**Journal of Informatics and Mathematical Sciences**
Vol. 17, No. 1, pp. 1–7, 2025
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
DOI: 10.26713/jims.v17i1.3104

http://www.rgnpublications.com

Research Article

# A Note on Value Distribution of the Product of a Meromorphic Function and Its Higher Order Derivatives

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**Received:** February 23, 2025 **Accepted:** April 3, 2025 **Published:** April 30, 2025

**Abstract.** In the present note, we consider the value distribution of the product of a meromorphic function and its higher order derivatives and generalise a result of Lahiri and Dewan (Value distribution of the product of a meromorphic function and its derivative, *Kodai Mathematical Journal* 26 (2003), 95 – 100).

**Keywords.** Meromorphic functions, Small functions, Derivatives, Value distribution

**Mathematics Subject Classification (2020).** 30D30



## 1. Introduction, Definitions and Notations

We first need some important definitions by assuming standard notations of the value distribution theory which are available in [4] (Hayman).

**Definition 1.1** ([4]). Let $f(z)$ be a non-constant meromorphic function in the complex plane and $a$ be any complex number. The deficiency of $a$ with respect to $f(z)$ is defined by

$$\delta(a)=\delta(a;f)=\varliminf_{r\to\infty}\frac{m(r,a)}{T(r,f)}=1-\varlimsup_{r\to\infty}\frac{N(r,a)}{T(r,f)}.$$

It is obvious that $0\le\delta(a;f)\le 1$.

If $\delta(a;f)>0$, then the complex number $a$ is called a deficient value of $f(z)$. The deficient value is also called exceptional value in the sense of Nevanlinna.

**Definition 1.2** ([4]). Let $f(z)$ be a non-constant meromorphic function in the complex plane and $a$ be any complex number. We define

$$\Theta(a)=\Theta(a;f)=1-\overline{\lim_{r\to\infty}}\frac{\bar{N}(r,a)}{T(r,f)}$$

and

$$\theta(a)=\theta(a;f)=\underline{\lim}_{r\to\infty}\frac{N(r,a)-\bar{N}(r,a)}{T(r,f)}.$$

It is evident that $0\le\Theta(a;f)\le 1$ and $0\le\theta(a;f)\le 1$.
Thus, we may write that $\delta(a;f)+\theta(a;f)\le\Theta(a;f)$.

**Definition 1.3** ([7]). Let $m$ be a positive integer. We denote by $N(r,a;f\mid\le m)$ the counting function of those $a$-points of $f$ whose multiplicities are not greater than $m$, where each $a$-point is counted according to its multiplicity. Similarly, we denote by $N(r,a;f\mid\ge m)$ the counting function of those $a$-points of $f$ whose multiplicities are not less than $m$, where each $a$-point is counted according to its multiplicity.
Similarly, we define $N(r,a;f\mid<m)$ and $N(r,a;f\mid>m)$. Also, $\bar{N}(r,a;f\mid\le m)$, $\bar{N}(r,a;f\mid\ge m)$, $\bar{N}(r,a;f\mid<m)$ and $\bar{N}(r,a;f\mid>m)$ are defined similarly ignoring the multiplicities.
Finally, we take $\bar{N}(r,a;f\mid\le\infty)\equiv\bar{N}(r,a;f)$ and $N(r,a;f\mid\le\infty)\equiv N(r,a;f)$.

**Definition 1.4** ([4]). A meromorphic function $a(z)$ is called a small function with respect to a meromorphic function $f(z)$ if $T(r,a)=S(r,f)$, i.e.,

$$T(r,a)=o(T(r,f)).$$

So,

$$\frac{T(r,a)}{T(r,f)}\to 0,\quad \text{as } r\to\infty$$

possibly outside a set of finite linear measure.

For a transcendental meromorphic function $f$ defined in the open complex plane C, Hayman [6] proved the following theorem.

