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Research Article

A Contemporary Conjecture for the Riemann Hypothesis

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Abstract. We will present two new results for the "Dirichlet eta" function $S(s) = \sum_{n\geq 1} \frac{(-1)^n}{n^s}$ which would lead us to announce a new conjecture equivalent to that of the Rieman Hypothesis.

Keywords. Riemann zeta-function, Hardy's functional equation, Adherent point, Complex number

Mathematics Subject Classification (2020). Primary 11E45; Secondary 11M41

1. Introduction

The Riemann hypothesis is a conjecture formulated in 1859 by the mathematician Bernhard Riemann, according to which the non-trivial and infinity zeros of Riemann's zeta function all have a real part equal to $\frac{1}{2}$.

Its demonstration would improve knowledge of the distribution of prime numbers and open up new areas for mathematics.

Riemann's article [4] on the distribution of prime numbers is his only text dealing with number theory, he develops the properties of the function zeta $C(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ and prove the prime number theorem by admitting to passing several results including what is now called the Riemann Hypothesis (RH). Then, Hardy demonstrated that there exist an infinity of zeros on the critical line (see, Hardy [1], and Hardy and Littlewood [2]), this gives us an esperance that the RH would be true.

Let $S(s)=\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s}=-\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}$, thus $S(s)=\rho(s)e^{i\theta(s)}S(1-s)$

Remark 1.1 (Functional equation of Hardy)**.** We have $\forall\ s\in\mathbb{C}$ such $\mathrm{Re}(s)\in\,]0,1[$

$$S(s)=\varphi(s)S(1-s)$$

with $\varphi(s)=2\frac{1-2^{s-1}}{1-2^s}\pi^{s-1}\sin\left(\frac{s}{2}\pi\right)\Gamma(1-s)=\rho(s)e^{i\theta(s)}$ (see [1] and [2]).

2. Preliminaries

Proposition 2.1. *Let $s=r+ic$, thus*

$$S=\sum_{n=1}^{+\infty}(-1)^n\frac{e^{-i\ln(n)c}}{n^r}=C_1-C_2$$

$$=\sum_{n=1}^{+\infty}(-1)^n\frac{e^{i\alpha_n}}{n^r}=R'+iI'$$

and

$$C=\sum_{n=1}^{+\infty}\frac{1}{n^s}=\sum_{n=1}^{+\infty}\frac{e^{-i\ln(n)c}}{(2n)^r}=\sum_{n=1}^{+\infty}\frac{e^{i\alpha_n}}{(2n)^r}$$

$$=C_1+C_2=R+iI$$

with $\alpha_n=-\ln(n)c$,

$$C_1=\sum_{n=1}^{+\infty}\frac{1}{(2n)^s}=\sum_{n=1}^{+\infty}\frac{e^{-i\ln(2n)c}}{(2n)^r}=\sum_{n=1}^{+\infty}\frac{e^{i\alpha_{2n}}}{(2n)^r}=R_1+iI_1,$$

$$C_2=\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^s}=\sum_{n=1}^{+\infty}\frac{e^{-i\ln(2n-1)c}}{(2n-1)^r}=\sum_{n=1}^{+\infty}\frac{e^{i\alpha_{2n-1}}}{(2n-1)^r}=R_2+iI_2$$

and

$$R_1=\sum_{n=1}^{+\infty}\frac{\cos(\alpha_{2n})}{(2n)^r},$$

$$I_1=\sum_{n=1}^{+\infty}\frac{\sin(\alpha_{2n}}{(2n)^r},$$

$$R_2=\sum_{n=1}^{+\infty}\frac{\cos(\alpha_{2n-1})}{(2n-1)^r},$$

$$I_2=\sum_{n=1}^{+\infty}\frac{\sin(\alpha_{2n-1})}{(2n-1)^r},$$

$$R=\sum_{n=1}^{+\infty}\frac{\cos(\alpha_n)}{(2n)^r}=R_1+R_2,$$

$$I=\sum_{n=1}^{+\infty}\frac{\sin(\alpha_n)}{(2n)^r}=I_1+I_2,$$

$$R'=\sum_{n=1}^{+\infty}(-1)^n\frac{\cos(\alpha_n)}{n^r},$$

$$I' = \sum_{n=1}^{+\infty} (-1)^n \frac{\sin(\alpha_n)}{n^r},$$
$$R' = R_1 - R_2,$$
$$I' = I_1 - I_2.$$

