Journal of Informatics and Mathematical Sciences
Vol. 17, No. 1, pp. 59–64, 2025
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
DOI: 10.26713/jims.v17i1.3065

http://www.rgnpublications.com

Research Article

Some Fixed Point Theorems Using Different Spaces in Banach Algebras

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Received: February 27, 2025 **Accepted:** March 18, 2025 **Published:** April 30, 2025

Abstract. In this paper, it expands and generalize several fixed point theorems on multiple spaces and prove the existence and some operations of fixed points for nonlinear operators over Banach algebras. Results are extensions, proofs and generalizations of well-known findings from the literature. These findings represent Das *et al*. [7] generalized findings.

Keywords. Banach algebra, Fixed point theorem, Projective tensor operator

Mathematics Subject Classification (2020). 47H10, 46B28, 47A80

1. Introduction

The first fixed point theorem was presented by Brouwer in 1912 [4]. However, S. Banach [1] is credited for this idea. It is also known as the Banach fixed point theorem. In 1922, S. Banach [1] developed a fixed point theorem for contraction mapping in entire metric space. Krasnosel'skii [15] established a fixed point theorem for a sum of two mappings $\hat{S}$ and T on a non-empty closed convex bounded subset of a Banach space χ in 1955. Many academics, including Vijayaraju [25], Edmunds [10], Nashed and Wong [17], Sehgal and Singh [22], and others, generalized and expanded the above theorem. Some fixed point theorems for $T+\hat{S}$ on a

Banach space χ were developed by Olga Hadzic in 1982 ([11,12]). Two fixed point theorem for sum and product of two operators were discussed by Dhage in 2003 [9].

Three Banach spaces χ, γ, z and the projective tensor product $\chi \otimes_y \gamma$, $\gamma \otimes_y z$, $\chi \otimes_y z$ are examined in this paper. Taking into consideration a triplet of mappings $T_1 : \chi \otimes_y \gamma \to \chi$, $T_2 : \gamma \otimes_y z \to \gamma$, $T_3 : z \otimes_y \chi \to z$, we create a self-mapping T on $\chi \otimes_y \gamma$, $\gamma \otimes_y z$, $\chi \otimes_y z$. Assume that A, B and C are three subsets of χ, γ and z respectively and that $\hat{S}$ is a self-mapping on $A \otimes B$, $B \otimes C$, $C \otimes A$. For $T + T\hat{S} + \hat{S}$ in the subset $A \otimes B$ of $\chi \otimes_y \gamma$, $B \otimes C$ of $\gamma \otimes_y z$, $C \otimes A$ of $\chi \otimes_y z$, we get some fixed point theorems. To demonstrate the reliability of the results which we achieved, whose examples are also provided.

2. Preliminaries

Definition 2.1. Let χ and γ are two normed spaces. A mapping $T : \chi \to \gamma$ is called non-expansive iff $\|Tm - Tn\| \le \|m - n\|$, $\forall\ m, n \in \chi$. It is said to be demi-closed, if its graph is sequentially closed in the product of weak topology on χ, with the norm topology on γ (refer to Mishra [16]).

A mapping $T : \chi \to \gamma$ is called contraction iff

$$\|Tm - Tn\| \le r\|m - n\|,$$

where r is a real number with $0 \le r < 1$, $\forall\ m, n \in \chi$.

Definition 2.2 ([4])**.** Let χ be a Banach space and f be a continuous mapping of χ into itself. The mapping f is said to be completely continuous, if the image under f of each bounded set of χ is contained in a compact.

Definition 2.3 ([4])**.** Let f be completely continuous self-mapping on a Banach space χ. If for some positive integer $p, f^p(\chi)$ be bounded, then f has a fixed point.

Definition 2.4 ([4])**.** Let K be a non-empty, convex and compact subset of a normal space. Any continuous mapping $T : K \to K$ has at least one fixed point.