**Theorem 1.1.** *For an integer $n$ $(\ge 3)$, $f^nf'$ assumes all finite values, except possibly zero, infinitely many times.*

Further, Hayman conjectured [5] that the above theorem remains valid even if $n=1$ or 2. Later on Mues [8] proved the result for $n=2$ and the case $n=1$ was proved by Bergweiler and Eremenko [1] and independently by Chen and Fang [3].

Naturally the question arised here was about the value distribution of $ff'-a$, where $a=a(z)$ is a non-zero small function of $f(z)$. In this context, we first mention the following theorem which was proved by Bergweiler [2].

**Theorem 1.2.** *If $f$ is of finite order and $a$ is a polynomial then $ff'-a$ has infinitely many zeros.*

We can see that Bergweiler imposed two restrictions to get the desired result. One is on $f$ which has to be of finite order and the other one is on $a(z)$ which has to be a polynomial instead of being arbitrary small function of $f(z)$.

Yu [10] proved the following general case but instead of a single small function he considered a small function and its negative to achieve the result.

**Theorem 1.3.** *If $a$ $(\not\equiv 0,\infty)$ is a small function of $f$ then at least one of $ff'-a$ and $ff'+a$ will have infinitely many zeros.*

To improve the Theorem 1.3, in 2003, Lahiri and Dewan [7] proved the following theorem first.

**Theorem 1.4.** *Let $\psi=(f)^{n_0}(f^{(k)})^{n_1}$, where $n_0$ $(\geq 2)$, $n_1$ and $k$ are positive integers such that $n_0(n_0-1)+(1+k)(n_0n_1-n_0-n_1)>0$. Then*

$$\left[1-\frac{1+k}{n_0+k}-\frac{n_0(1+k)}{(n_0+k)\{n_0+(1+k)n_1\}}\right]T(r,\psi)\leq \bar{N}(r,a;\psi)+S(r,\psi), \tag{1.1}$$

*for any small function $a$ $(\not\equiv 0,\infty)$ of $f$.*

And, then using the result by Lahiri and Dewan [7, eq. (1.1)] proved the following result which improved general version of Theorem 1.3.

**Theorem 1.5.** *Let $F=ff^{(k)}$, where $k$ is a positive integer. Then for any small function $a$ $(\not\equiv 0,\infty)$ of $f$*

$$\Theta(a;F)+\Theta(-a;F)\leq 2-\frac{2}{(2+k)^2}. \tag{1.2}$$

In this paper, we prove more general result that improves Theorem 1.4 and Theorem 1.5. In fact we prove the above two results to investigate the value distribution of the product of a meromorphic function with its all finite order derivatives instead of just only $k$th order derivative. Throughout the paper, unless otherwise mentioned, we denote by $f$ a transcendental meromorphic function defined in the open complex plane C.

## 2. Preliminary Results

To reach the main results of this paper we need the following lemmas.

**Lemma 2.1** ([4]). *For a non-constant meromorphic function $f$, $n$ being a positive integer*

$$T(r,f^{(n)})\leq (1+n)T(r,f)+S(r,f). \tag{2.1}$$

**Lemma 2.2** ([7]). *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}|f\neq 0)\leq k\bar{N}(r,\infty;f)+N(r,0;f|<k)+k\bar{N}(r,0;f|\geq k)+S(r,f). \tag{2.2}$$

## 3. Main Results

In this section, we present the main result of the paper.

**Theorem 3.1.** *Let $\psi=(f)^{n_0}(f^{(1)}f^{(2)}\dots f^{(k)})^{n_1}$, where $n_0(\geq 2)$, $n_1$ and $k$ are positive integers such that $2[n_0(n_0-2)+n_1k(n_0-1)]+k(k+1)[n_0n_1-n_0-n_1]>0$. Then,*

$$\left[1-\frac{2+k(k+1)}{2n_0+k(k+1)}-\frac{2n_0\{2+k(k+1)\}}{\{2n_0+k(k+1)\}\{2(n_0+n_1k)+n_1k(k+1)\}}\right]T(r,\psi)\leq \bar{N}(r,a;\psi)+S(r,\psi), \tag{3.1}$$