Proposition 2.2. *Let* $s = r + ic = r + i\frac{\alpha}{\ln(2)}$ *(since* $\alpha = \ln(2)c$*),*

$$C_1 = \frac{e^{-i\alpha}}{2^r} C,$$
$$C_2 = \left(1 - \frac{e^{-i\alpha}}{2^r}\right) C,$$
$$S = (2^{1-r} e^{-i\alpha} - 1)C.$$

Proof. $\alpha = \ln(2)c \implies e^{-i\ln(2)c} = e^{-i\alpha}$. Therefore,

$$C_1 = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n)c}}{(2n)^r} = \frac{e^{-i\ln(2)c}}{2^r} \sum_{n=1}^{+\infty} \frac{e^{-i\ln(n)c}}{n^r},$$
$$C_1 = \frac{e^{-i\ln(2)c}}{2^r} C = \frac{e^{-i\alpha}}{2^r} C,$$
$$C_2 = C - C_1 = C - \frac{e^{-i\alpha}}{2^r} C \implies C_2 = \left(1 - \frac{e^{-i\alpha}}{2^r}\right) C$$

and

$$S = C_1 - C_2 = \frac{e^{-i\alpha}}{2^r} C - \left(1 - \frac{e^{-i\alpha}}{2^r}\right) C \implies S = (2^{1-r} e^{-i\alpha} - 1)C.$$ □

3. Contributions

Theorem 3.1 (Adherent Point and Closure)**.** *Let* X *be a topological space and* $A \subseteq X$ *be a subset. A point* $x \in X$ *is said to be an adherent point of* A *if every open neighborhood of* x *intersects* A*. The closure of* A*, denoted by* $\overline{A}$*, consists of all adherent points of* A*.*

Theorem 3.2 (Adherent Point and Existence of Convergent Sequences)**.** *Let* X *be a topological space and* $A \subseteq X$ *be a subset. A point* $x \in X$ *is an adherent point of* A *if and only if there exists a sequence* x_n *in* A *such that*

$$\lim_{n \Longrightarrow +\infty} x_n = x.$$

3.1 The First Announcement

Lemma 3.1. *Assuming that there exists an* s_1 *with* $r_1 = \mathrm{Re}[s_1] \in \left]0, \frac{1}{2}\right[$ *and* $\alpha = \ln(2)c > 0$ *such that* $S^2(s_1) \in \mathbb{R}$*, so*

(i) $\exists\, V(s_1) \subset \mathbb{C}$ *such* $\forall\, s \in V(s_1) - \{s_1\}$, $S^2(s) \notin \mathbb{R}$,

(ii) $\exists\, u_n \in V(s_1) - \{s_1\}$ *such* $\lim u_n = s_1$ *(since* $s_1 \in \overline{V(s_1) - \{s_1\}}$ *with* $\overline{A}$ *is the adherant of* A*).*

Proof. Obvious.

(i) reasoning through the absurd.

(ii) using (i) and the last theorem. □

Lemma 3.2. *Let* $D_1 = \{z \in \mathbb{C}/\mathrm{Re}(z) \in\,]0,1[,\ \mathrm{Re}(z) \neq \frac{1}{2}$ *and* $\mathrm{Im}(z) \neq 0\}$, *so* $\forall\ s \in D_1$,

$$S^2(s) \in \mathbb{R} \iff S^2(1-s) \in \mathbb{R}.$$

Proof. Since the first lemma: Assuming that there exists an s_1 with $r_1 = \mathrm{Re}[s_1] \in \left]0, \frac{1}{2}\right[$ and $\alpha = \ln(2)c > 0$ such that $S^2(s_1) \in \mathbb{R}$, so

(i): $\exists\ V(s_1) \subset \mathbb{C}$ such $\forall\ s \in V(s_1) - \{s_1\}$, $S^2(s) \notin \mathbb{R}$.

