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Research Article

A Note on q-Sălăgean Type Functions

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Abstract. In the present paper, we define certain classes of analytic function in the open disc by using the generalized linear operator $D^n_{\lambda,\delta}$ with help of q calculus. For functions belonging to these classes we provide coefficient inequalities, distortion theorem.

Keywords. Analytic function, Ruscheweyh derivative, Al-Oboudi operator, Coefficient estimates

Mathematics Subject Classification (2020). Primary 30C45; Secondary30C50

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic and univalent functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

defined in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$.

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathcal{U}$, consisting of analytic functions whose non-zero coefficients from the second terms onward are negative, that is, an analytic function $f \in \mathcal{T}$ if it has a Taylor series expansion of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \tag{1.2}$$

which are analytic in the open disc $\mathcal{U}$.

For functions $f \in \mathcal{A}$ of the form (1.1), Dileep and Rashmi [4] defined Sălăgean q-differential operator,

$$\mathcal{S}_q^0 f(z) = f(z),\ \mathcal{S}_q^1 f(z) = z\partial_q f(z),\ \cdots,\ \mathcal{S}_q^n f(z) = z\partial_q(\mathcal{S}_q^{n-1} f(z)).$$

A simple calculation implies

$$\mathcal{S}_q^n f(z) = f(z) * G_{q,n}(z),$$

where

$$G_{q,n}(z) = z + \sum_{m=2}^{\infty} [m]_q^n z^m \quad (n \in \mathbb{N}),$$

where $[m]_q = \frac{1-q^m}{1-q}$.

The power series of $\mathcal{S}_q^n f(z)$ for functions $f \in \mathcal{A}$ of the form (1.1) is given by

$$\mathcal{S}_q^n f(z) = z + \sum_{m=2}^{\infty} [m]_q^n a_m z^m. \tag{1.3}$$

Note that

$$\lim_{q \to 1^-} G_{q,n}(z) = z + \sum_{m=2}^{\infty} m^n z^m$$

and

$$\lim_{q \to 1^-} \mathcal{S}_q^n f(z) = f(z) * \left(z + \sum_{m=2}^{\infty} m^n z^m \right)$$

which is the familiar Sălăgean derivative [8].

Now we define the following subclass of $\mathcal{T}$ by using Sălăgean q-differential operator. Let $\mathcal{T}_q^n(b, \alpha, \lambda)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the conditions

$$\Re \left\{ \frac{(b-2)S_q^n f + 2S_q^{n+1} f}{(b-2\lambda)S_q^n f + 2\lambda S_q^{n+1} f} \right\} > \alpha, \tag{1.4}$$

for some $\alpha \ge 0$, $\lambda > 1$, b is a non-zero real numbers and $z \in \mathcal{U}$. For different parametric values of q and b we get the classes defined by Mostafa [5] and Ravikumar *et al*. [6].

2. Main Results

Now, we prove the sufficient conditions for the class $\mathcal{T}_q^n(b, \alpha, \lambda)$.

Theorem 2.1. *A function $f(z)$ defined by* (1.2) *is in the class $\mathcal{T}_q^n(b, \alpha, \lambda)$ if and only if*

$$\sum_{m=2}^{\infty} 2[m]_q^n a_m (1 - \alpha\lambda)([m]_q - 1) \le b(1 - \alpha). \tag{2.1}$$

Proof. Let

$$\Re \left\{ \frac{(b-2)S_q^n f + 2S_q^{n+1} f}{(b-2\lambda)S_q^n f + 2\lambda S_q^{n+1} f} \right\} > \alpha$$

which implies

$$\Re \left\{ \frac{(b-2)\left[z - \sum_{m=2}^{\infty}[m]_q^n a_m z^m\right] + 2\left[z - \sum_{m=2}^{\infty}[m]_q^{n+1} a_m z^m\right]}{(b-2\lambda)\left[z - \sum_{m=2}^{\infty}[m]_q^n a_m z^m\right] + 2\lambda\left[z - \sum_{m=2}^{\infty}[m]_q^{n+1} a_m z^m\right]} \right\} > \alpha,$$

$$\Re\left\{\frac{bz-\sum_{m=2}^{\infty}[m]_q^n a_m z^m[b+2[m]_q-2]}{bz-\sum_{m=2}^{\infty}[m]_q^n a_m z^m[2\lambda[m]_q+b-2\lambda]}\right\}>\alpha.$$

Letting $z \to 1$, then, we get

$$b-\sum_{m=2}^{\infty}[m]_q^n a_m(b+2[m]_q^n-2)>\alpha\left(b-\sum_{m=2}^{\infty}[m]_q a_m[2\lambda[m]_q+b-2\lambda]\right),$$

$$\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda)([m]_q+1)<b(1-\alpha).$$