*for any small function $a$ $(\not\equiv 0,\infty)$ of $f$.*

*Proof.* First we note that from [9],

$$T(r,f)+S(r,f)\le CT(r,\psi)+S(r,\psi) \tag{3.2}$$

where $C$ is a constant.
Again using Lemma 2.1,

$$\begin{aligned}
T(r,\psi) &\le T(r,(f)^{n_0})+T(r,(f^{(1)})^{n_1})+T(r,(f^{(2)})^{n_1})+\ldots+T(r,(f^{(k)})^{n_1})\\
&\le n_0T(r,f)+n_1T(r,f^{(1)})+n_1T(r,f^{(2)})+\ldots+n_1T(r,f^{(k)})\\
&\le n_0T(r,f)+n_1(1+1)T(r,f)+n_1(1+2)T(r,f)+\ldots+n_1(1+k)T(r,f)+S(r,f)\\
&= n_0T(r,f)+n_1[2+3+\ldots+(1+k)]T(r,f)+S(r,f)\\
&= (n_0-n_1)T(r,f)+n_1[1+2+3+\ldots+(1+k)]T(r,f)+S(r,f)\\
&= \left[(n_0-n_1)+\frac{n_1(k+1)(k+2)}{2}\right]T(r,f)+S(r,f).
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3) it is evident that $a$ $(\not\equiv 0,\infty)$ is a small function of $f$ if and only if $a$ is also a small function of $\psi$.
Now by Nevanlinna's three small functions theorem [4], we get

$$T(r,\psi)\le \bar{N}(r,0;\psi)+\bar{N}(r,\infty;\psi)+\bar{N}(r,a;\psi)+S(r,\psi), \tag{3.4}$$

where $\bar{N}(r,a;\psi)=\bar{N}(r,0;\psi-a)$.
Now by Lemma 2.2, we get

$$\begin{aligned}
\bar{N}(r,0;\psi) &\le \bar{N}(r,0;f)+N(r,0;f^{(1)}|f\ne 0)+N(r,0;f^{(2)}|f\ne 0,f^{(1)}\ne 0)+\ldots\\
&\quad +N(r,0;f^{(k)}|f\ne 0,f^{(1)}\ne 0,f^{(2)}\ne 0,\ldots,f^{(k-1)}\ne 0)\\
&\le \bar{N}(r,0;f)+N(r,0;f^{(1)}|f\ne 0)+N(r,0;f^{(2)}|f\ne 0)+\ldots+N(r,0;f^{(k)}|f\ne 0)\\
&\le \bar{N}(r,0;f)+[1.\bar{N}(r,\infty;f)+N(r,0;f|<1)+1\cdot\bar{N}(r,0;f|\ge 1)]\\
&\quad +[2\cdot\bar{N}(r,\infty;f)+N(r,0;f|<2)+2\cdot\bar{N}(r,0;f|\ge 2)]+\ldots\\
&\quad +[k\cdot\bar{N}(r,\infty;f)+N(r,0;f|<k)+k\cdot\bar{N}(r,0;f|\ge k)]+S(r,f)\\
&\le \bar{N}(r,0;f)+[1\cdot\bar{N}(r,\infty;f)+1\cdot\bar{N}(r,0;f|<1)+1\cdot\bar{N}(r,0;f|\ge 1)]\\
&\quad +[2\cdot\bar{N}(r,\infty;f)+2\cdot\bar{N}(r,0;f|<2)+2\cdot\bar{N}(r,0;f|\ge 2)]+\ldots\\
&\quad +[k\cdot\bar{N}(r,\infty;f)+k\cdot\bar{N}(r,0;f|<k)+k\cdot\bar{N}(r,0;f|\ge k)]+S(r,f)\\
&= \bar{N}(r,0;f)+[1\cdot\bar{N}(r,\infty;f)+1\cdot\bar{N}(r,0;f)]+[2\cdot\bar{N}(r,\infty;f)+2\cdot\bar{N}(r,0;f)]+\ldots\\
&\quad +[k\cdot\bar{N}(r,\infty;f)+k\cdot\bar{N}(r,0;f)]+S(r,f)\\
&= \bar{N}(r,0;f)+[1+2+\ldots+k]\bar{N}(r,\infty;f)+[1+2+\ldots+k]\bar{N}(r,0;f)+S(r,f)\\
&= \left[1+\frac{k(k+1)}{2}\right]\bar{N}(r,0;f)+\frac{k(k+1)}{2}\bar{N}(r,\infty;f)+S(r,f).
\end{aligned} \tag{3.5}$$