Definition 2.5 ([1])**.** Let (χ, d) be a complete metric space, $c \in (0,1)$ and $T : \chi \to \chi$ be a mapping such that each for $m, n \in \chi$, $d(Tm, Tn) \le cd(m, n)$.
Then T has a unique fixed point $\alpha \in \chi$, such that for each $m \in \chi$, $\lim\limits_{n \to \infty} T_n m = \alpha$.

Projective Tensor Product 2.6 ([1])**.** Given normed spaces χ and γ, then projective tensor norm on $\chi \otimes_y \gamma$ is defined by

$$\|h\| = \inf\{\Sigma_i \|m_i\| - \|n_i\| : h = \Sigma_i m_i \otimes_y n_i\},$$

where the infimum is taken over all (finite) representation of h.

Definition 2.7. Let A and B be two bounded subsets of the Banach spaces χ and γ, respectively. A pair of mappings $T_1 : A \otimes B \to A$, $T_2 : A \otimes B \to B$ is called (k, k') contraction mapping if:

(i) $\|T_1 m - T_1 n\| \le \frac{k}{U_2}\|m - n\|$, $\frac{k}{U_2} < 1$,

(ii) $\|T_2 m - T_2 n\| \le \frac{k'}{U_1}\|m - n\|$, $\frac{k'}{U_1} < 1$,

(iii) $\|T_1 m\| \le U_1$, $\|T_2 m\| \le U_2$, $\forall\ m, n \in A \otimes B$, where $A \otimes B$ is bounded by $U_1 U_2$.

3. Main Results

Theorem 3.1. *Let A,B and C are three subsets of χ, γ and z respectively and let $T_1 : \chi \otimes_y \gamma \to \chi$, $T_2 : \gamma \otimes_y z \to \gamma$, $T_3 : z \otimes_y \chi \to z$ are three continuous mapping such that $T_1(A \otimes B) \subseteq A$, $T_2(B \otimes C) \subseteq B$, $T_3(C \otimes A) \subseteq C$. We define $T : \chi \otimes_y \gamma \to \chi \otimes_y \gamma$ by $T(s) = T_1(s) \otimes T_2(s)$, $s \in \chi \otimes_y \gamma$. Similarly, can be define for $T : \gamma \otimes_y z \to \gamma \otimes_y z$, $T : z \otimes_y \chi \to z \otimes_y \chi$. Let $\hat{S}$ is a completely additive self-mapping on $A \otimes B$, $B \otimes C$, $C \otimes A$ such that $\hat{S}$ and T commute on $A \otimes B$, $B \otimes C$, $C \otimes A$. Let for every $\Sigma_i a_i \otimes b_i$ in $A \otimes B$, there exist one and only one solution $(\Sigma_i a_i \otimes b_i)^0$ in $A \otimes B$ of the equation*

$$\Sigma_i \alpha_i \otimes \beta_i = T(\Sigma_i \alpha_i \otimes \beta_i) + T(\Sigma_i a_i \otimes b_i) + \Sigma_i a_i \otimes b_i\,, \tag{3.1}$$

where $\Sigma_i \alpha_i \otimes \beta_i \in \chi \otimes_y \gamma$.
Then $(T + T\hat{S} + \hat{S})$ has a fixed point in $A \otimes B$. It is true for $B \otimes C$ and $C \otimes A$.

Proof. Let us define $\tau : A \otimes B \to A \otimes B$ by $\tau(\Sigma_i a_i \otimes b_i) = (\Sigma_i a_i \otimes b_i)^0$.
Firstly, we have to show that τ is continuous. Let $\{\Sigma_i a_{in} \otimes b_{in}\}_n$ be a sequence in $A \otimes B$ such that $\Sigma_i a_{in} \otimes b_{in} \to \Sigma_i a_i \otimes b_i$ as $n \to \infty$.

