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Common Fixed Point Theorem in Complex Valued b-Metric Space for Rational Contractions

Research Article

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Abstract. In this paper we prove the common fixed point theorem in complex valued b-metric space for rational contractions. Our results extend, generalize and improve the corresponding result of Uthayakumar and Prabakar [16].

Keywords. Rational expressions; Complex valued b-metric space; Common fixed point

MSC. 47H10; 54H25

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1. Introduction

The Banach contraction principle was used to establish the existence of a unique solution for a nonlinear integral equation [4]. There are many generalizations of the Banach contraction principle particularly in metric spaces. In 1978, Fisher and Khan [9] generalized the Banach contraction principle with rational expressions and proved some fixed and common fixed point theorems. Huang and Zhange introduced the notion of cone metric space which is a generalization of metric spaces. Subsequently, many authors have generalized the results of Huang and Zhang and have studied fixed point theorems in cone metric spaces.

The fixed point theorems regarding rational contractive conditions cannot be extended in cone metric spaces, since the definition of these spaces is based on a Banach space which is not a division ring. Recently, Azam et al. [2] introduced the notion of complex valued metric spaces and proved some common fixed point theorems for mapping satisfying rational inequality which are not meaningful in cone metric spaces.

In the same way, various authors have studied and proved the fixed point results for mapping satisfying different type contractive conditions in the framework of complex valued metric spaces (see [15], [8], [5], [18], [17]).

In 2013, Rao et al. [13] introduced the concept of complex valued b-metric space which was more general than the well known complex valued metric space. In sequel, A.A. Mukheimer [11] obtained common fixed point results satisfying certain rational expressions in complex valued b-metric spaces.

The aim of this paper is to extend the results of Uthayakumar and Prabakar [16] and prove common fixed point theorem satisfying rational expressions in complex valued b-metric space.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{if and only if} \quad \text{Re}(z_1) \leq \text{Re}(z_2),\ \text{Im}(z_1) \leq \text{Im}(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$,

(ii) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$,

(iii) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$,

(iv) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that

(a) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$,

(b) If $z_1 \precsim z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$,

(c) If $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}$.

The following definition is recently introduced by Rao et al. [9].

Definition 2.1. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b-metric space.

Example 2.2 ([13]). If $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$.
Then (X, d) is complex valued b-metric space with $s = 2$.

Definition 2.3 ([13]). Let (X,d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$, $B(x,r) \cap (A - \{x\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.

(v) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x,r) : x \in X$ and $0 \prec r\}$.

Definition 2.4 ([13]). Let (X,d) be a complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec r$ then there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Lemma 2.5 ([13]). *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 2.6 ([13]). *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence in X if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Now we are ready to state and prove our main results.

3. Main Results

Definition 3.1 ([1]). Let (X,d) be a cone metric space. A self mapping T on X is called an almost Jaggi contraction if it satisfies the following condition:

$$d(Tx,Ty) \le \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) + \text{Lmin}\{d(x,Ty), d(y,Tx)\} \tag{3.1}$$

for all $x, y \in X$, where $L \ge 0$ and $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$.

Theorem 3.2. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$. Let $T : X \to X$ be an almost Jaggi contraction, for all $x, y \in X$, where $L \ge 0$ and α, β are nonnegative reals with $s(\alpha+\beta) < 1$. Then T has a unique fixed point in X.*

Proof. For any arbitrary point $x_0 \in X$, define sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n)\\
&\precsim \frac{\alpha d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)} + \beta d(x_{n-1},x_n) + \text{Lmin}\{d(x_{n-1},Tx_n), d(x_n,Tx_{n-1})\}\\
&\precsim \frac{\alpha d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)} + \beta d(x_{n-1},x_n) + \text{Lmin}\{d(x_{n-1},x_{n+1}), d(x_n,x_n)\}\\
&\precsim \frac{\alpha d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)} + \beta d(x_{n-1},x_n)
\end{aligned}$$

so that

$$\begin{aligned}
|d(x_n,x_{n+1})| &\le \frac{\alpha |d(x_{n-1},x_n)|\,|d(x_n,x_{n+1})|}{|d(x_{n-1},x_n)|} + \beta |d(x_{n-1},x_n)|\\
&\le \alpha |d(x_n,x_{n+1})| + \beta |d(x_{n-1},x_n)|
\end{aligned}$$

