ and also let $C = D \cup B = \{(x, TC(x), IC(x), FC(x))/x \in R\}$, where $\max\{T_D(x), T_B(x)\} = TC(x)$, $\max\{I_D(x), I_B(x)\} = IC(x)$, $\min\{F_D(x), F_B(x)\} = FC(x)$. Now,

(i) (a)
$$\begin{aligned} TC(x+y) &= \max\{T_D(x+y), T_B(x+y)\} \\ &\le \max\{\max\{T_D(x), T_D(y)\}, \max\{T_B(x), T_B(y)\}\} \\ &= \max\{\max\{T_D(x), T_B(x)\}, \max\{T_D(y), T_B(y)\}\} \\ &= \max\{TC(x), TC(y)\}. \end{aligned}$$

Accordingly, $TC(x+y) \le \max\{TC(x), TC(y)\}$, universally for x and y in R.

(b)
$$\begin{aligned} IC(x+y) &= \max\{I_D(x+y), I_B(x+y)\} \\ &\le \max\{\max\{I_D(x), I_D(y)\}, \max\{I_B(x), I_B(y)\}\} \end{aligned}$$

$= \max\{\max\{I_D(x), I_B(x)\}, \max\{I_D(y), I_B(y)\}\}$
$= \max\{IC(x), IC(y)\}$.

Accordingly, $IC(x+y) \leq \max\{IC(x), IC(y)\}$, universally for x and y in R.

(c) $FC(x+y) = \min\{F_D(x+y), F_B(x+y)\}$
$\geq \min\{\min\{F_D(x), F_D(y)\}, \min\{F_B(x), F_B(y)\}\}$
$= \min\{\min\{F_D(x), F_B(x)\}, \min\{F_D(y), F_B(y)\}\}$
$= \min\{FC(x), FC(y)\}$.

Accordingly, $FC(x+y) \geq \min\{FC(x), FC(y)\}$, universally for x and y in R.

(ii) (a) $TC(xy) = \max\{T_D(xy), T_B(xy)\}$
$\leq \max\{\max\{T_D(x), T_D(y)\}, \max\{T_B(x), T_B(y)\}\}$
$= \max\{\max\{T_D(x), T_B(x)\}, \max\{T_D(y), T_B(y)\}\}$
$= \max\{TC(x), TC(y)\}$.

Accordingly, $TC(xy) \leq \max\{TC(x), TC(y)\}$, universally for x and y in R.

(b) $IC(xy) = \max\{I_D(xy), I_B(xy)\}$
$\leq \max\{\max\{I_D(x), I_D(y)\}, \max\{I_B(x), I_B(y)\}\}$
$= \max\{\max\{I_D(x), I_B(x)\}, \max\{I_D(y), I_B(y)\}\}$
$= \max\{IC(x), IC(y)\}$.

Accordingly, $IC(xy) \leq \max\{IC(x), IC(y)\}$, universally for x and y in R.

(c) $FC(xy) = \min\{F_D(xy), F_B(xy)\}$
$\geq \min\{\min\{F_D(x), F_D(y)\}, \min\{F_B(x), F_B(y)\}\}$
$= \min\{\min\{F_D(x), F_B(x)\}, \min\{F_D(y), F_B(y)\}\}$
$= \min\{FC(x), FC(y)\}$.

Accordingly, $FC(xy) \geq \min\{FC(x), FC(y)\}$, universally for x and y in R.

Accordingly, C is an NAFSSR of a SRR. Hence the union of any two NAFSSR's of a SRR is an NAFSSR of R. □

Theorem 3.2. *The union of a family of NAFSSR's of SRR is an NAFSSR of R.*

Proof. Let $\{V_i : i \in I\}$ be a family of NAFSSRR and let $D = \bigcup_{i \in I} V_i$. Let x and y in R. Then,

(i) (a) $T_D(x+y) = TV_i(x+y)$
$\leq \max\{TV_i(x), TV_i(y)\}$
$= \max\{TV_i(x), TV_i(y)\}$
$= \max\{T_D(x), T_D(y)\}$.

Accordingly, $T_D(x+y) \leq \max\{T_D(x), T_D(y)\}$, universally for x and y in R.

(b) $I_D(x+y) = IV_i(x+y)$
$\leq \max\{IV_i(x), IV_i(y)\}$
$= \max\{IV_i(x), IV_i(y)\}$
$= \max\{I_D(x), I_D(y)\}$.

Accordingly, $I_D(x+y) \leq \max\{I_D(x), I_D(y)\}$, universally for x and y in R.

(c) $F_D(x+y) = FV_i(x+y)$
$\geq \min\{FV_i(x), FV_i(y)\}$
$= \min\{FV_i(x), FV_i(y)\}$

$= \min\{F_D(x), F_D(y)\}$.

Accordingly, $F_D(x+y) \geq \min\{F_D(x), F_D(y)\}$, universally for x and y in R.

(ii) (a) $T_D(xy) = TV_i(xy)$
$\leq \max\{TV_i(x), TV_i(y)\}$
$= \max\{TV_i(x), TV_i(y)\}$
$= \max\{T_D(x), T_D(y)\}$.

Accordingly, $T_D(xy) \leq \max\{T_D(x), T_D(y)\}$, universally for x and y in R.

(b) $I_D(xy) = IV_i(xy)$
$\leq \max\{IV_i(x), IV_i(y)\}$
$= \max\{IV_i(x), IV_i(y)\}$
$= \max\{I_D(x), I_D(y)\}$.

Accordingly, $I_D(xy) \leq \max\{I_D(x), I_D(y)\}$, universally for x and y in R.

(c) $F_D(xy) = FV_i(xy)$
$\geq \min\{FV_i(x), FV_i(y)\}$
$= \min\{FV_i(x), FV_i(y)\}$
$= \min\{F_D(x), F_D(y)\}$.