(ii): $\exists\ u_n \in V(s_1) - \{s_1\}$ such $\lim u_n = s_1$ (since $s_1 \in \overline{V(s_1) - \{s_1\}}$ with $\overline{A}$ is the adherant of A).

$$
\begin{aligned}
& u_n \in V(s_1) - \{s_1\} \\
\implies\ & S^2(u_n) \notin \mathbb{R} \\
\implies\ & (S(u_n), \overline{S(u_n)}) \text{ is a basis of } \mathbb{C} \\
\implies\ & \nexists\ (a_n, b_n) \in \mathbb{R}^2 \text{ such } S(1-u_n) = a_n S(u_n) + b_n \overline{S(u_n)} \\
& S(s) = \varphi(s) S(1-s) \\
\implies\ & S(1-u_n) = a_n \varphi(u_n) S(1-u_n) + b_n \overline{\varphi(u_n) S(1-u_n)} \\
\implies\ & [1 - a_n \varphi(u_n)] S(1-u_n) = [b_n \overline{\varphi(u_n)}]\overline{S(1-u_n)} \\
\implies\ & [1 - a_n \overline{\varphi(u_n)}]\overline{S(1-u_n)} = [b_n \varphi(u_n)] S(1-u_n) \\
\implies\ & b_n \varphi(u_n) S^2(1-u_n) = [1 - a_n \overline{\varphi(u_n)}] |S(1-u_n)|^2 \\
& \varphi(s)\varphi(1-s) = 1 \quad \forall\ s \\
\implies\ & b_n S^2(1-u_n) = \varphi(1-u_n)[1 - a_n \overline{\varphi(u_n)}] |S(1-u_n)|^2 \\
\implies\ & b_n S^2(1-u_n) = [\varphi(1-u_n) - a_n \overline{\varphi(u_n)} \varphi(1-u_n)] |S(1-u_n)|^2 \\
& \varphi(s)\varphi(1-s) = 1 \\
\implies\ & |\varphi(s)|^2 \varphi(1-s) = \overline{\varphi(s)} \\
\implies\ & \overline{\varphi(s)} = \rho^2 \varphi(1-s) \\
\implies\ & b_n S^2(1-u_n) = |S(1-u_n)|^2 [\varphi(1-u_n) - a_n \rho_n^2 \varphi^2(1-u_n)] \\
& \qquad\qquad\qquad = |S(1-u_n)|^2 \varphi(1-u_n)[1 - a_n \rho_n^2 \varphi(1-u_n)] \\
\implies\ & |b_n| = |\varphi(1-u_n)|\, |1 - a_n \rho_n^2 \varphi(1-u_n)| \text{ or } |S(1-u_n)| = 0
\end{aligned}
$$

We have

$$
\begin{aligned}
& S^2(u_n) \notin \mathbb{R} \\
\implies\ & S(u_n) \neq 0 \\
\implies\ & |S(1-u_n)| \neq 0 \\
\implies\ & |b_n| = |\varphi(1-u_n)|\, |1 - a_n \rho_n^2 \varphi(1-u_n)|
\end{aligned}
$$