therefore

$$|d(x_n,x_{n+1})| \le \frac{\beta}{1-\alpha}|d(x_{n-1},x_n)|.$$

Since $s(\alpha+\beta) < 1$ and $s \ge 1$, we get $\alpha+\beta < 1$. Therefore, with $h = \frac{\beta}{1-\alpha} < 1$ and for all $n \ge 0$ and consequently, we have

$$|d(x_n,x_{n+1})| \le h|d(x_{n-1},x_n)| \le h^2|d(x_{n-2},x_{n-1})| \le \cdots \le h^n|d(x_0,x_1)|.$$

i.e.

$$|d(x_{n+1},x_{n+2})| \le h^{n+1}|d(x_0,x_1)|. \tag{3.2}$$

Thus for any $m > n$, $m,n \in \mathbb{N}$ and since

$$sh = \frac{s\beta}{1-\alpha} < 1,$$

we get

$$\begin{aligned}
|d(x_n,x_m)| &\le s|d(x_n,x_{n+1})| + s|d(x_{n+1},x_m)|\\
&\le s|d(x_n,x_{n+1})| + s^2|d(x_{n+1},x_{n+2})| + s^2|d(x_{n+2},x_m)|\\
&\le s|d(x_n,x_{n+1})| + s^2|d(x_{n+1},x_{n+2})| + s^3|d(x_{n+2},x_{n+3})| + s^3|d(x_{n+3},x_m)|\\
&\vdots\\
&\le s|d(x_n,x_{n+1})| + s^2|d(x_{n+1},x_{n+2})| + s^3|d(x_{n+2},x_{n+3})|\\
&\quad + \cdots + s^{m-n-1}|d(x_{m-2},x_{m-1})| + s^{m-n}|d(x_{m-1},x_m)|.
\end{aligned}$$

By (3.2), we get,

$$\begin{aligned}|d(x_n,x_m)| &\le sh^n|d(x_0,x_1)|+s^2h^{n+1}|d(x_0,x_1)|+s^3h^{n+2}|d(x_0,x_1)|\\ &\quad+\cdots+s^{m-n-1}h^{m-2}|d(x_0,x_1)|+s^{m-n}h^{m-1}|d(x_0,x_1)|\\ &=\sum_{i=1}^{m-n}s^ih^{i+n-1}|d(x_0,x_1)|.\end{aligned}$$

Therefore,

$$\begin{aligned}|d(x_n,x_m)| &\le \sum_{i=1}^{m-n}s^{i+n-1}h^{i+n-1}|d(x_0,x_1)|=\sum_{t=n}^{m-1}s^th^t|d(x_0,x_1)|\\ &\le\sum_{t=n}^{\infty}(sh)^t|d(x_0,x_1)|=\frac{(sh)^n}{1-sh}|d(x_0,x_1)|\end{aligned}$$

and hence

$$|d(x_n,x_m)|\le\frac{(sh)^n}{1-sh}|d(x_0,x_1)|\to 0 \text{ as } m,n\to\infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u\in X$ such that $x_n\to u$ as $n\to\infty$. Assume not, then there exists $z\in X$ such that

$$|d(u,Tu)|=|z|>0. \tag{3.3}$$

So by using the triangular inequality and (3.1), we get

$$\begin{aligned}z=d(u,Tu)&\precsim sd(u,x_{n+1})+sd(x_{n+1},Tu)\\ &\precsim sd(u,x_{n+1})+sd(Tx_n,Tu)\\ &\precsim sd(u,x_{n+1})+\frac{s\alpha d(x_n,Tx_n)d(u,Tu)}{d(x_n,u)}+s\beta d(x_n,u)+s\,\mathrm{L}\min\{d(x_n,Tu),d(u,Tx_n)\}\\ &=sd(u,x_{n+1})+\frac{s\alpha d(x_n,x_{n+1})d(u,T_u)}{d(x_n,u)}+s\beta d(x_n,u)+s\,\mathrm{L}\min\{d(x_n,Tu),d(u,x_{n+1})\}\end{aligned}$$

which implies that

$$\begin{aligned}|z|&=|d(u,Tu)|\\ &\le s|d(u,x_{n+1})|+\frac{s\alpha|d(x_n,x_{n+1})||z|}{|d(x_n,u)|}+s\beta|d(x_n,u)|+s|\,\mathrm{L}\min\{d(x_n,Tu),d(u,x_{n+1})\}|.\end{aligned} \tag{3.4}$$

Taking the limit of (3.4) as $n\to\infty$, we get

$$|z|=|d(u,Tu)|\le 0,$$

a contradiction with (3.3).

