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Research Article

Distortion of Orbits Under Fluctuating Gravitational Forces

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Abstract. The motion of a celestial body under the influence of a central force is considered. The influence of other bodies on the gravitational attraction force of the two-body problem is modeled as fluctuating force perturbations about a mean value. The ordinary differential equation determining the orbits is derived in polar coordinates. The exact analytical solution is given. Three cases, namely the elliptic, parabolic and hyperbolic cases are investigated. The distortion of the orbits under the influence of small perturbational forces are depicted. The equation determining the escape angle is solved numerically by the Newton-Raphson method. Effects of fluctuation frequencies on the distortion of orbits as well as escape angles are studied in detail. It is found that the force fluctuations may alter the nature of the orbits and destroy their symmetries. When the fluctuation frequency is an integer number, the orbits are distorted but the escape angles remain unaffected for the parabolic case. For non-integer frequencies, the escape angles may decrease in magnitude.

Keywords. Celestial mechanics, Orbits, Gravitational forces, Escape angles, Analytical solutions, Numerical calculations

Mathematics Subject Classification (2020). 70F15, 70M20, 34A05

1. Introduction

Kepler is the first one to point out that planets are travelling on elliptic paths around the Sun and changed the wrong belief that all celestial bodies are moving around the Earth in circular paths. Newton later verified the axioms of Kepler by mathematical formulations. The problem of determining the orbits of celestial bodies reduces to the problem of a mass moving under the influence of a central gravitational attraction force. The mechanical energy and angular momentum are the conserved quantities for the problem. The analytical solutions reveal three distinct cases for the orbits, namely the elliptic, parabolic and hyperbolic paths. While the elliptic paths are stable, for the parabolic and hyperbolic paths, the mass escapes from the influence of the main gravitational attracting body which itself is much larger than the rotating mass around it. The limiting case is the parabolic path for which the escape angles are $\mp\pi$ radians and for the hyperbolic paths, they are lower than these values (Beer *et al*. [1], and Meriam and Kraige [3]).

The two-body system is subject to influences from the other celestial bodies. For a planet rotating around a star, a comet rotating around the planet may influence the orbits. Other celestial bodies in the outer space of the orbital area may influence the main attractional forces between the bodies as well. In this work, the small variations in the central gravitational force due to external bodies are modelled by a fluctuating force about a mean value. The governing equations of motion are expressed in polar coordinates and the differential equation determining the orbits is derived. The exact solution of the equation is given. The unperturbed orbits and the distortion of the orbits due to fluctuations are depicted in figures. Three different cases, the elliptical, parabolic and hyperbolic paths are investigated in detail. It is shown that the symmetry of the paths is spoiled due to the fluctuations in the gravitational forces. As a result, one branch may express a stable path while the other may express an unstable path. The effect of fluctuation frequencies on the escape angles are studied in detail. For originally parabolic paths, the fluctuation frequency may stabilize one branch with no escape angles and destabilizing the other path with lower escape angles. The elliptic paths with high eccentricity may also be destabilized partially under the influence of fluctuations. The hyperbolic paths may also alter their nature under the fluctuations. The escape angles are doomed to change under the influence of the fluctuations.

Some of the relevant study on the subject is given: Solution of the Kepler problem was given by taking the energy equation in a suitable two term square form and solving them algebraically. An additional potential energy term is considered to explain the presession of Mercury around Sun (Moriconi [4]). A perturbing central force on the elliptic Keplerian orbit was considered by Davies [2] and applied to the explanation of perihelion presession of Mercury. A quadratic nonlinear model including the relativistic effects on the precession of the orbit of Mercury is treated by Pakdemirli [5] using the newly developed Multiple Scales Lindstedt Poincare method. The commonly referred Kepler's equation is a transcendental equation which relates the dependence of position with time. For solution of the Kepler's equation numerically for nearly parabolic orbits, see, e.g., Rasheed [6], and Serafin [8]. To the best of the authors' knowledge, the effect of gravitational force fluctuations on the orbits of celestial bodies was not considered before which is addressed in this study for the first time.