Conversely, suppose $f\in\mathcal{T}_q^n(b,\alpha,\lambda)$ satisfies (2.1). Equivalently

$$\left|\left\{\frac{(b-2)S_q^n f+2S_q^{n+1}f}{(b-2\lambda)S_q^n f+2\lambda S_q^{n+1}f}\right\}-1\right|\le\alpha,$$

$$\left|\left\{\frac{bz-\sum_{m=2}^{\infty}[m]_q^n a_m z^m[b+2[m]_q-2]}{bz-\sum_{m=2}^{\infty}[m]_q^n a_m z^m[2\lambda[m]_q+b-2\lambda]}\right\}-1\right|\le 1-\alpha.$$

As $z\to 1$,

$$\left|\frac{\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\lambda)([m]_q+1)}{b-\sum_{m=2}^{\infty}2[m]_q^n a_m[2\lambda[m]_q+b-2\lambda]}\right|\le 1-\alpha.$$

This expression is bounded by $1-\alpha$, if

$$\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\lambda)([m]_q+1)\le(1-\alpha)\left(b-\sum_{m=2}^{\infty}2[m]_q^n a_m[2\lambda[m]_q+b-2\lambda]\right),$$

which is true by hypothesis. □

As $q\to 1$, we get the following result

Corollary 2.2. *A function f defined* (1.2) *is in the class $\mathcal{T}^n(b,\alpha,\lambda)$ if and only if*

$$\sum_{m=2}^{\infty}2m^n a_m(1-\alpha\lambda)(m-1)\le b(1-\alpha). \tag{2.2}$$

Corollary 2.3. *If $f\in\mathcal{T}_q^n(b,\alpha,\lambda)$, then*

$$|a_m|\le\frac{b(1-\alpha)}{2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)}.$$

Theorem 2.4. *Let $0\le\alpha<1$, $0\le\lambda_1\le\lambda_2<1$, then $\mathcal{T}_q^n(b,\alpha,\lambda_2)\subset\mathcal{T}_q^n(b,\alpha,\lambda_1)$.*

Proof. From Theorem 2.1,

$$\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda_2)([m]_q-1)\le\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda_1)([m]_q-1)\le b(1-\alpha).$$

For $f\in\mathcal{T}_q^n(b,\alpha,\lambda_2)$. Hence $f\in\mathcal{T}_q^n(b,\alpha,\lambda_1)$. □

Theorem 2.5. *Let $f\in\mathcal{T}_q^n(b,\alpha,\lambda)$ and define $f_1(z)=z$, and*

$$f_n(z)=z-\frac{b(1-\alpha)}{2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)}z^n.$$

Let $f\in\mathcal{T}_q^n(b,\alpha,\lambda)$ if and only if f can be expressed as $f(z)=\sum_{m=1}^{\infty}\mu_m f_m(z)$, where $\mu_m\ge 0$ and $\sum_{m=1}^{\infty}\mu_m=1$.

Proof. If $f(z)=\sum_{m=1}^{\infty}\mu_m f_m(z)$, with $\mu_m\ge 0$ and $\sum_{m=1}^{\infty}\mu_m=1$. Then

$$\frac{\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)b(1-\alpha)}{\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)}=\sum_{m=2}^{\infty}\mu_m b(1-\alpha)=(1-\mu_1)b(1-\alpha)\le b(1-\alpha).$$

Hence $f\in\mathcal{T}_q^n(b,\alpha,\lambda)$.

Conversely, $f(z)=z-\sum_{m=2}^{\infty}a_m z^m\in\mathcal{T}_q^n(b,\alpha,\lambda)$. Define

$$\mu_m=\frac{\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)|a_m|}{b(1-\alpha)}$$

and define $\mu_1=1-\sum_{m=2}^{\infty}\mu_m$, from Theorem 2.1, $\sum_{m=2}^{\infty}\mu_m\le 1$ and $\mu_1\ge 0$. Since $\sum_{m=1}^{\infty}\mu_m f_m(z)=z-\sum_{m=2}^{\infty}a_m z^m=f(z)$. □

Theorem 2.6. *The class $\mathcal{T}_q^n(b,\alpha,\lambda)$ is closed under convex linear combination.*

Proof. Let $f,g\in\mathcal{T}_q^n(b,\alpha,\lambda)$, and let

$$f(z)=z-\sum_{m=2}^{\infty}a_m z^m,\quad g(z)=z-\sum_{m=2}^{\infty}b_m z^m.$$

For η such that $0\le\eta\le 1$, it is enough to show that the function defined by

$$h(z)=(1-\eta)f(z)+\eta g(z)\in\mathcal{T}_q^n(b,\alpha,\lambda).$$

Now

$$h(z)=z-\sum_{m=2}^{\infty}[(1-\eta)a_m+\eta b_m]z^m.$$

Applying Theorem 2.1 to $f,g\in\mathcal{T}_q^n(b,\alpha,\lambda)$, we have

$$\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)[(1-\eta)a_m+\eta b_m]\le(1-\eta)b(1-\alpha)+\eta b(1-\alpha)=b(1-\alpha).$$