So $|z|=0$. Hence $Tu=u$, u is a fixed point of T.

To prove the uniqueness of fixed point, let $u^\star$ in X be another fixed point of T that is $u^\star = Tu^\star$. Then

$$\begin{aligned} d(u,u^\star) &= d(Tu,Tu^\star) \\ &\precsim \frac{\alpha d(u,Tu)d(u^\star,Tu^\star)}{d(u,u^\star)} + \beta d(u,u^\star) + \text{Lmin}\{d(u,Tu^\star),d(u^\star,Tu)\} \\ &= \frac{\alpha d(u,u)d(u^\star,u^\star)}{d(u,u^\star)} + \beta d(u,u^\star) + \text{Lmin}\{d(u,u^\star),d(u^\star,u)\} \end{aligned}$$

so that

$$|d(u,u^\star)| \le \beta |d(u,u^\star)| + |\text{Lmin}\{d(u,u^\star),d(u^\star,u)\}|.$$

Since $|\text{Lmin}\{d(u,u^\star),d(u^\star,u)\}| = L|d(u,u^\star)|$, therefore $|d(u,u^\star)| \le (\beta + L)|d(u,u^\star)|$, which is a contradiction, so that $u = u^\star$. This completes the proof. □

Definition 3.3 ([10])**.** Let (X,d) be a cone metric space. A self mapping T on X is called Jaggi contraction if it satisfies the following condition:

$$d(Tx,Ty) \le \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \tag{3.5}$$

for all $x,y \in X$ and $\alpha,\beta \in [0,1)$ with $\alpha + \beta < 1$.

Corollary 3.4. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$. Let $T : X \to X$ be a Jaggi contraction*

$$d(Tx,Ty) \precsim \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \tag{3.6}$$

for all $x,y \in X$, and α, β are nonnegative reals such that $s(\alpha+\beta) < 1$. Then T has a unique fixed point in X.

Proof. By applying $L = 0$ in Theorem 3.2, we get the required result. □

Theorem 3.5. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$. Suppose the mappings S, T are called an almost Jaggi contraction if it satisfies the following condition:*

$$d(Sx,Ty) \precsim \frac{\alpha d(x,Sx)d(y,Ty)}{d(x,y)} + \beta d(x,y) + \text{Lmin}\{d(x,Ty),d(y,Sx)\} \tag{3.7}$$

for all $x,y \in X$, where $L \ge 0$ and α, β are nonnegative reals with $s(\alpha+\beta) < 1$. Then the maps S and T have a unique common fixed point.

Proof. For any arbitrary point $x_0 \in X$, define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}, \quad \text{for } n = 0,1,2,\dots .$$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (3.7), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \frac{\alpha d(x_{2n}, Sx_{2n}) d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}) \\ &\quad + \mathrm{Lmin}\{d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n})\} \\ &= \frac{\alpha d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}) \\ &\quad + \mathrm{Lmin}\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\} \\ d(x_{2n+1}, x_{2n+2}) &\precsim \frac{\alpha d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}) \end{aligned}$$

so that

$$|d(x_{2n+1}, x_{2n+2})| \leq \alpha |d(x_{2n+1}, x_{2n+2})| + \beta |d(x_{2n}, x_{2n+1})|$$

and hence

$$|d(x_{2n+1}, x_{2n+2})| \leq \frac{\beta}{1-\alpha} |d(x_{2n}, x_{2n+1})|. \tag{3.8}$$

Similarly, we obtain

$$|d(x_{2n+2}, x_{2n+3})| \leq \frac{\beta}{1-\alpha} |d(x_{2n+1}, x_{2n+2})|. \tag{3.9}$$