Since $\psi=(f)^{n_0}(f^{(1)}f^{(2)}\ldots f^{(k)})^{n_1}$, a zero of $f$ with multiplicity $q$ will be a zero of $\psi$ with multiplicity

$$\begin{aligned}
&= n_0q+n_1(q-1)+n_1(q-2)+\ldots+n_1(q-k)\\
&= n_0q+n_1[kq-(1+2+\ldots+k)]\\
&= (n_0+n_1k)q-\frac{n_1k(k+1)}{2}.
\end{aligned}$$

Therefore, a zero of $f$ with multiplicity $q \geq (k+1)$ becomes a zero of $\psi$ with multiplicity

$$\geq (n_0 + n_1 k)(k+1) - \frac{n_1 k(k+1)}{2}$$
$$= (k+1)\left(n_0 + \frac{n_1 k}{2}\right).$$

Thus, we see that

$$N(r,0;\psi) - \bar{N}(r,0;\psi) \geq \left[(k+1)\left(n_0 + \frac{n_1 k}{2}\right) - 1\right]\bar{N}(r,0;f| \geq k+1) + (n_0 - 1)\bar{N}(r,0;f| \leq k)$$

which gives

$$\bar{N}(r,0;f| \leq k) \leq \frac{1}{n_0 - 1}\left[N(r,0;\psi) - \bar{N}(r,0;\psi) - \left\{(k+1)\left(n_0 + \frac{n_1 k}{2}\right) - 1\right\}\bar{N}(r,0;f| \geq k+1)\right]. \quad (3.6)$$

Hence from (3.5) and (3.6), we get

$$\begin{aligned}
\bar{N}(r,0;\psi) \leq & \left[1 + \frac{k(k+1)}{2}\right]\bar{N}(r,0;f| \geq k+1) + \left[1 + \frac{k(k+1)}{2}\right]\bar{N}(r,0;f| \leq k) \\
& + \frac{k(k+1)}{2}\bar{N}(r,\infty;f) + S(r,f) \\
\leq & \left[1 + \frac{k(k+1)}{2}\right]\bar{N}(r,0;f| \geq k+1) + \frac{1}{n_0 - 1}\left[1 + \frac{k(k+1)}{2}\right][N(r,0;\psi) - \bar{N}(r,0;\psi) \\
& - \left\{(k+1)\left(n_0 + \frac{n_1 k}{2}\right) - 1\right\}\bar{N}(r,0;f| \geq k+1)] + \frac{k(k+1)}{2}\bar{N}(r,\infty;f) + S(r,f),
\end{aligned}$$

which gives

$$\begin{aligned}
& \left[1 + \frac{1}{n_0 - 1}\left\{1 + \frac{k(k+1)}{2}\right\}\right]\bar{N}(r,0;\psi) \\
& \leq \frac{1}{n_0 - 1}\left\{1 + \frac{k(k+1)}{2}\right\}N(r,0;\psi) \\
& \quad - \left\{1 + \frac{k(k+1)}{2}\right\}\left[\frac{1}{n_0 - 1}\left\{(k+1)\left(n_0 + \frac{n_1 k}{2}\right) - 1\right\} - 1\right]\bar{N}(r,0;f| \geq k+1) \\
& \quad + \frac{k(k+1)}{2}\bar{N}(r,\infty;f) + S(r,f) \\
& \leq \frac{1}{n_0 - 1}\left\{1 + \frac{k(k+1)}{2}\right\}N(r,0;\psi) + \frac{k(k+1)}{2}\bar{N}(r,\infty;f) + S(r,f).
\end{aligned}$$