2. Equation of Motion and the Orbital Equation

The motion of a particle moving in an orbit under the influence of a central force only can be more practically expressed in polar coordinates (Beer *et al*. [1], and Meriam and Kraige [3]),

$$-F = m(\ddot{r} - r\dot{\theta}^2), \tag{1}$$

$$0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}), \tag{2}$$

where the parentheses on the right-hand sides express the radial and tangential accelerations in polar coordinates. The second equation can be integrated to yield

$$r^2\dot{\theta} = h, \tag{3}$$

where h is the angular momentum per unit mass which is a constant of conserved quantity. Expressing the first and second derivatives

$$\dot{r} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right), \quad \ddot{r} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \tag{4}$$

and substituting the latter into (1) yields the ordinary differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \tag{5}$$

with

$$u = \frac{1}{r}. \tag{6}$$

One may now assume that the central force is a gravitational force with small fluctuations about a mean value

$$F = \frac{GMm}{r^2}(1 + \alpha\sin\Omega\theta) = GMmu^2(1 + \alpha\sin\Omega\theta), \tag{7}$$

where α is the fluctuation amplitude ($\alpha \ll 1$) and Ω is the fluctuation frequency. The fluctuations may represent the influences of other external bodies (Comets rotating around the small mass, external celestial objects outside the orbital area of the two masses, etc.). In view of (7), equation (5) reduces to

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}(1 + \alpha\sin\Omega\theta). \tag{8}$$

The initial conditions for the problem can be written as follows

$$u(0) = u_0 = \frac{1}{r_0}, \quad \frac{du}{d\theta}(0) = 0 \tag{9}$$

with r_0 being the initial distance between the center of masses. The exact solution of (8) subject to the conditions (9) is

$$u = \frac{1}{r} = \frac{GM}{h^2}\left(1 + \varepsilon\cos\theta + \frac{\alpha}{1-\Omega^2}(\sin\Omega\theta - \Omega\sin\theta)\right). \tag{10}$$

When the fluctuations are absent, i.e., $\alpha = 0$, the solution reduces to the well-known solution of conical orbits (Beer *et al*. [1], and Meriam and Kraige [3]),

$$u = \frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon\cos\theta). \tag{11}$$

Note that

$$\varepsilon = \frac{h^2}{GMr_0} - 1 \tag{12}$$

is called the eccentricity of the orbit. For $\varepsilon < 1$, the orbit is elliptic, for $\varepsilon = 1$, it is parabolic and for $\varepsilon > 1$, it is hyperbolic. Only the elliptic orbit is stable and the mass does not escape from the gravitational attraction of the main body. The three cases will be investigated in the next section with respect to distortions in the orbits and the escape angles if present.

3. Numerical Results

The orbits and the escape angles will be treated for all cases numerically in this section. The dimensionless distance $\bar{r}$ is used in all graphics of orbits

$$\bar{r} = \frac{GM}{h^2} r. \tag{13}$$

3.1 Elliptic Orbits ($\varepsilon < 1$)

The eccentricity is less than one in this special case. The Earth's orbit around Sun is almost circular with $\varepsilon = 0.0167$ whereas for Mercury, the eccentricity is higher ($\varepsilon = 0.2056$) being more elliptic (Saari [7]). For perfect circular orbits $\varepsilon = 0$. For this case, the solution is

$$u = \frac{1}{r} = \frac{GM}{h^2}\left(1 + \frac{\alpha}{1-\Omega^2}(\sin\Omega\theta - \Omega\sin\theta)\right) \tag{14}$$

and keeping in mind that $\frac{\alpha}{1-\Omega^2} \ll 1$ if Ω is away from 1, the larger parenthesis does not vanish and the orbit is stable (Figure 1).