Since $s(\alpha+\beta) < 1$ and $s \geq 1$, we get $\alpha + \beta < 1$. Therefore, with $h = \frac{\beta}{1-\alpha} < 1$ and for all $n \geq 0$ and consequently, we have

$$|d(x_{2n+1}, x_{2n+2})| \leq h|d(x_{2n}, x_{2n+1})| \leq h^2 |d(x_{2n-1}, x_{2n})| \leq \cdots \leq h^{2n+1} |d(x_0, x_1)|$$

i.e.

$$|d(x_{n+1}, x_{n+2})| \leq h^{n+1} |d(x_0, x_1)|. \tag{3.10}$$

Thus for any $m > n$, $m, n \in \mathbb{N}$ and since $sh = \frac{s\beta}{1-\alpha} < 1$, we get

$$\begin{aligned} |d(x_n, x_m)| &\leq s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^3 |d(x_{n+2}, x_{n+3})| \\ &\quad + \cdots + s^{m-n-1} |d(x_{m-2}, x_{m-1})| + s^{m-n} |d(x_{m-1}, x_m)|. \end{aligned}$$

By (3.10), we get

$$|d(x_n, x_m)| \leq \sum_{i=1}^{m-n} s^i h^{i+n-1} |d(x_0, x_1)|.$$

Therefore,

$$|d(x_n,x_m)| \leq \sum_{i=1}^{m-n} s^{i+n-1}h^{i+n-1}|d(x_0,x_1)| = \sum_{t=n}^{m-1} s^t h^t |d(x_0,x_1)|$$

$$\leq \sum_{t=n}^{\infty}(sh)^t|d(x_0,x_1)| = \frac{(sh)^n}{1-sh}|d(x_0,x_1)|$$

and hence

$$|d(x_n,x_m)| \leq \frac{(sh)^n}{1-sh}|d(x_0,x_1)| \to 0 \text{ as } m,n \to \infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Assume not, then there exists $z \in X$ such that

$$|d(u,Su)| = |z| > 0. \tag{3.11}$$

So by using the triangular inequality and (3.7), we get

$$\begin{aligned}
z &= d(u,Su)\\
&\precsim sd(u,x_{2n+2}) + sd(x_{2n+2},Su)\\
&= sd(u,x_{2n+2}) + sd(Su,Tx_{2n+1})\\
&\precsim sd(u,x_{2n+2}) + \frac{s\alpha d(u,Su)d(x_{2n+1},Tx_{2n+1})}{d(u,x_{2n+1})}\\
&\quad + s\beta d(u,x_{2n+1}) + s\,\mathrm{L}\min\{d(u,Tx_{2n+1}),d(x_{2n+1},Su)\}\\
&= sd(u,x_{2n+2}) + \frac{s\alpha d(u,Su)d(x_{2n+1},x_{2n+2})}{d(u,x_{2n+1})}\\
&\quad + s\beta d(u,x_{2n+1}) + s\,\mathrm{L}\min\{d(u,x_{2n+2}),d(x_{2n+1},Su)\}
\end{aligned}$$

which implies that

$$\begin{aligned}
|z| &= |d(u,Su)|\\
&\leq s|d(u,x_{2n+2})| + \frac{s\alpha|d(u,Su)||d(x_{2n+1},x_{2n+2})|}{|d(u,x_{2n+1})|}\\
&\quad + s\beta|d(u,x_{2n+1})| + s|\,\mathrm{L}\min\{d(u,x_{2n+2}),d(x_{2n+1},Su)\}\,|.
\end{aligned} \tag{3.12}$$

Taking the limit of (3.12) as $n \to \infty$, we get $|z| = |d(u,Su)| \leq 0$, a contradiction with (3.11). So $|z| = 0$. Hence $Su = u$. Similarly, it can easily be proved that $Tu = u$.