This implies that $h\in\mathcal{T}_q^n(b,\alpha,\lambda)$. □

Theorem 2.7. *Let $j=1,2,\cdots,n$, $f_j(z)=z-\sum_{m=2}^{\infty}a_{m,j}z^m\in\mathcal{T}_q^n(b,\alpha,\lambda)$, such that $\sum_{j=1}^{m}\lambda_j=1$, then the function $F(z)$ is defined by*

$$F(z)=\sum_{j=1}^{m}\lambda_j f_j(z)\in\mathcal{T}_q^n(b,\alpha,\lambda).$$

Proof. For each $j\in\{1,2,\cdots,n\}$, we obtain

$$\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)|a_m|<b(1-\alpha),$$

$$\begin{aligned}F(z)&=\sum_{j=1}^{m}\lambda_j\left[z-\sum_{m=2}^{\infty}a_{m,j}z^m\right]\\&=z-\sum_{m=2}^{\infty}\left[\sum_{j=1}^{m}a_{m,j}\right]z^m\sum_{m=2}^{\infty}2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)\left[\sum_{j=1}^{m}a_{m,j}\right]\\&=\sum_{j=1}^{m}\lambda_j\left[2[m]_q^n a_m(1-\alpha\lambda)([m]_q-1)\left[\sum_{j=1}^{m}a_{m,j}\right]\right]<b(1-\alpha).\end{aligned}$$

Therefore, $F(z)\in\mathcal{T}_q^n(b,\alpha,\lambda)$. □

Theorem 2.8. *Let $f \in \mathcal{T}_q^n(b,\alpha,\lambda)$ then for every $0 \le \beta < 1$, the function*

$$H_\beta = (1-\beta)f(z) + \beta \int_0^z \frac{f(t)}{t} d_q t$$

then $H_\beta(z) \in \mathcal{T}_q^n(b,\alpha,\lambda)$.

Proof. We have

$$H_\beta(z) = z - \sum_{m=2}^{\infty} \left(\beta + \frac{\beta}{m} - \beta\right) a_m z^m.$$

Since $\left(\beta + \frac{\beta}{m} - \beta\right) < 1$, $m \ge 2$, so by Theorem 2.1,

$$\left(\beta + \frac{\beta}{m} - \beta\right) \sum_{m=2}^{\infty} \left[2[m]_q^n a_m (1-\alpha\lambda)([m]_q - 1)\right] a_m$$
$$< \sum_{m=2}^{\infty} \left[2[m]_q^n a_m (1-\alpha\lambda)([m]_q - 1)\right] a_m < b(1-\alpha).$$

Hence $H_\beta(z) \in \mathcal{T}_q^n(b,\alpha,\lambda)$. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] F. M. Al-Oboudi On univalent functions defined by a generalized Sălăgean operator, *International Journal of Mathematics and Mathematical Sciences* **2004**(25-28) (2004), 1429 – 1436, URL: https://eudml.org/doc/51673.

[2] O. Altintas and S. Owa, On subclasses of univalent functions with negative coefficients, *Pusan Kyongnam Mathematical Journal* **4** (1988), 41 – 56.

[3] O. Crisan, Differential subordinations obtained by using Al-Oboudi and Ruscheweyh operators, *Studia Universitatis Babeş-Bolyai, Mathematica* **56**(3) (2011), 45 – 51.

[4] L. Dileep and S. V. D. Rashmi, Certain subclasses of analytic function by Sălăgean q-differential operator, *TWMS Journal of Applied and Engineering Mathematics* **13**(4) (2023), 46 – 52.

[5] A. O. Mostafa, A study on starlike and convex properties for hypergeometric functions, *Journal of Inequalities in Pure and Applied Mathematics* **10**(3) (2009), 8 pages.

[6] N. Ravikumar, L. Dileep and S. Latha, A note on Al-Oboudi type functions, *Journal of Rajasthan Academy of Physical Sciences* **9**(2) (2010), 155 – 164.

[7] S. Ruscheweyh, New criteria for univalent functions, *Proceedings of the American Mathematical Society* **49**(1) (1975), 109 – 115, URL: https://www.ams.org/journals/proc/1975-049-01/S0002-9939-1975-0367176-1/S0002-9939-1975-0367176-1.pdf.

[8] G. S. Sălăgean, Subclasses of univalent functions, in: *Complex Analysis – Fifth Romanian-Finnish Seminar*, C. A. Cazacu, N. Boboc, M. Jurchescu and I. Suciu (editors), Lecture Notes in Mathematics, Vol. 1013, Springer, Berlin — Heidelberg, pp. 362 – 372, (1983) DOI: 10.1007/BFb0066543.