$$\begin{aligned}
\tau(\Sigma_i a_{in} \otimes b_{in}) &= T(\Sigma_i a_{in} \otimes b_{in})^0 + T(\Sigma_i a_{in} \otimes b_{in}) + \Sigma_i a_{in} \otimes b_{in},\\
\lim_{n\to\infty} \tau(\Sigma_i a_{in} \otimes b_{in}) &= T(\lim_{n\to\infty} (\Sigma_i a_{in} \otimes b_{in})^0) + T(\lim_{n\to\infty} \Sigma_i a_{in} \otimes b_{in}) + \Sigma_i a_i \otimes b_i\\
&\qquad\qquad (T \text{ is a continuous as } T_1 \text{ and } T_2)\\
&= T(\lim_{n\to\infty} \tau(\Sigma_i a_{in} \otimes b_{in})) + T(\Sigma_i a_i \otimes b_i) + \Sigma_i a_i \otimes b_i\,.
\end{aligned}$$

So, $\lim_{n\to\infty} \tau(\Sigma_i a_{in} \otimes b_{in})$ is a solution of the equation (3.1).
Therefore, $\lim_{n\to\infty} \tau(\Sigma_i a_{in} \otimes b_{in}) = (\Sigma_i a_i \otimes b_i)^0 = \tau(\Sigma_i a_i \otimes b_i)$.
So, that τ is continuous.
For $m \in A \otimes B$, $\tau m = m^0 = T(m^0) + T(m) + m$, using equation (3.1).
Now, $\hat{S}(\tau m) = \hat{S}(m^0) = T(\hat{S}(m^0)) + T(\hat{S}(m)) + \hat{S}(m)$.
Therefore, $\hat{S}(m^0)$ is a solution of the equation (3.1) for $\hat{S}(m)$ in $A \otimes B$.
Hence, $\hat{S}(m^0) = (\hat{S}(m))^0$, i.e., $\hat{S}(\tau m) = \tau(\hat{S} m)$.
Then $\hat{S}$ and τ are commute.
Now, we define

$$K : A \otimes B \to A \otimes B$$

by, $K(m) = \hat{S}(m^0) = \hat{S}(\tau m)$, for $m \in A \otimes B$.
Since τ is continuous and $\hat{S}$ is completely continuous, so the mapping K is completely continuous. Since A and B are bounded subsets, so $A \otimes B$ is bounded subset of $\chi \otimes_y \gamma$. Now, $K^n(A \otimes B) = \hat{S}^n \tau^n(A \otimes B)$ is bounded for $n \in N$. So, using Definition 2.3, we get K has fixed point, say a in $A \otimes B$.
Therefore,

$$a = K(a) = \hat{S}\tau(a) = \tau\hat{S}(a) = (\hat{S}(a))^0 = T((\hat{S}(a))^0) + T(\hat{S}(a)) + \hat{S}(a) = T(a) + T(\hat{S}(a)) + \hat{S}(a).$$

So, a is a fixed point for $T + T\hat{S} + \hat{S}$ in $A \otimes B$ and it is true for $B \otimes C$ and $C \otimes A$. □

Theorem 3.2. *Let $T_1 : \chi \otimes_y \gamma \to \chi$, $T_2 : \gamma \otimes_y z \to \gamma$, $T_3 : z \otimes_y \chi \to z$ are three continuous mapping and $T : \chi \otimes_y \gamma \to \chi \otimes_y \gamma$ be define by $T(s) = T_1(s) \otimes T_2(s)$, $s \in \chi \otimes_y \gamma$. Similarly, $T : \gamma \otimes_y z \to \gamma \otimes_y z$, $T : z \otimes_y \chi \to z \otimes_y \chi$ be define by $T(s) = T_2(s) \otimes T_3(s)$, $s \in \gamma \otimes_y z$ and $T(s) = T_3(s) \otimes T_1(s)$, respectively. Let $\hat{S}$ is a completely additive self-mapping on $\chi \otimes_y \gamma$ such that $\hat{S}T = T\hat{S}$ and for some $u > 1$, $\hat{S}^u(\chi \otimes_y \gamma)$ is bounded.*

Let for every $\Sigma_i m_i \otimes n_i$ in $\chi \otimes_y \gamma$, there exist exactly one solution $(\Sigma_i m_i \otimes n_i)^0$ in $\chi \otimes_y \gamma$, of the equation

$$\Sigma_i \alpha_i \otimes \beta_i = T(\Sigma_i \alpha_i \otimes \beta_i) + T(\Sigma_i m_i \otimes n_i) + \Sigma_i m_i \otimes n_i,$$

where $\Sigma_i \alpha_i \otimes \beta_i \in \chi \otimes_y \gamma$.