Also $\rho_n \neq 0$ since

$$
\begin{aligned}
& |S(u_n)| = \rho_n |S(1-u_n)| \\
\implies\ & |b_n| = \frac{1}{\rho_n} |1 - a_n \rho_n^2 \varphi(1-u_n)|
\end{aligned}
$$

$$\Longrightarrow \quad b_n^2\rho_n^2 = |1 - a_n\rho_n^2\varphi(1-u_n)|^2$$
$$\Longrightarrow \quad b_n^2\rho_n^2 = 1 + a_n^2\rho_n^2 - 2a_n\rho_n\cos(\theta_n), \tag{1}$$
$$S(1-u_n) = a_nS(u_n) + b_n\overline{S(u_n)}$$
$$\Longrightarrow \quad |S(1-u_n)|^2 = (a_n^2+b_n^2)|S(u_n)|^2 + a_nb_n(S^2(u_n)+\overline{S^2(u_n)})$$
$$\Longrightarrow \quad \rho_n^2|S(1-u_n)|^2 = (a_n^2\rho_n^2+b_n^2\rho_n^2)|S(u_n)|^2 + a_nb_n\rho_n^2(S^2(u_n)+\overline{S^2(u_n)})$$
$$\Longrightarrow \quad |S(u_n)|^2 = (a_n^2\rho_n^2+1+a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n))|S(u_n)|^2 + a_nb_n\rho_n^2(S^2(u_n)+\overline{S^2(u_n)})$$
$$\Longrightarrow \quad 0 = (2a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n))|S(u_n)|^2 + a_nb_n\rho_n^2(S^2(u_n)+\overline{S^2(u_n)})$$
$$\Longrightarrow \quad a_n\rho_n[2(a_n\rho_n-\cos(\theta_n))|S(u_n)|^2 + b_n\rho_n(S^2(u_n)+\overline{S^2(u_n)})] = 0$$
$$\Longrightarrow \quad a_n\rho_n = 0 \text{ or } 2[a_n\rho_n-\cos(\theta_n)]|S(u_n)|^2 + b_n\rho_n[S^2(u_n)+\overline{S^2(u_n)}] = 0,$$

$$2[a_n\rho_n-\cos(\theta_n)]|S(u_n)|^2 + b_n\rho_n[S^2(u_n)+\overline{S^2(u_n)}] = 0$$
$$\Longrightarrow \quad b_n\rho_n[S^2(u_n)+\overline{S^2(u_n)}] = -2[a_n\rho_n-\cos(\theta_n)]|S(u_n)|^2$$
$$\Longrightarrow \quad b_n^2\rho_n^2[S^2(u_n)+\overline{S^2(u_n)}]^2 = 4[a_n\rho_n-\cos(\theta_n)]^2|S(u_n)|^4$$
$$\Longrightarrow \quad \left(1+a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\right)[S^2(u_n)+\overline{S^2(u_n)}]^2 = 4[a_n\rho_n-\cos(\theta_n)]^2|S(u_n)|^4$$
$$\Longrightarrow \quad [(a_n\rho_n-\cos(\theta_n))^2+\sin^2(\theta_n)][S^2(u_n)+\overline{S^2(u_n)}]^2 = 4[a_n\rho_n-\cos(\theta_n)]^2|S(u_n)|^4$$
$$\Longrightarrow \quad \sin^2(\theta_n)[S^2(u_n)+\overline{S^2(u_n)}]^2 = [a_n\rho_n-\cos(\theta_n)]^2[4|S(u_n)|^4-(S^2(u_n)+\overline{S^2(u_n)})^2]$$
$$\Longrightarrow \quad \sin^2(\theta_n)[S^2(u_n)+\overline{S^2(u_n)}]^2 = [a_n\rho_n-\cos(\theta_n)]^2[2|S(u_n)|^4-S^4(u_n)-\overline{S^4(u_n)}]$$
$$\Longrightarrow \quad \sin^2(\theta_n)[S^2(u_n)+\overline{S^2(u_n)}]^2 = -[a_n\rho_n-\cos(\theta_n)]^2[S^2(u_n)-\overline{S^2(u_n)}]^2$$
$$\Longrightarrow \quad [i\sin(\theta_n)]^2[S^2(u_n)+\overline{S^2(u_n)}]^2 = [a_n\rho_n-\cos(\theta_n)]^2[S^2(u_n)-\overline{S^2(u_n)}]^2$$
$$\Longrightarrow \quad [a_n\rho_n-\cos(\theta_n)][S^2(u_n)-\overline{S^2(u_n)}] = \pm i\sin(\theta_n)[S^2(u_n)+\overline{S^2(u_n)}]$$
$$\Longrightarrow \quad [a_n\rho_n-\cos(\theta_n)\mp i\sin(\theta_n)]S^2(u_n) = [a_n\rho_n-\cos(\theta_n)\pm i\sin(\theta_n)]\overline{S^2(u_n)} = Z$$