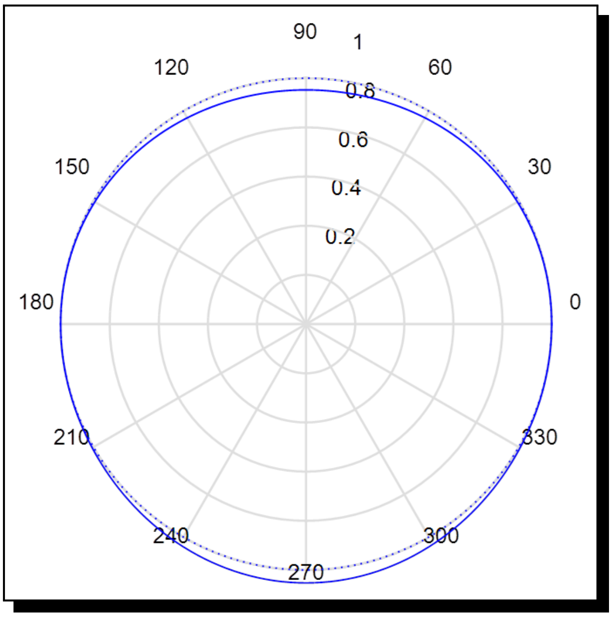

Figure 1. Unperturbed (dotted) and perturbed (solid) orbits ($\varepsilon = 0$, $\alpha = 0.1$, $\Omega = 3$)

Distortions in the circular orbit can be seen from Figure 1. For the given specific parameter values, the distances between the mass centers become shorter for the upper half of the graph and longer for the lower half.

Distortions of an elliptic orbit is given in Figure 2 for $\varepsilon = 0.5$ (equation (10)).

Slight deviations from the elliptic orbit are observed which does not spoil the stability of the orbit.

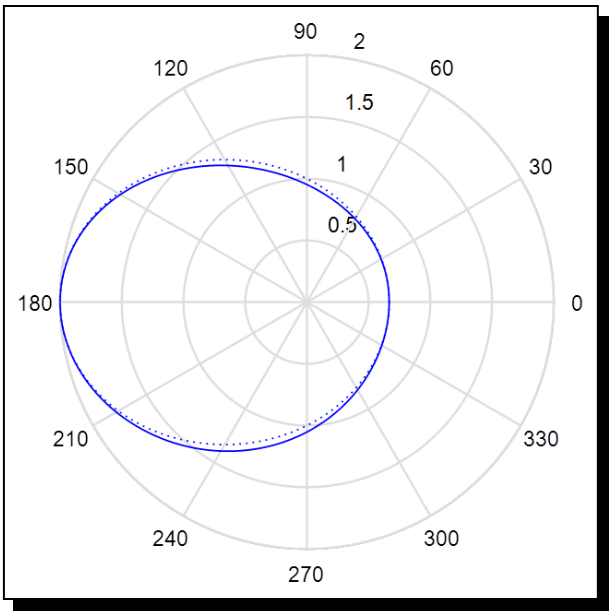

Figure 2. Unperturbed (dotted) and perturbed (solid) orbits ($\varepsilon = 0.5$, $\alpha = 0.1$, $\Omega = 3$)

Nearly parabolic orbits have eccentricity values $0.99 \leq \varepsilon \leq 1.01$ (Serafin [8]). When the original elliptic orbit is nearly parabolic, the distance may tend to infinity with fluctuations. A sample case is depicted in Figure 3.

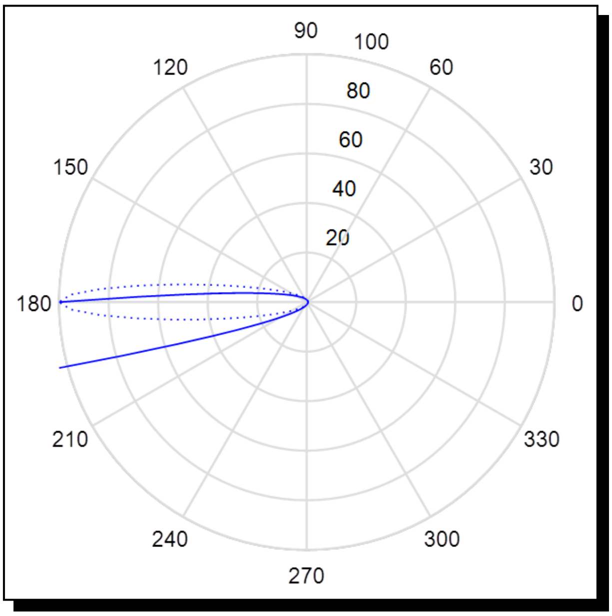

Figure 3. Unperturbed (dotted) and perturbed (solid) orbits ($\varepsilon = 0.99$, $\alpha = 0.1$, $\Omega = 2$)

From Figure 3, it is clear that the lower branch tends to infinity for the specific parameter values taken making the orbit unstable.