Accordingly, $F_D(xy) \geq \min\{F_D(x), F_D(y)\}$, universally for x and y in R, that is, D is an NAFSSR of a SRR.

Hence, the union of a family of NAFSSR's of R is an NAFSSR of R. □

Theorem 3.3. *If D and B are any two NAFSSR's of the SR's R_1 and R_2 respectively, then anti-product $D \times B$ is an NAFSSR of $R_1 \times R_2$.*

Proof. Let D and B be two NAFSSR's of the SR's R_1 and R_2, respectively. Let x_1 and x_2 be in R_1, y_1 and y_2 be in R_2. Then (x_1, y_1) and (x_2, y_2) are in $R_1 \times R_2$. Now,

(i) (a) $T_D \times B[(x_1, y_1) + (x_2, y_2)] = T_D \times B(x_1 + x_2, y_1 + y_2)$
$= \max\{T_D(x_1 + x_2), T_B(y_1 + y_2)\}$
$\leq \max\{\max\{T_D(x_1), T_D(x_2)\}, \max\{T_B(y_1), T_B(y_2)\}\}$
$= \max\{\max\{T_D(x_1), T_B(y_1)\}, \max\{T_D(x_2), T_B(y_2)\}\}$
$= \max\{T_D \times B(x_1, y_1), T_D \times B(x_2, y_2)\}$.

Accordingly, $T_D \times B[(x_1, y_1) + (x_2, y_2)] \leq \max\{T_D \times B(x_1, y_1), T_D \times B(x_2, y_2)\}$.

(b) $I_D \times B[(x_1, y_1) + (x_2, y_2)] = I_D \times B(x_1 + x_2, y_1 + y_2)$
$= \max\{I_D(x_1 + x_2), I_B(y_1 + y_2)\}$
$\leq \max\{\max\{I_D(x_1), I_D(x_2)\}, \max\{I_B(y_1), I_B(y_2)\}\}$
$= \max\{\max\{I_D(x_1), I_B(y_1)\}, \max\{I_D(x_2), I_B(y_2)\}\}$
$= \max\{I_D \times B(x_1, y_1), I_D \times B(x_2, y_2)\}$.

Accordingly, $I_D \times B[(x_1, y_1) + (x_2, y_2)] \leq \max\{I_D \times B(x_1, y_1), I_D \times B(x_2, y_2)\}$.

(c) $F_D \times B[(x_1, y_1) + (x_2, y_2)] = F_D \times B(x_1 + x_2, y_1 + y_2)$
$= \min\{F_D(x_1 + x_2), F_B(y_1 + y_2)\}$
$\geq \min\{\min\{F_D(x_1), F_D(x_2)\}, \min\{F_B(y_1), F_B(y_2)\}\}$
$= \min\{\min\{F_D(x_1), F_B(y_1)\}, \min\{F_D(x_2), F_B(y_2)\}\}$
$= \min\{F_D \times B(x_1, y_1), F_D \times B(x_2, y_2)\}$.

Accordingly, $F_D \times B[(x_1,y_1)+(x_2,y_2)] \geq \min\{F_D \times B(x_1,y_1), F_D \times B(x_2,y_2)\}$.

(ii) (a)
$$\begin{aligned} T_D \times B[(x_1,y_1)(x_2,y_2)] &= T_D \times B(x_1x_2, y_1y_2) \\ &= \max\{T_D(x_1x_2), T_B(y_1y_2)\} \\ &\leq \max\{\max\{T_D(x_1), T_D(x_2)\}, \max\{T_B(y_1), T_B(y_2)\}\} \\ &= \max\{\max\{T_D(x_1), T_B(y_1)\}, \max\{T_D(x_2), T_B(y_2)\}\} \\ &= \max\{T_D \times B(x_1,y_1), T_D \times B(x_2,y_2)\}. \end{aligned}$$
Accordingly, $T_D \times B[(x_1,y_1)(x_2,y_2)] \leq \max\{T_D \times B(x_1,y_1), T_D \times B(x_2,y_2)\}$.

(b)
$$\begin{aligned} I_D \times B[(x_1,y_1)(x_2,y_2)] &= I_D \times B(x_1x_2, y_1y_2) \\ &= \max\{I_D(x_1x_2), I_B(y_1y_2)\} \\ &\leq \max\{\max\{I_D(x_1), I_D(x_2)\}, \max\{I_B(y_1), I_B(y_2)\}\} \\ &= \max\{\max\{I_D(x_1), I_B(y_1)\}, \max\{I_D(x_2), I_B(y_2)\}\} \\ &= \max\{I_D \times B(x_1,y_1), I_D \times B(x_2,y_2)\}. \end{aligned}$$
Accordingly, $I_D \times B[(x_1,y_1)(x_2,y_2)] \leq \max\{I_D \times B(x_1,y_1), I_D \times B(x_2,y_2)\}$.

(c)
$$\begin{aligned} F_D \times B[(x_1,y_1)(x_2,y_2)] &= F_D \times B(x_1x_2, y_1y_2) \\ &= \min\{F_D(x_1x_2), F_B(y_1y_2)\} \\ &\geq \min\{\min\{F_D(x_1), F_D(x_2)\}, \min\{F_B(y_1), F_B(y_2)\}\} \\ &= \min\{\min F_D(x_1), F_B(y_1), \min\{F_D(x_2), F_B(y_2)\}\} \\ &= \min\{F_D \times B(x_1,y_1), F_D \times B(x_2,y_2)\}. \end{aligned}$$
Accordingly, $F_D \times B[(x_1,y_1)(x_2,y_2)] \geq \min\{F_D \times B(x_1,y_1), F_D \times B(x_2,y_2)\}$.