As $Z=\overline{Z}$ so $Z=[a_n\rho_n-\cos(\theta_n)\mp i\sin(\theta_n)]S^2(u_n)\in\mathbb{R}$,

$$\Longrightarrow \quad (a_n\rho_n-e^{\pm i\theta_n})S^2(u_n)\in\mathbb{R}$$
$$\Longrightarrow \quad S^2(u_n) = K_n(a_n\rho_n-e^{\mp i\theta_n})$$

with $K_n\in\mathbb{R}$,

$$\Longrightarrow \quad \operatorname{Im}[S^2(u_n)] = \pm K_n\sin(\theta_n)$$

Since $\lim u_n = s_1$ and $S^2(s_1)\in\mathbb{R}^*$ so

$$\lim[\sin(\theta_n)] = \sin(\theta(s_1)) = 0$$
$$\Longrightarrow \quad \theta(s_1)\equiv 0\ [\pi]$$

As

$$S(s) = \varphi(s)S(1-s) \quad \text{and} \quad \varphi(s) = \rho(s)e^{i\theta(s)}$$
$$\Longrightarrow \quad S(s_1) = \rho(s_1)e^{i\theta(s_1)}S(1-s_1)$$
$$\Longrightarrow \quad S^2(s_1) = \rho^2(s_1)e^{2i\theta(s_1)}S^2(1-s_1),$$

$$\theta(s_1) \equiv 0\ [\pi]$$
$$\Longrightarrow \quad S^2(s_1) = \rho^2(s_1) S^2(1-s_1).$$

Thus,

$$S^2(s) \in \mathbb{R}$$
$$\Longleftrightarrow \quad S^2(1-s) \in \mathbb{R}$$

Another proof if $\lim a_n = a$ and $\lim b_n = b$, and as we have

$$b_n S^2(1-u_n) = |S(1-u_n)|^2 \left[\varphi(1-u_n) - a_n \rho_n^2 \varphi^2(1-u_n)\right]$$

with $\rho_n = |\varphi(u_n)|$, where $n \longrightarrow +\infty$ we would have

$$bS^2(1-s_1) = |S(1-s_1)|^2[\varphi(1-s_1) - a\overline{\varphi(s_1)}\varphi(1-s_1)]$$
$$\Longrightarrow \quad bS^2(1-s_1) = |S(1-s_1)|^2[\varphi(1-s_1) - a\rho^2\varphi^2(1-s_1)]$$

with

$$S(s_1) = \varphi(s_1)S(1-s_1) = \rho e^{i\theta(s_1)} S(1-s_1)$$
$$\Longrightarrow \quad S^2(1-s_1) = \rho^{-2} e^{-i2\theta(s_1)} S^2(s_1) \quad (\text{or } \rho = 0)$$
$$(S(s_1) \neq 0 \Longleftrightarrow \rho \neq 0)$$
$$\Longrightarrow \quad b\rho^{-2} e^{-i2\theta(s_1)} S^2(s_1) = |S(1-s_1)|^2[\varphi(1-s_1) - a\rho^2\varphi^2(1-s_1)] \ \text{ or } \ S^2(s_1) = S^2(1-s_1) = 0$$