3.2 Parabolic Orbits ($\varepsilon = 1$)

Parabolic orbits are the limiting cases where the orbits transform from a stable to unstable configuration or vice versa. The escape angles where the distances tend to infinity are from

$1+\cos\theta = 0$ or $\theta_e = \mp\pi$. When fluctuation effects are included, these angles may be lower in magnitude. The equation to be solved numerically is

$$1+\cos\theta + \frac{\alpha}{1-\Omega^2}(\sin\Omega\theta - \Omega\sin\theta) = 0. \tag{15}$$

For the fluctuation frequencies being integer numbers, i.e., $\Omega = n$, $n = 1,2,3\dots$, the escape angles are still the same with the unperturbed case, $\theta_e = \mp\pi$. However, for non-integer values, they are less in magnitudes. The frequencies versus escape angles are given in Figure 4 for $2 < \Omega < 3$.

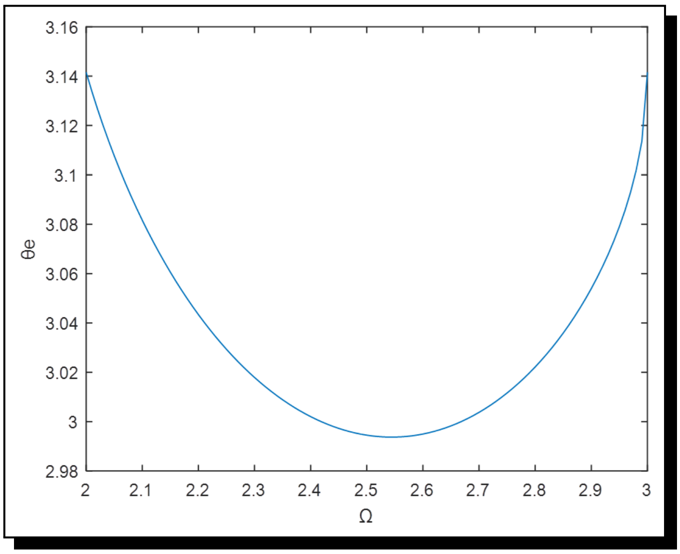

Figure 4. Frequency versus escape angles ($\varepsilon = 1$, $\alpha = 0.1$, $2 < \Omega < 3$)

Note that, in this parametric range, only positive roots exist for equation (15) leading to a divergence in the upper branch only. For the range of $3 < \Omega < 4$, only negative roots exist as depicted in Figure 5.

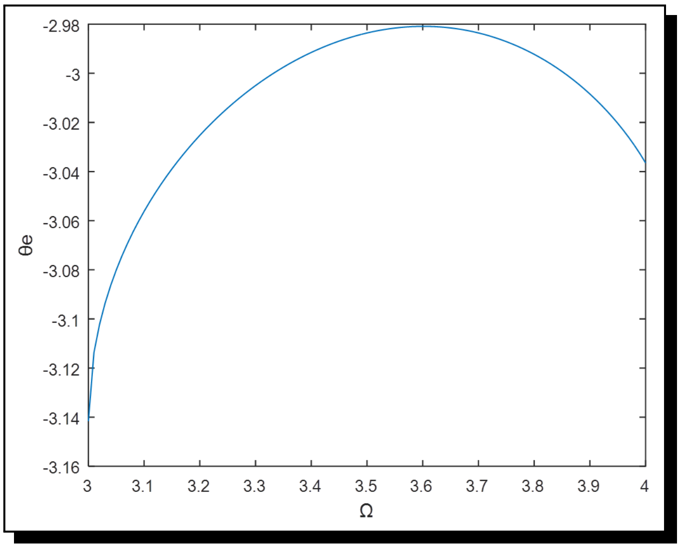

Figure 5. Frequency versus escape angles ($\varepsilon = 1$, $\alpha = 0.1$, $3 < \Omega < 4$)

As a general rule, if the integer part of the frequency is even, the escape angles are positive and if the integer part is odd, the escape angles are negative.