On simplification, we get

$$\left[\frac{n_0}{n_0 - 1} + \frac{k(k+1)}{2(n_0 - 1)}\right]\bar{N}(r,0;\psi) \leq \frac{2 + k(k+1)}{2(n_0 - 1)}N(r,0;\psi) + \frac{k(k+1)}{2}\bar{N}(r,\infty;f) + S(r,f)$$

i.e.,

$$\bar{N}(r.0;\psi) \leq \frac{2 + k(k+1)}{2n_0 + k(k+1)}N(r,0;\psi) + \frac{k(k+1)(n_0 - 1)}{2n_0 + k(k+1)}\bar{N}(r,\infty;f) + S(r,f). \quad (3.7)$$

Again since $\psi = (f)^{n_0}(f^{(1)}f^{(2)} \dots f^{(k)})^{n_1}$, a pole of $f$ with multiplicity $p$ will be a pole of $\psi$ with multiplicity

$$= n_0 p + n_1(p+1) + n_1(p+2) + \dots + n_1(p+k)$$
$$= n_0 p + n_1[pk + (1 + 2 + \dots + k)]$$

$$
\begin{aligned}
&= (n_0+n_1k)p + \frac{n_1k(k+1)}{2} \\
&\geq (n_0+n_1k) + \frac{n_1k(k+1)}{2}.
\end{aligned}
$$

Hence

$$N(r,\infty;\psi) \geq \left[(n_0+n_1k)+\frac{n_1k(k+1)}{2}\right]\bar{N}(r,\infty;\psi). \tag{3.8}$$

Since $\bar{N}(r,\infty;\psi)=\bar{N}(r,\infty;f)$ and $S(r,\psi)=S(r,f)$, from (3.4), (3.7) and (3.8), we get

$$
\begin{aligned}
T(r,\psi) &\leq \frac{2+k(k+1)}{2n_0+k(k+1)}N(r,0;\psi)+\frac{k(k+1)(n_0-1)}{2n_0+k(k+1)}\bar{N}(r,\infty;f)+\bar{N}(r,\infty;\psi) \\
&\quad +\bar{N}(r,a;\psi)+S(r,\psi) \\
&= \frac{2+k(k+1)}{2n_0+k(k+1)}N(r,0;\psi)+\left\{1+\frac{k(k+1)(n_0-1)}{2n_0+k(k+1)}\right\}\bar{N}(r,\infty;\psi)+\bar{N}(r,a;\psi)+S(r,\psi) \\
&= \frac{2+k(k+1)}{2n_0+k(k+1)}N(r,0;\psi)+\frac{n_0\{2+k(k+1)\}}{2n_0+k(k+1)}\bar{N}(r,\infty;\psi)+\bar{N}(r,a;\psi)+S(r,\psi) \\
&\leq \frac{2+k(k+1)}{2n_0+k(k+1)}N(r,0;\psi)+\frac{n_0\{2+k(k+1)\}}{2n_0+k(k+1)}\cdot\frac{2}{2(n_0+n_1k)+n_1k(k+1)}N(r,\infty;\psi) \\
&\quad +\bar{N}(r,a;\psi)+S(r,\psi) \\
&\leq \frac{2+k(k+1)}{2n_0+k(k+1)}T(r,\psi)+\frac{2n_0\{2+k(k+1)\}}{\{2n_0+k(k+1)\}\cdot\{2(n_0+n_1k)+n_1k(k+1)\}}T(r,\psi) \\
&\quad +\bar{N}(r,a;\psi)+S(r,\psi)
\end{aligned}
$$

which gives the required result (3.1). □

**Theorem 3.2.** *Let $F=ff^{(1)}f^{(2)}\dots f^{(k)}$, where $k$ is a positive integer. Then, for any small function $a(\not\equiv 0,\infty)$ of $f$*