Hence $D \times B$ is an NAFSSR of SR of $R_1 \times R_2$. □

Theorem 3.4. *If D_i are NAFSSR's of the SR's R_i, then ID_i is an NAFSSR of R_i.*

Proof. It is trivial. □

Theorem 3.5. *Let D be a NFS of a SRR and V be the strongest NAF relation of R. Then D is an NAFSSR of R if and only if V is an NAFSSR of $R \times R$.*

Proof. Suppose that D is an NAFSSR of a SRR.

Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $R \times R$. We have

(i) (a)
$$\begin{aligned} TV(x+y) &= TV[(x_1,x_2)+(y_1,y_2)] \\ &= TV(x_1+y_1, x_2+y_2) \\ &= \max\{T_D(x_1+y_1), T_D(x_2+y_2)\} \\ &\leq \max\{\max\{T_D(x_1), T_D(y_1)\}, \max\{T_D(x_2), T_D(y_2)\}\} \\ &= \max\{\max\{T_D(x_1), T_D(x_2)\}, \max\{T_D(y_1), T_D(y_2)\}\} \\ &= \max\{TV(x_1,x_2), TV(y_1,y_2)\} = \max\{TV(x), TV(y)\}. \end{aligned}$$
Accordingly, $TV(x+y) \leq \max\{TV(x), TV(y)\}$, universally for x and y in $R \times R$.

(b)
$$\begin{aligned} IV(x+y) &= IV[(x_1,x_2)+(y_1,y_2)] \\ &= IV(x_1+y_1, x_2+y_2) \\ &= \max\{I_D(x_1+y_1), I_D(x_2+y_2)\} \\ &\leq \max\{\max\{I_D(x_1), I_D(y_1)\}, \max\{I_D(x_2), I_D(y_2)\}\} \\ &= \max\{\max\{I_D(x_1), I_D(x_2)\}, \max\{I_D(y_1), I_D(y_2)\}\} \end{aligned}$$

$= \max\{IV(x_1,x_2), IV(y_1,y_2)\}$
$= \max\{IV(x), IV(y)\}.$

Accordingly, $IV(x+y) \leq \max\{IV(x), IV(y)\}$, universally for x and y in $R \times R$.

(c) $FV(x+y) = FV[(x_1,x_2)+(y_1,y_2)]$
$= FV(x_1+y_1, x_2+y_2)$
$= \min\{F_D(x_1+y_1), F_D(x_2+y_2)\}$
$\geq \min\{\min\{F_D(x_1), F_D(y_1)\}, \min\{F_D(x_2), F_D(y_2)\}\}$
$= \min\{\min\{F_D(x_1), F_D(x_2)\}, \min\{F_D(y_1), F_D(y_2)\}\}$
$= \min\{FV(x_1,x_2), FV(y_1,y_2)\}$
$= \min\{FV(x), FV(y)\}.$

Accordingly, $FV(x+y) \geq \min\{FV(x), FV(y)\}$, universally for x and y in $R \times R$.

(ii) (a) $TV(xy) = TV[(x_1,x_2)(y_1,y_2)]$
$= TV(x_1y_1, x_2y_2)$
$= \max\{T_D(x_1y_1), T_D(x_2y_2)\}$
$\leq \max\{\max\{T_D(x_1), T_D(y_1)\}, \max\{T_D(x_2), T_D(y_2)\}\}$
$= \max\{\max\{T_D(x_1), T_D(x_2)\}, \max\{T_D(y_1), T_D(y_2)\}\}$
$= \max\{TV(x_1,x_2), TV(y_1,y_2)\}$
$= \max\{TV(x), TV(y)\}.$

Accordingly, $TV(xy) \leq \max\{TV(x), TV(y)\}$, universally for x and y in $R \times R$.

(b) $IV(xy) = IV[(x_1,x_2)(y_1,y_2)]$
$= IV(x_1y_1, x_2y_2)$
$= \max\{I_D(x_1y_1), I_D(x_2y_2)\}$
$\leq \max\{\max\{I_D(x_1), I_D(y_1)\}, \max\{I_D(x_2), I_D(y_2)\}\}$
$= \max\{\max\{I_D(x_1), I_D(x_2)\}, \max\{I_D(y_1), I_D(y_2)\}\}$
$= \max\{IV(x_1,x_2), IV(y_1,y_2)\}$
$= \max\{IV(x), IV(y)\}.$

Accordingly, $IV(xy) \leq \max\{IV(x), IV(y)\}$, universally for x and y in $R \times R$.

(c) $FV(xy) = FV[(x_1,x_2)(y_1,y_2)]$
$= FV(x_1y_1, x_2y_2)$
$= \min\{F_D(x_1y_1), F_D(x_2y_2)\}$
$\geq \min\{\min\{F_D(x_1), F_D(y_1)\}, \min\{F_D(x_2), F_D(y_2)\}\}$
$= \min\{\min\{F_D(x_1), F_D(x_2)\}, \min\{F_D(y_1), F_D(y_2)\}\}$
$= \min\{FV(x_1,x_2), FV(y_1,y_2)\}$
$= \min\{FV(x), FV(y)\}.$

Accordingly, $FV(xy) \geq \min\{FV(x), FV(y)\}$, universally for x and y in $R \times R$.