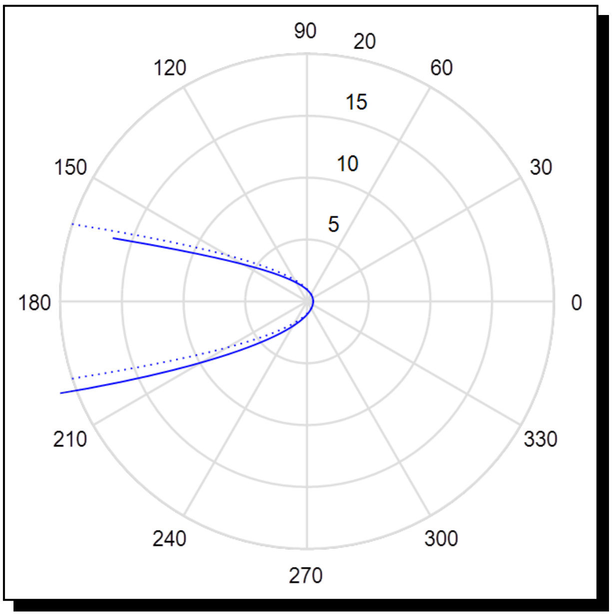

Figure 6. Unperturbed (dotted) and perturbed (solid) orbits ($\varepsilon = 1$, $\alpha = 0.1$, $\Omega = 2.3$)

A sample plot is given to compare the unperturbed and perturbed solutions in Figure 6.

3.3 Hyperbolic Orbits ($\varepsilon > 1$)

For hyperbolic unperturbed orbits, the eccentricity is larger than 1 and the orbit is unstable with the distances tending to infinity when $1+\varepsilon\cos\theta = 0$ or

$$\theta_e = \arccos\left(-\frac{1}{\varepsilon}\right). \tag{16}$$

For the perturbed orbits, the equation has to be solved numerically for the escape angles

$$1+\varepsilon\cos\theta + \frac{\alpha}{1-\Omega^2}(\sin\Omega\theta - \Omega\sin\theta) = 0. \tag{17}$$

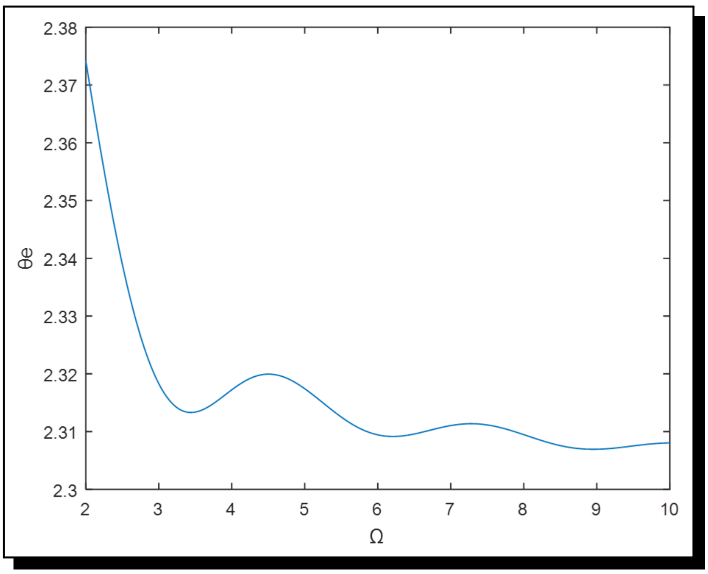

Figure 7. Frequency versus positive escape angles ($\varepsilon = 1.5$, $\alpha = 0.1$, $2 < \Omega < 10$)