Moreover

$$S^2(1-s_1) = \rho^{-2} e^{-i2\theta(s_1)} S^2(s_1) \ \text{ and } \ S^2(s_1) \in \mathbb{R}$$
$$\Longrightarrow \quad |S(1-s_1)|^2 = \pm \rho^{-2} S^2(s_1)$$
$$\Longrightarrow \quad \pm b e^{-i2\theta(s_1)} = \varphi(1-s_1) - a\rho^2\varphi^2(1-s_1) \ \text{ or } \ S^2(s_1) = S^2(1-s_1) = 0,$$
$$\varphi(s_1) = \rho e^{i\theta(s_1)} \ \text{ and } \ \varphi(s_1)\varphi(1-s_1) = 1$$
$$\Longrightarrow \quad \pm \rho^2 = \varphi(s_1) - a\rho^2 \ \text{ or } \ S^2(s_1) = S^2(1-s_1) = 0$$
$$\Longrightarrow \quad (a \pm b)\rho^2 = \varphi(s_1) \ \text{ or } \ S^2(s_1) = S^2(1-s_1) = 0,$$
$$(a \pm b)\rho^2 = \varphi(s_1) = \rho e^{i\theta(s_1)}$$
$$\Longrightarrow \quad (|a \pm b|\rho = 1 \ \text{ and } \ \theta(s_1) \equiv 0\ [\pi]) \ \text{ or } \ \rho = 0$$
$$\Longrightarrow \quad S^2(1-s_1) = \rho^{-2} e^{-i2\theta(s_1)} S^2(s_1) = \rho^{-2} S^2(s_1) \in \mathbb{R} \ \text{ or } \ S(s_1) = 0$$

Hence

$$S^2(s_1) \in \mathbb{R}$$
$$\Longrightarrow \quad S^2(1-s_1) \in \mathbb{R} \ \text{ or } \ S^2(s_1) = S^2(1-s_1) = 0.$$

□

Remark 3.1. It's obvious if $s \in \mathbb{R}$, $S(s) \in \mathbb{R}$ and $S(1-s) \in \mathbb{R}$.

Lemma 3.3. *Let* $D = \{z \in \mathbb{C} / \mathrm{Re}(z) \in]0,1[\}$, *so* $\forall\ s \in D$,

$$S^2(s) \in \mathbb{R}$$
$$\Longleftrightarrow \quad S^2(1-s) \in \mathbb{R}$$

Proof. Since $D - D_1 = \{z \in \mathbb{C}/\mathrm{Re}(z) = \frac{1}{2} \text{and} \, \mathrm{Im}(z) \neq 0\}$,

$$\mathrm{Re}(s) = \frac{1}{2}$$

$$\Longrightarrow \quad 1 - s = \overline{s}$$

So

$$S(1-s) = S(\overline{s}) = \overline{S(s)}$$

and

$$S^2(s) \in \mathbb{R}$$

$$\Longleftrightarrow \quad S^2(1-s) \in \mathbb{R}$$

□

Claim 1. Let $D = \{z \in \mathbb{C}/\mathrm{Re}(z) \in]0,1[\}$, so $\forall \ s \in D$,

$$S(s) \in \mathbb{R} \Leftrightarrow S(1-s) \in \mathbb{R},$$

$$S(s) \in i\mathbb{R} \Leftrightarrow S(1-s) \in i\mathbb{R}.$$

Proof. $S(s) \in \mathbb{R}$ or $S(s) \in i\mathbb{R} \Longrightarrow S^2(s) \in \mathbb{R} \Longrightarrow \theta(s) \equiv 0 \ [\pi]$,

$$S(s) = \varphi(s)S(1-s) = \rho e^{i\theta(s)} S(1-s_1) = \pm\rho S(1-s_1).$$

□

3.2 Other Results

Let be $S = S(s)$ and $S' = S(1-s)$ such $S(s) \in \mathbb{R}$, thus

$$C_1 = \frac{e^{-i\alpha}}{2^r}C, \quad C_2 = \left(1 - \frac{e^{-i\alpha}}{2^r}\right)C \quad \text{and} \quad S = (2^{1-r}e^{-i\alpha} - 1)C$$