This proves that V is an NAFSSR of $R \times R$. Conversely, assume that V is an NAFSSR of $R \times R$, then for any $x = (x_1,x_2)$ and $y = (y_1,y_2)$ are in $R \times R$, we have

(i) (a) $\max\{T_D(x_1+y_1), T_D(x_2+y_2)\} = TV(x_1+y_1, x_2+y_2)$
$= TV[(x_1,x_2)+(y_1,y_2)]$
$= TV(x+y) \leq \max\{TV(x), TV(y)\}$
$= \max\{TV(x_1,x_2), TV(y_1,y_2)\}$

$= \max\{\max\{T_D(x_1), T_D(x_2)\}, \max\{T_D(y_1), T_D(y_2)\}\}.$

(b) $\max\{I_D(x_1+y_1), I_D(x_2+y_2)\} = IV(x_1+y_1, x_2+y_2)$
$= IV[(x_1,x_2)+(y_1,y_2)]$
$= IV(x+y) \leq \max\{IV(x), IV(y)\}$
$= \max\{IV(x_1,x_2), IV(y_1,y_2)\}$
$= \max\{\max\{I_D(x_1), I_D(x_2)\}, \max\{I_D(y_1), I_D(y_2)\}\}.$

(c) $\min\{F_D(x_1+y_1), F_D(x_2+y_2)\} = FV(x_1+y_1, x_2+y_2)$
$= FV[(x_1,x_2)+(y_1,y_2)]$
$= FV(x+y) \geq \min\{FV(x), FV(y)\}$
$= \min\{FV(x_1,x_2), FV(y_1,y_2)\}$
$= \min\{\min\{F_D(x_1), F_D(x_2)\}, \min\{F_D(y_1), F_D(y_2)\}\}.$

If $T_D(x_1+y_1) \geq T_D(x_2+y_2)$, $I_D(x_1+y_1) \geq I_D(x_2+y_2)$, $F_D(x_1+y_1) \leq F_D(x_2+y_2)$, $T_D(x_1) \geq T_D(x_2)$, $I_D(x_1) \geq I_D(x_2)$, $F_D(x_1) \leq F_D(x_2)$, $T_D(y_1) \geq T_D(y_2)$, $I_D(y_1) \geq I_D(y_2)$, $F_D(y_1) \leq F_D(y_2)$, we get $T_D(x_1+y_1) \leq \max\{T_D(x_1), T_D(y_1)\}$, $I_D(x_1+y_1) \leq \max\{I_D(x_1), I_D(y_1)\}$, $F_D(x_1+y_1) \geq \min\{F_D(x_1), F_D(y_1)\}$ universally for x_1 and y_1 in R.

(ii) (a) $\max T_D(x_1y_1), T_D(x_2y_2) = TV(x_1y_1, x_2y_2)$
$= TV[(x_1,x_2)(y_1,y_2)]$
$= TV(xy)$
$\leq \max\{TV(x), TV(y)\}$
$= \max\{TV(x_1,x_2), TV(y_1,y_2)\}$
$= \max\{\max\{T_D(x_1), T_D(x_2)\}, \max\{T_D(y_1), T_D(y_2)\}\}.$

(b) $\max\{I_D(x_1y_1), I_D(x_2y_2)\} = IV(x_1y_1, x_2y_2)$
$= IV[(x_1,x_2)(y_1,y_2)]$
$= IV(xy)$
$\leq \max\{IV(x), IV(y)\}$
$= \max\{IV(x_1,x_2), IV(y_1,y_2)\}$
$= \max\{\max\{I_D(x_1), I_D(x_2)\}, \max\{I_D(y_1), I_D(y_2)\}\}.$

(c) $\min\{F_D(x_1y_1), F_D(x_2y_2)\} = FV(x_1y_1, x_2y_2)$
$= FV[(x_1,x_2)(y_1,y_2)]$
$= FV(xy) \geq \min\{FV(x), FV(y)\}$
$= \min\{FV(x_1,x_2), FV(y_1,y_2)\}$
$= \min\{\min\{F_D(x_1), F_D(x_2)\}, \min\{F_D(y_1), F_D(y_2)\}\}.$

If $T_D(x_1y_1) \geq T_D(x_2y_2), I_D(x_1y_1) \geq I_D(x_2y_2)$, $F_D(x_1y_1) \leq F_D(x_2y_2)$, $T_D(x_1) \geq T_D(x_2)$, $I_D(x_1) \geq I_D(x_2)$, $F_D(x_1) \leq F_D(x_2)$, $T_D(y_1) \geq T_D(y_2)$, $I_D(y_1) \geq I_D(y_2)$, $F_D(y_1) \leq F_D(y_2)$, we get $T_D(x_1y_1) \leq \max\{T_D(x_1), T_D(y_1)\}, I_D(x_1y_1) \leq \max\{I_D(x_1), I_D(y_1)\}, F_D(x_1y_1) \geq \min\{F_D(x_1), F_D(y_1)\}$ universally for x_1 and y_1 in R. Accordingly D is an NAFSSR of R. □

Theorem 3.6. *D is an NAFSSR of a SR $(R,+,\cdot)$ if and only if $T_D(x+y) \leq \max\{T_D(x), T_D(y)\}$, $I_D(x+y) \leq \max\{I_D(x), I_D(y)\}$, $F_D(x+y) \geq \min\{F_D(x), F_D(y)\}$, and $T_D(xy) \leq \max\{T_D(x), T_D(y)\}$, $I_D(xy) \leq \max\{I_D(x), I_D(y)\}$, $F_D(xy) \geq \min\{F_D(x), F_D(y)\}$, universally for x and y in R.*

Proof. It is trivial. □

Theorem 3.7. *If D is an NAFSSR of a SR $(R,+,\cdot)$, then $H = \{(x, T_D, I_D, F_D)/x \in R : T_D(x) = 0, I_D(x) = 0, F_D(x) = 0\}$ is either empty or is a SSR of R.*

Proof. If no element satisfies this condition, then H is empty. If x and y in H, then

(i) (a) $T_D(x+y) \le \max\{T_D(x), T_D(y)\} = \max\{0,0\} = 0$.
Accordingly, $T_D(x+y) = 0$.
(b) $I_D(x+y) \le \max\{I_D(x), I_D(y)\} = \max\{0,0\} = 0$.
Accordingly, $I_D(x+y) = 0$.
(c) $F_D(x+y) \ge \min\{F_D(x), F_D(y)\} = \min\{0,0\} = 0$.
Accordingly, $F_D(x+y) = 0$.