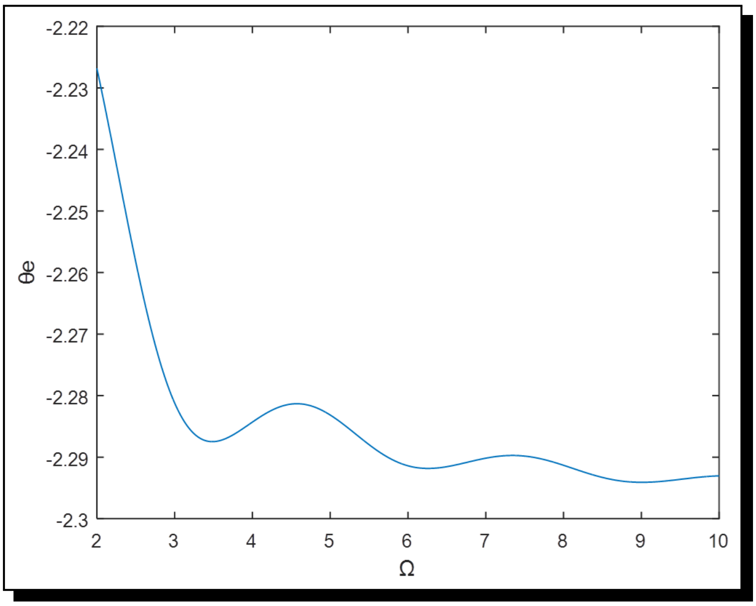

Figure 8. Frequency versus negative escape angles ($\varepsilon = 1.5$, $\alpha = 0.1$, $2 < \Omega < 10$)

In Figure 7, the positive escape angles and in Figure 8, the negative escape angles are given for various fluctuation frequencies. From (16), the escape angles for the unperturbed equation are $\theta_e = \mp 2.301$ radians. For the given parameters, the positive escape angles are larger than the unperturbed case whereas, the negative escape angles are smaller in magnitude than the unperturbed case. A sample plot of the unperturbed and perturbed orbits is given in Figure 9.

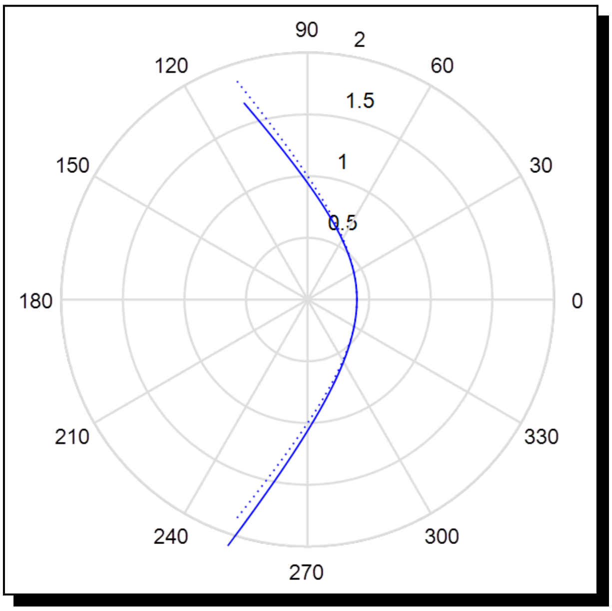

Figure 9. Unperturbed (dotted) and perturbed (solid) orbits ($\varepsilon = 1.5$, $\alpha = 0.1$, $\Omega = 2.3$)

4. Concluding Remarks

The effects of perturbed gravitational forces on the orbits of celestial bodies are investigated in detail. The perturbations are modeled as harmonic fluctuations about a mean value.

The fluctuations distort the symmetry of the unperturbed orbits and results in deviations from them. Depending on the physical parameters, they may alter the nature of the orbits also. The escape angles are also affected by the fluctuations and a detailed analysis is given for the escape angles.

Availability of Supporting Data

Additional data available upon request from the author.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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