with $C_2 = C - C_1$, $S = C_1 - C_2$ and

$$C'_1 = \frac{e^{i\alpha}}{2^{1-r}}C', \ C'_2 = \left(1 - \frac{e^{i\alpha}}{2^{1-r}}\right)C' \quad \text{and} \quad S' = (2^r e^{i\alpha} - 1)C'$$

with $C'_2 = C' - C'_1$, $S' = C'_1 - C'_2$.
As $1 - s = r' + ic' = 1 - r - ic$,

$$\alpha = \ln(2)c$$

$$\Longrightarrow \quad r' = 1 - r,$$

$$c' = -c$$

$$\Longrightarrow \quad r' = 1 - r, \ \alpha' = -\alpha$$

(1) $2C_1C'_1 = CC'$,

(2) $C_1\overline{C}' = 2^{1-2r}C\overline{C}'_1$,

(3) $2C_1\overline{C}'_1 = e^{-i2\alpha}C\overline{C}'$,

(4) $2C_1C'_2 = \overline{S}C' = SC'$,

(5) $2C_2C'_2 = SS' \in \mathbb{R}$,

(6) $SC'_1 = CC'_2$.

Proof. (1) $C_1C'_1 = \dfrac{e^{-i\alpha}}{2^r}C\dfrac{e^{i\alpha}}{2^{1-r}}C' = \dfrac{CC'}{2}$,

(2) $C'_1 = \dfrac{e^{i\alpha}}{2^{1-r}} C' \Longrightarrow C' = 2^{1-r} e^{-i\alpha} C'_1$

$$C_1 \overline{C}' = \left(\frac{e^{-i\alpha}}{2^r} C\right) \overline{(2^{1-r} e^{-i\alpha} C'_1)}$$

$$C_1 \overline{C}' = \left(\frac{e^{-i\alpha}}{2^r} C\right) (2^{1-r} e^{i\alpha} \overline{C'_1}) = 2^{1-2r} C \overline{C}'_1$$

(3) $2C_1 \overline{C}'_1 = 2\left(\dfrac{e^{-i\alpha}}{2^r} C\right) \left(\overline{\dfrac{e^{i\alpha}}{2^{1-r}} C'}\right) = 2 \frac{e^{-i\alpha}}{2^r} C \frac{e^{-i\alpha}}{2^{1-r}} \overline{C'}$

$$2C_1 \overline{C}'_1 = e^{-i2\alpha} C \overline{C}'$$

(4) We have $S = 2C_1 - C = 2\overline{C}_1 - \overline{C} = \overline{S} \in \mathbb{R}$

$\Longrightarrow \quad 2C_1 + \overline{C} = 2\overline{C}_1 + C$

$\Longrightarrow \quad 2C_1 C' + \overline{C} C' = 2\overline{C}_1 C' + C C'$

$\Longrightarrow \quad 2C_1 C' + \overline{C} C' = 2\overline{C}_1 C' + 2C_1 C'_1$

$\Longrightarrow \quad 2C_1 C' - 2C_1 C'_1 = 2\overline{C}_1 C' - \overline{C} C'$

$\Longrightarrow \quad 2C_1 (C' - C'_1) = (2\overline{C}_1 - \overline{C}) C'$

$\Longrightarrow \quad 2C_1 C'_2 = \overline{S} C' = S C'.$

(5) $2C_1 C'_2 = S C' \Longrightarrow 2(S + C_2) C'_2 = S C'$

$\Longrightarrow \quad 2S C'_2 + 2C_2 C'_2 = S C' \Longrightarrow S(C' - S') + 2C_2 C'_2 = S C'$

$\Longrightarrow \quad S C' - S S' + 2C_2 C'_2 = S C'$

$\Longrightarrow \quad S S' = 2C_2 C'_2.$

(6) is (4)+(5). □

Lemma 3.4. $S(s) \in \mathbb{R} \Longrightarrow C_2(s) C_2(1-s) \in \mathbb{R}$.