(ii) (a) $T_D(xy) \le \max\{T_D(x), T_D(y)\} = \max\{0,0\} = 0$.
Accordingly, $T_D(xy) = 0$.
(b) $I_D(xy) \le \max\{I_D(x), I_D(y)\} = \max\{0,0\} = 0$.
Accordingly, $I_D(xy) = 0$.
(c) $F_D(xy) \ge \min\{F_D(x), F_D(y)\} = \min\{0,0\} = 0$.
Accordingly, $F_D(xy) = 0$.

We get $x+y$, xy in H. Accordingly, H is a SSR of R. Hence H is either empty or is a SSR of R. □

Theorem 3.8. *If D be an NAFSSR of a SR$(R,+,\cdot)$, then if $T_D(x+y) = 1$, $I_D(x+y) = 1$, $F_D(x+y) = 1$ then either $T_D(x) = 1$ or $T_D(y) = 1$, $I_D(x) = 1$ or $I_D(y) = 1$, $F_D(x) = 1$ or $F_D(y) = 1$, universally for x and y in R.*

Proof. Let x and y in R. By the definition $T_D(x+y) \le \max\{T_D(x), T_D(y)\}$, which implies that $1 \le \max\{T_D(x), T_D(y)\}$. $I_D(x+y) \le \max\{I_D(x), I_D(y)\}$, which implies that $1 \le \max\{I_D(x), I_D(y)\}$. $F_D(x+y) \ge \min\{F_D(x), F_D(y)\}$, which implies that $1 \ge \min\{F_D(x), F_D(y)\}$. Accordingly, either $T_D(x) = 1$ or $T_D(y) = 1$, $I_D(x) = 1$ or $I_D(y) = 1$, $F_D(x) = 1$ or $F_D(y) = 1$. □

In the following theorem is the composition operation of functions:

Theorem 3.9. *Let D be an NAFSSR of a SR H and f is an isomorphism from a SRR onto H. Then $D \circ f$ is an NAFSSR of R.*

Proof. Let x and y in R and D be an NAFSSR of a SR H. Then, we have

(i) (a)
$$\begin{aligned}(T_D \circ f)(x+y) &= T_D(f(x+y)) \\ &= T_D(f(x)+f(y)) \\ &\le \max\{T_D(f(x)), T_D(f(y))\} \\ &\le \max\{(T_D \circ f)(x), (T_D \circ f)(y)\}.\end{aligned}$$
Consequently,
$$(T_D \circ f)(x+y) \le \max\{(T_D \circ f)(x), (T_D \circ f)(y)\}.$$
(b)
$$\begin{aligned}(I_D \circ f)(x+y) &= I_D(f(x+y)) \\ &= I_D(f(x)+f(y))\end{aligned}$$

$$\leq \max\{I_D(f(x)), I_D(f(y))\}$$
$$\leq \max\{(I_D \circ f)(x), (I_D \circ f)(y)\}.$$

Consequently,

$$(I_D \circ f)(x+y) \leq \max\{(I_D \circ f)(x), (I_D \circ f)(y)\}.$$

(c) $(F_D \circ f)(x+y) = F_D(f(x+y))$
$$= F_D(f(x)+f(y)$$
$$\geq \min\{F_D(f(x)), F_D(f(y))\}$$
$$\geq \min\{(F_D \circ f)(x), (F_D \circ f)(y)\}.$$

Consequently,

$$(F_D \circ f)(x+y) \geq \min\{(F_D \circ f)(x), (F_D \circ f)(y)\}.$$

(ii) (a) $(T_D \circ f)(xy) = T_D(f(xy))$
$$= T_D(f(x)f(y))$$
$$\leq \max\{T_D(f(x)), T_D(f(y))\}$$
$$\leq \max\{(T_D \circ f)(x), (T_D \circ f)(y)\}.$$

Consequently,

$$(T_D \circ f)(xy) \leq \max\{(T_D \circ f)(x), (T_D \circ f)(y)\}.$$

(b) $(I_D \circ f)(xy) = I_D(f(xy))$
$$= I_D(f(x)f(y))$$
$$\leq \max\{I_D(f(x)), I_D(f(y))\}$$
$$\leq \max\{(I_D \circ f)(x), (I_D \circ f)(y)\}.$$

Consequently,

$$(I_D \circ f)(xy) \leq \max\{(I_D \circ f)(x), (I_D \circ f)(y)\}.$$

(c) $(F_D \circ f)(xy) = F_D(f(xy))$
$$= F_D(f(x)f(y))$$
$$\geq \min\{F_D(f(x)), F_D(f(y))\}$$
$$\geq \min\{(F_D \circ f)(x), (F_D \circ f)(y)\}.$$

Consequently,

$$(F_D \circ f)(xy) \geq \min\{(F_D \circ f)(x), (F_D \circ f)(y)\}.$$

Accordingly $(D \circ f)$ is an NAFSSR of a SRR. □

Theorem 3.10. *Let D be an NAFSSR of a SR H and f is an anti-isomorphism from a SRR onto H. Then $D \circ f$ is an NAFSSR of R.*

Proof. Let x and y in R and D be an NAFSSR of a SR H. Then we have,

(i) (a) $(T_D \circ f)(x+y) = T_D(f(x+y))$
$$= T_D(f(y)+f(x))$$
$$\leq \max\{T_D(f(x)), T_D(f(y))\}$$
$$\leq \max\{(T_D \circ f)(x), (T_D \circ f)(y)\}.$$