Then $T + T\hat{S} + \hat{S}$ has a fixed point in $\chi \otimes_y \gamma$. It is true for $\gamma \otimes_y z$ and $z \otimes_y \chi$.

Lemma 3.3. *Let the pair (T_1, T_2), (T_2, T_3), (T_3, T_1) be defined as (k, k') contraction, then the mapping $T : A \otimes B \to A \otimes B$ defined by $T(m) = T_1(m) \otimes T_2(m)$, $T(m) = T_2(m) \otimes T_3(m)$, $T(m) = T_3(m) \otimes T_1(m)$, $m \in A \otimes B$ has a fixed point if $(k + k') < 1$.*

Example 3.4. Let $\psi_i^1 \otimes_y d$ be a subset of $i^1 \otimes_y d$ bounded by a constant p. We define $T_1 : \psi_i^1 \otimes_y d \to \psi_i^1$ by $T_1(\alpha_i \otimes m_i) = 1/2p.\Sigma_i\{\alpha_{in} m_i\}_n$ where $\alpha_i = \{\alpha_{in}\}_n$ and $T_2 : \psi_i^1 \otimes_y d \to \psi_d$ by $T_2(\alpha_i \otimes m_i) = 1/4.\Sigma_i \|\alpha_i\|.|m_i|$, where ψ_i^1 and ψ_d are bounded subset of i^1 and d, respectively. Then (T_1, T_2) is a pair of (k, k') contraction mapping with $(k + k') < 1$. Therefore, the mapping $T : \psi_i^1 \otimes_y d \to \psi_i^1 \otimes_y d$ defined by $T(\alpha_i \otimes m_i) = 1/8p.\Sigma_i\{N\alpha_{in} m_i\}_n$ where $n = \|\alpha_i\|.|m_i|$ has a unique fixed point in $\psi_i^1 \otimes_y d$.

Lemma 3.5. *Let (T_1, T_2), (T_2, T_3), (T_3, T_1) be a pair of (k, k') contraction mappings and T be defined as $T : A \otimes B \to A \otimes B$ defined by $T(m) = T_1(m) \otimes T_2(m)$, $T(m) = T_2(m) \otimes T_3(m)$, $T(m) = T_3(m) \otimes T_1(m)$. Let $\hat{S}$ is a completely additive self-mapping on $A \otimes B$ such that $\hat{S}T = T\hat{S}$. If $(k + k') < 1$, then $T + T\hat{S} + \hat{S}$ has a fixed point in $A \otimes B$.*

Theorem 3.6. *Let A,B and C be non-empty compact subsets of Banach spaces χ, γ and z, respectively. Let (T_1, T_2), (T_2, T_3), (T_3, T_1) be a pair of (k, k') contraction mappings with $(k + k') < 1$ and mapping T be defined as $T : A \otimes B \to A \otimes B$ defined by $T(m) = T_1(m) \otimes T_2(m)$, $T(m) = T_2(m) \otimes T_3(m)$, $T(m) = T_3(m) \otimes T_1(m)$. Let $\hat{S}$ is a continuous self-mapping on $A \otimes B$ such that $Tm + T\hat{S}a + \hat{S}a \in A \otimes B$ for all $m, a \in A \otimes B$.*

Then $T + T\hat{S} + \hat{S}$ has a fixed point in $A \otimes B$. It is true for $B \otimes C$, $C \otimes A$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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