Proof. Since the last claim $\forall\ s \in D = \{z \in \mathbb{C} / \text{Re}(z) \in]0,1[\}$,

$$S(s) \in \mathbb{R} \Longleftrightarrow S(1-s) \in \mathbb{R}$$

$$S(s) \in i\mathbb{R} \Longleftrightarrow S(1-s) \in i\mathbb{R}$$

$\Longrightarrow \quad S S' = S(s) S(1-s) \in \mathbb{R}$

and from (5)

$$2C_2 C'_2 = S S' \in \mathbb{R}.$$ □

3.3 The Second Announcement

Claim 2. $\exists\ s_0 / S(s_0) \in i\mathbb{R} \Longleftrightarrow \exists\ s_1 \in (r_0, s_0] / S(s_1) = 0$ such that

$$r_0 = \text{Re}(s_0) \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[\quad \text{and} \quad (r_0, s_0] = \{r_0 + ic \in \mathbb{C} / 0 < c \leq c_0\}.$$

Proof. Assuming that $\exists\ s' = r' + ic'$ such $S(s') \in i\mathbb{R}^*$ with $r' \in]0, \frac{1}{2}[$.
Let

$$c_0 = \min\{c \in \mathbb{R}^+ / \exists n \in \mathbb{N}^*, S^n(r' + ic) \in i\mathbb{R}^*\}. \tag{*}$$

So $\exists\ m \in \mathbb{N}^*, S^m(s_0) \in i\mathbb{R}^*$ with $s_0 = r' + ic_0$

$\Longrightarrow \quad S^{2m}(s_0) \in \mathbb{R}_*^-$

without forgetting $S^{2m}(r') \in \mathbb{R}_*^+$, since

$$S(r) = \sum_{n\geq 1} \frac{(-1)^n}{n^r} \in \mathbb{R}, \quad \forall\ r \in \mathbb{R}_*^+.$$

Let now $S^{2m}(s) = R(s) + iI(s)$, we have $R(s_0) \prec 0$ and $R(r') \succ 0$, thus $\exists\ s_1 = r' + ic_1 \in (r', s_0)$ such

$$R(s_1) = 0,$$

$\Longrightarrow \quad S^{2m}(s_1) \in i\mathbb{R}$ with $0 \prec c_1 \prec c_0$,

$$0 \prec c_1 \prec c_0$$

$\Longrightarrow \quad S^{2m}(s_1) \notin i\mathbb{R}_*, \quad$ (Since $(*)$)

$$S^{2m}(s_1) \in i\mathbb{R} \text{ and } S^{2m}(s_1) \notin i\mathbb{R}_*$$

$\Longrightarrow \quad S^{2m}(s_1) = 0$

$\Longrightarrow \quad S(s_1) = 0$

Thus,

$$\exists\ s_0/S(s_0) \in i\mathbb{R} \Longrightarrow S(s_0) \in i\mathbb{R}^* \quad \text{or} \quad S(s_0) = 0$$

$\Longrightarrow \quad \exists\ s_1 \in (r_0, s_0]/S(s_1) = 0$

The other implication is obvious:

$$\exists\ s_1 \in (r_0, s_0]/S(s_1) = 0$$

$\Longrightarrow \quad \exists\ s_1 \in (r_0, s_1]/S(s_1) = 0$

with $s_1 = s_0$,

$$S(s_0) = S(s_1) = 0$$

$\Longrightarrow \quad \exists\ s_0/S(s_0) \in i\mathbb{R}.$ □

Conjecture 1. $\forall\ r \in]0,1[$,

$$r \neq \frac{1}{2} \Longrightarrow \operatorname{Re}[S(s)] \neq 0.$$

4. Conclusions

A new conjecture that is based on Riemann's Hypothesis, and therefor it is a new way to see if this hypothesis is just, otherwise we also have a new useful to determine a counter example.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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