Consequently,

$$(T_D \circ f)(x+y) \le \max\{(T_D \circ f)(x), (T_D \circ f)(y)\}.$$

(b) $(I_D \circ f)(x+y) = I_D(f(x+y))$
$= I_D(f(y)+f(x))$
$\le \max\{I_D(f(x)), I_D(f(y))\}$
$\le \max\{(I_D \circ f)(x), (I_D \circ f)(y)\}.$

Consequently,

$$(I_D \circ f)(x+y) \le \max\{(I_D \circ f)(x), (I_D \circ f)(y)\}.$$

(c) $(F_D \circ f)(x+y) = F_D(f(x+y))$
$= F_D(f(y)+f(x))$
$\ge \min\{F_D(f(x)), F_D(f(y))\}$
$\ge \min\{(F_D \circ f)(x), (F_D \circ f)(y)\}.$

Consequently,

$$(F_D \circ f)(x+y) \ge \min\{(F_D \circ f)(x), (F_D \circ f)(y)\}.$$

(ii) (a) $(T_D \circ f)(xy) = T_D(f(xy))$
$= T_D(f(y)f(x))$
$\le \max\{T_D(f(x)), T_D(f(y))\}$
$\le \max\{(T_D \circ f)(x), (T_D \circ f)(y)\}.$

Consequently,

$$(T_D \circ f)(xy) \le \max\{(T_D \circ f)(x), (T_D \circ f)(y)\}.$$

(b) $(I_D \circ f)(xy) = I_D(f(xy))$
$= I_D(f(y)f(x))$
$\le \max\{I_D(f(x)), I_D(f(y))\}$
$\le \max\{(I_D \circ f)(x), (I_D \circ f)(y)\}.$

Consequently,

$$(I_D \circ f)(xy) \le \max\{(I_D \circ f)(x), (I_D \circ f)(y)\}.$$

(c) $(F_D \circ f)(xy) = F_D(f(xy))$
$= F_D(f(y)f(x))$
$\ge \min\{F_D(f(x)), F_D(f(y))\}$
$\ge \min\{(F_D \circ f)(x), (F_D \circ f)(y)\}.$

Consequently,

$$(F_D \circ f)(xy) \ge \min\{(F_D \circ f)(x), (F_D \circ f)(y)\}.$$

Accordingly $D \circ f$ is an NAFSSR of a SRR. □

Theorem 3.11. *Let D be an NAFSSR of a SR$(R,+,\cdot)$, then the pseudo NAF coset $(aD)^p$ is an NAFSSR of a SRR, for every a in R.*

Proof. Let D be an NAFSSR of a SRR. For every x and y in R, we have

(i) (a) $((aT_D)^p)(x+y) = p(a)T_D(x+y)$
$\leq p(a)\max\{(T_D(x), T_D(y))\}$
$= \max\{p(a)T_D(x), p(a)T_D(y)\}$
$= \max\{((aT_D)^p)(x), ((aT_D)^p)(y)\}.$

Accordingly, $((aT_D)^p)(x+y) \leq \max\{((aT_D)^p)(x), ((aT_D)^p)(y)\}$.

(b) $((aI_D)^p)(x+y) = p(a)I_D(x+y)$
$\leq p(a)\max\{(I_D(x), I_D(y)\}$
$= \max\{p(a)I_D(x), p(a)I_D(y)\}$
$= \max\{((aI_D)^p)(x), ((aI_D)^p)(y)\}.$

Accordingly, $((aI_D)^p)(x+y) \leq \max\{((aI_D)^p)(x), ((aI_D)^p)(y)\}$.

(c) $((aF_D)^p)(x+y) = p(a)F_D(x+y)$
$\geq p(a)\min\{(F_D(x), F_D(y)\}$
$= \min\{p(a)F_D(x), p(a)F_D(y)\}$
$= \min\{((aF_D)^p)(x), ((aF_D)^p)(y)\}.$

Accordingly, $((aF_D)^p)(x+y) \geq \min\{((aF_D)^p)(x), ((aF_D)^p)(y)\}$.

(ii) (a) $((aT_D)^p)(xy) = p(a)T_D(xy)$
$\leq p(a)\max\{T_D(x), T_D(y)\}$
$= \max\{p(a)T_D(x), p(a)T_D(y)\}$
$= \max\{((aT_D)^p)(x), ((aT_D)^p)(y)\}.$

Accordingly, $((aT_D)^p)(xy) \leq \max\{((aT_D)^p)(x), ((aT_D)^p)(y)\}$.

(b) $((aI_D)^p)(xy) = p(a)I_D(xy)$
$\leq p(a)\max\{I_D(x), I_D(y)\}$
$= \max\{p(a)I_D(x), p(a)I_D(y)\}$
$= \max\{((aI_D)^p)(x), ((aI_D)^p)(y)\}.$

Accordingly, $((aI_D)^p)(xy) \leq \max\{((aI_D)^p)(x), ((aI_D)^p)(y)\}$.

(c) $((aF_D)^p)(xy) = p(a)F_D(xy)$
$\geq p(a)\min\{F_D(x), F_D(y)\}$
$= \min\{p(a)F_D(x), p(a)F_D(y)\}$
$= \min\{((aF_D)^p)(x), ((aF_D)^p)(y)\}.$

Accordingly, $((aF_D)^p)(xy) \geq \min\{((aF_D)^p)(x), ((aF_D)^p)(y)\}$.