$$\Theta(a;F)+\Theta(-a;F)\leq 2-\frac{8k}{(k+1)(k+2)(k^2+k+4)}. \tag{3.9}$$

*Proof.* Since $a$ is a small function of $f$, $a^2$ will also be a small function of $f$. Thus, when we consider the case $n_0=n_1=2$ in Theorem 3.1, from (3.1) we get

$$
\begin{aligned}
&\left[1-\frac{2+k(k+1)}{4+k(k+1)}-\frac{4\{2+k(k+1)\}}{\{4+k(k+1)\}\{2(2+2k)+2k(k+1)\}}\right]T(r,F^2) \\
&\leq \bar{N}(r,a^2;F^2)+S(r,F),
\end{aligned}
$$

i.e.,

$$2\left[1-\frac{2+k(k+1)}{4+k(k+1)}-\frac{2\{2+k(k+1)\}}{\{4+k(k+1)\}\{2(1+k)+k(k+1)\}}\right]T(r,F)\leq\bar{N}(r,a;F)+\bar{N}(r,-a;F)+S(r,F)$$

i.e.,

$$2\left[1-\frac{2+k(k+1)}{4+k(k+1)}\left\{1+\frac{2}{(k+1)(k+2)}\right\}\right]T(r,F)\leq\bar{N}(r,a;F)+\bar{N}(r,-a;F)+S(r,F).$$

Now dividing both sides of above inequality by $T(r,F)$ and letting $r\to\infty$, we get

$$\Theta(a,F)+\Theta(-a,F)\leq\frac{2\{2+k(k+1)\}}{4+k(k+1)}\left\{1+\frac{2}{(k+1)(k+2)}\right\}$$

$$
\begin{aligned}
&= 2\left[\frac{k^2+k+2}{k^2+k+4}\cdot\frac{k^2+3k+4}{k^2+k+2}\right] \\
&= 2\left[\frac{k^4+4k^3+9k^2+10k+8}{k^4+4k^3+9k^2+14k+8}\right] \\
&= 2\left[1-\frac{4k}{k^4+4k^3+9k^2+14k+8}\right] \\
&= 2-\frac{8k}{(k+1)(k+2)(k^2+k+4)}. \qquad \square
\end{aligned}
$$

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

## References


[1] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Revista Matemática Iberoamericana* **11** (1995), 355 – 373, DOI: 10.4171/RMI/176.

[2] W. Bergweiler, On the product of a meromorphic function and its derivative, *Bulletin of the Hong Kong Mathematical Society* **1** (1997), 97 – 101.

[3] H. H. Chen and M. L. Fang, The value distribution of $f^n f'$, *Science China. Mathematics* **38** (1995), 789 – 798.

[4] W. K. Hayman, *Meromorphic Functions*, The Claredon Press, Oxford, (1964).

[5] W. K. Hayman, *Research Problems in Function Theory*, The Athlone Press University of London, London, viii + 284 pages (1967).

[6] W. K. Hayman, Picard value of meromorphic functions and their derivatives, *Annals of Mathematics* **70**(1) (1959), 9 – 42, URL: https://doi.org/10.2307/1969890.

[7] I. Lahiri and S. Dewan, Value distribution of the product of a meromorphic function and its derivative, *Kodai Mathematical Journal* **26** (2003), 95 – 100, DOI: 10.2996/kmj/1050496651.

[8] E. Mues, Über ein Problem von Hayman, *Mathematische Zeitschrift* **164** (1979), 239 – 259, DOI: 10.1007/BF01182271.

[9] A. P. Singh, On order of homogeneous differential polynomials, *Indian Journal of Pure and Applied Mathematics* **16** (1985), 791 – 795.

[10] K.-W. Yu, A note on the product of meromorphic functions and its derivatives, *Kodai Mathematical Journal* **24** (2001), 339 – 343, DOI: 10.2996/kmj/1106168808.