Hence $(aD)^p$ is an NAFSSR of a SRR. □

Theorem 3.12. *Let $(R,+,\cdot)$ and $(R_1,+,\cdot)$ be any two SR's. The homomorphic image of an NAFSSR of R is an NAFSSR of R_1.*

Proof. Let $(R,+,\cdot)$ and $(R_1,+,\cdot)$ be any two SR's. Let $V = f(D)$, where D is an NAFSSR of R. We have to prove that V is an NAFSSR of R_1. Now, for $f(x)$, $f(y)$ in R_1,

(i) (a) $Tv(f(x)+f(y)) = Tv(f(x+y))$
$\leq T_D(x+y)$
$\leq \max\{T_D(x), T_D(y)\}.$

Consequently,

$$Tv(f(x)+f(y)) \leq \max\{Tv(f(x)), Tv(f(y))\}.$$

(b) $Iv(f(x)+f(y)) = Iv(f(x+y))$
$\leq I_D(x+y)$
$\leq \max\{I_D(x), I_D(y)\}$.

Consequently,

$$Iv(f(x)+f(y)) \leq \max\{Iv(f(x)), Iv(f(y))\}.$$

(c) $Fv(f(x)+f(y)) = Fv(f(x+y))$
$\geq F_D(x+y)$
$\geq \min\{F_D(x), F_D(y)\}$.

Consequently,

$$Fv(f(x)+f(y)) \geq \min\{Fv(f(x)), Fv(f(y))\}.$$

(ii) (a) $Tv(f(x)f(y)) = Tv(f(xy))$
$\leq T_D(xy)$
$\leq \max\{T_D(x), T_D(y)\}$.

Consequently,

$$Tv(f(x)f(y)) \leq \max\{Tv(f(x)), Tv(f(y))\}.$$

(b) $Iv(f(x)f(y)) = Iv(f(xy))$
$\leq I_D(xy)$
$\leq \max\{I_D(x), I_D(y)\}$.

Consequently,

$$Iv(f(x)f(y)) \leq \max\{Iv(f(x)), Iv(f(y))\}.$$

(c) $Fv(f(x)f(y)) = Fv(f(xy))$
$\geq F_D(xy)$
$\geq \min\{F_D(x), F_D(y)\}$.

Consequently,

$$Fv(f(x)f(y)) \geq \min\{Fv(f(x)), Fv(f(y))\}.$$

Hence V is an NAFSSR of R_1. □

Theorem 3.13. *Let $(R,+,\cdot)$ and $(R_1,+,\cdot)$ be any two SR's. The homomorphic preimage of an NAFSSR of R_1 is an NAFSSR of R.*

Proof. Let $(R,+,\cdot)$ and $(R_1,+,\cdot)$ be any two SR's. Let $V = f(D)$, where V is an NAFSSR of R_1. We have to prove that D is an NAFSSR of R. Let x and y in R. Then,

(i) (a) $T_D(x+y) = Tv(f(x+y))$, since

$$\begin{aligned} Tv(f(x)) &= T_D(x) \\ &= Tv(f(x)+f(y)) \\ &\leq \max\{Tv(f(x)), Tv(f(y))\} \\ &= \max\{T_D(x), T_D(y)\}. \end{aligned}$$

Consequently,

$$T_D(x+y) \le \max\{T_D(x), T_D(y)\}.$$

(b) $I_D(x+y) = Iv(f(x+y))$, since

$$\begin{aligned} Iv(f(x)) = I_D(x) \\ &= Iv(f(x)+f(y)) \\ &\le \max\{Iv(f(x)), Iv(f(y))\} \\ &= \max\{I_D(x), I_D(y)\}. \end{aligned}$$

Consequently,

$$I_D(x+y) \le \max\{I_D(x), I_D(y)\}.$$

(c) $F_D(x+y) = Fv(f(x+y))$, since

$$\begin{aligned} Fv(f(x)) = F_D(x) \\ &= Fv(f(x)+f(y)) \\ &\ge \min\{Fv(f(x)), Fv(f(y))\} \\ &= \min\{F_D(x), F_D(y)\}. \end{aligned}$$

Consequently,

$$F_D(x+y) \ge \min\{F_D(x), F_D(y)\}.$$

(ii) (a) $T_D(xy) = Tv(f(xy))$, since

$$\begin{aligned} Tv(f(x)) = T_D(x) \\ &= Tv(f(x)f(y)) \\ &\le \max\{Tv(f(x)), Tv(f(y))\} \\ &= \max\{T_D(x), T_D(y)\}. \end{aligned}$$

Consequently,

$$T_D(xy) \le \max\{T_D(x), T_D(y)\}.$$

(b) $$\begin{aligned} I_D(xy) &= Iv(f(xy)) \\ &= Iv(f(x)f(y)) \\ &\le \max\{Iv(f(x)), Iv(f(y))\} \\ &= \max\{I_D(x), I_D(y)\}. \end{aligned}$$

Consequently,

$$I_D(xy) \le \max\{I_D(x), I_D(y)\}.$$

(c) $$\begin{aligned} F_D(xy) &= Fv(f(xy)) \\ &= Fv(f(x)f(y)) \\ &\ge \min\{Fv(f(x)), Fv(f(y))\} \\ &= \min\{F_D(x), F_D(y)\}. \end{aligned}$$

Consequently,

$$F_D(xy) \ge \min\{F_D(x), F_D(y)\}.$$

Hence D is an NAFSSR of R. □

Theorem 3.14. *Let $(R,+,\cdot)$ and $(R_1,+,\cdot)$ be any two SR's. The anti-homomorphic image of an NAFSSR of R is an NAFSSR of R_1.*

Proof. Let $(R,+,\cdot)$ and $(R_1,+,\cdot)$ be any two SR's. Let $V = f(D)$, where D is an NAFSSR of R. We have to prove that V is an NAFSSR of R_1. Now, for $f(x)$, $f(y)$ in R_1,

(i) (a)
$$\begin{aligned} Tv(f(x)+f(y)) &= Tv(f(y+x)) \\ &\leq T_D(y+x) \\ &\leq \max\{T_D(y),T_D(x)\} \\ &= \max\{T_D(x),T_D(y)\}. \end{aligned}$$

Consequently,
$$Tv(f(x)+f(y)) \leq \max\{Tv(f(x)),Tv(f(y))\}.$$

(b)
$$\begin{aligned} Iv(f(x)+f(y)) = Iv(f(y+x)) &\leq I_D(y+x) \\ &\leq \max\{I_D(y),I_D(x)\} \\ &= \max\{I_D(x),I_D(y)\}. \end{aligned}$$

Consequently,
$$Iv(f(x)+f(y)) \leq \max\{Iv(f(x)),Iv(f(y))\}.$$

(c)
$$\begin{aligned} Fv(f(x)+f(y)) &= Fv(f(y+x)) \\ &\geq F_D(y+x) \\ &\geq \min\{F_D(y),F_D(x)\} \\ &= \min\{F_D(x),F_D(y)\}. \end{aligned}$$

Consequently,
$$Fv(f(x)+f(y)) \geq \min\{Fv(f(x)),Fv(f(y))\}.$$

(ii) (a)
$$\begin{aligned} Tv(f(x)f(y)) &= Tv(f(yx)) \\ &\leq T_D(yx) \\ &\leq \max\{T_D(y),T_D(x)\} \\ &= \max\{T_D(x),T_D(y)\}. \end{aligned}$$

Consequently,
$$Tv(f(x)f(y)) \leq \max\{Tv(f(x)),Tv(f(y))\}.$$

(b)
$$\begin{aligned} Iv(f(x)f(y)) = Iv(f(yx)) &\leq I_D(yx) \\ &\leq \max\{I_D(y),I_D(x)\} \\ &= \max\{I_D(x),I_D(y)\}. \end{aligned}$$

Consequently,
$$Iv(f(x)f(y)) \leq \max\{Iv(f(x)),Iv(f(y))\}.$$

(c)
$$\begin{aligned} Fv(f(x)f(y)) = Fv(f(yx)) &\geq F_D(yx) \\ &\geq \min\{F_D(y),F_D(x)\} \\ &= \min\{F_D(x),F_D(y)\}. \end{aligned}$$

Consequently,
$$Fv(f(x)f(y)) \geq \min\{Fv(f(x)),Fv(f(y))\}.$$

Hence V is an NAFSSR of R_1. □

Theorem 3.15. *Let* $(R,+,\cdot)$ *and* $(R_1,+,\cdot)$ *be any two SR's. The anti-homomorphic preimage of an NAFSSR of* R_1 *is an NAFSSR of* R.

Proof. Let $(R,+,\cdot)$ and $(R_1,+,\cdot)$ be any two SR's. Let $V = f(D)$, where V is an NAFSSR of R_1. We have to prove that D is an NAFSSR of R. Let x and y in R. Then

(i) (a)
$$\begin{aligned} T_D(x+y) &= Tv(f(x+y)) \\ &= Tv(f(y)+f(x)) \\ &\le \max\{Tv(f(y)),Tv(f(x))\} \\ &= \max\{Tv(f(x)),Tv(f(y))\} \\ &= \max\{T_D(x),T_D(y)\}. \end{aligned}$$
Consequently,
$$T_D(x+y) \le \max\{T_D(x),T_D(y)\}.$$

(b)
$$\begin{aligned} I_D(x+y) &= Iv(f(x+y)) \\ &= Iv(f(y)+f(x)) \\ &\le \max\{Iv(f(y)),Iv(f(x))\} \\ &= \max\{Iv(f(x)),Iv(f(y))\} \\ &= \max\{I_D(x),I_D(y)\}. \end{aligned}$$
Consequently,
$$I_D(x+y) \le \max\{I_D(x),I_D(y)\}.$$

(c)
$$\begin{aligned} F_D(x+y) &= Fv(f(x+y)) \\ &= Fv(f(y)+f(x)) \\ &\ge \min\{Fv(f(y)),Fv(f(x))\} \\ &= \min\{Fv(f(x)),Fv(f(y))\} \\ &= \min\{F_D(x),F_D(y)\}. \end{aligned}$$
Consequently,
$$F_D(x+y) \ge \min\{F_D(x),F_D(y)\}.$$

(ii) (a)
$$\begin{aligned} T_D(xy) &= Tv(f(xy)) \\ &= Tv(f(y)f(x)) \\ &\le \max\{Tv(f(y)),Tv(f(x))\} \\ &= \max\{Tv(f(x)),Tv(f(y))\} \\ &= \max\{T_D(x),T_D(y)\}. \end{aligned}$$
Consequently,
$$T_D(xy) \le \max\{T_D(x),T_D(y)\}.$$

(b)
$$\begin{aligned} I_D(xy) &= Iv(f(xy)) \\ &= Iv(f(y)f(x)) \\ &\le \max\{Iv(f(y)),Iv(f(x))\} \\ &= \max\{Iv(f(x)),Iv(f(y))\} \\ &= \max\{I_D(x),I_D(y)\}. \end{aligned}$$

Consequently,

$$I_D(xy) \le \max\{I_D(x), I_D(y)\}.$$

(c)
$$\begin{aligned} F_D(xy) &= Fv(f(xy)) \\ &= Fv(f(y)f(x)) \\ &\ge \min\{Fv(f(y)), Fv(f(x))\} \\ &= \min\{Fv(f(x)), Fv(f(y))\} \\ &= \min\{F_D(x), F_D(y)\}. \end{aligned}$$

Consequently,

$$F_D(xy) \ge \min\{F_D(x), F_D(y)\}.$$

Hence D is an NAFSSR of R. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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