To prove the uniqueness of common fixed point, let $u^\star$ be another common fixed point of $S and T$ that is $u^\star = Su^\star = Tu^\star$. Then

$$\begin{aligned}
d(u,u^\star) &= d(Su,Tu^\star)\\
&\precsim \frac{\alpha d(u,Su)d(u^\star,Tu^\star)}{d(u,u^\star)} + \beta d(u,u^\star) + \mathrm{L}\min\{d(u,Tu^\star),d(u^\star,Su)\}\\
&= \frac{\alpha d(u,u)d(u^\star,u^\star)}{d(u,u^\star)} + \beta d(u,u^\star) + \mathrm{L}\min\{d(u,u^\star),d(u^\star,u)\},
\end{aligned}$$

so that

$$
\begin{aligned}
|d(u,u^\star)| &\le \beta|d(u,u^\star)| + L|d(u,u^\star)| \\
&\le (\beta+L)|d(u,u^\star)|,
\end{aligned}
$$

which is a contradiction, so that $u = u^\star$. This completes the proof. □

Definition 3.6 ([1])**.** Let (X,d) be a cone metric space. A self mapping T on X is called Dass-Gupta contraction if it satisfies the following condition:

$$
d(Tx,Ty) \le \frac{\alpha d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y) + \mathrm{Lmin}\{d(x,Tx), d(x,Ty), d(y,Tx)\} \tag{3.13}
$$

for all $x,y \in X$, where $L \ge 0$ and $\alpha,\beta \in [0,1)$ with $\alpha+\beta<1$.

Theorem 3.7. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$. Let $T : X \to X$ be a Dass-Gupta contraction satisfies the condition:*

$$
d(Tx,Ty) \precsim \frac{\alpha d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y) + \mathrm{Lmin}\{d(x,Tx), d(x,Ty), d(y,Tx)\} \tag{3.14}
$$

for all $x,y \in X$, where $L \ge 0$ and α, β are nonnegative reals with $s(\alpha+\beta)<1$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0$, $x_n = Tx_{n-1}$.

$$
\begin{aligned}
d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\
&\precsim \frac{\alpha d(x_n,Tx_n)[1+d(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)} + \beta d(x_{n-1},x_n) \\
&\quad + \mathrm{Lmin}\{d(x_{n-1},Tx_{n-1}), d(x_{n-1},Tx_n), d(x_n,Tx_{n-1})\} \\
&= \frac{\alpha d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)} + \beta d(x_{n-1},x_n) \\
&\quad + \mathrm{Lmin}\{d(x_{n-1},x_n), d(x_{n-1},x_{n+1}), d(x_n,x_n)\},
\end{aligned}
$$

so that

$$
\begin{aligned}
|d(x_n,x_{n+1})| &\le \frac{\alpha|d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]|}{|1+d(x_{n-1},x_n)|} + \beta|d(x_{n-1},x_n)| \\
&= \alpha|d(x_n,x_{n+1})| + \beta|d(x_{n-1},x_n)| \\
|d(x_n,x_{n+1})| &\le \frac{\beta}{1-\alpha}|d(x_{n-1},x_n)|.
\end{aligned}
$$

Since $s(\alpha+\beta)<1$ and $s \ge 1$, we get $\alpha+\beta<1$. Therefore with $h = \frac{\beta}{1-\alpha} < 1$ and for all $n \ge 0$ and consequently, we have

$$
|d(x_n,x_{n+1})| \le h|d(x_{n-1},x_n)| \le h^2|d(x_{n-2},x_{n-1})| \le \cdots \le h^n|d(x_0,x_1)|
$$

i.e.

$$|d(x_{n+1},x_{n+2})| \le h^{n+1}|d(x_0,x_1)|. \tag{3.15}$$

Thus for any $m > n$, $m,n \in \mathbb{N}$ and since $sh = \frac{s\beta}{1-\alpha} < 1$ we get

$$\begin{aligned}|d(x_n,x_m)| &\le s|d(x_n,x_{n+1})| + s^2|d(x_{n+1},x_{n+2})| + s^3|d(x_{n+2},x_{n+3})| \\ &\quad + \cdots + s^{m-n-1}|d(x_{m-2},x_{m-1})| + s^{m-n}|d(x_{m-1},x_m)|.\end{aligned}$$

By (3.15) we get

$$\begin{aligned}|d(x_n,x_m)| &\le sh^n|d(x_0,x_1)| + s^2h^{n+1}|d(x_0,x_1)| + s^3h^{n+2}|d(x_0,x_1)| \\ &\quad + \cdots + s^{m-n-1}h^{m-2}|d(x_0,x_1)| + s^{m-n}h^{m-1}|d(x_0,x_1)|. \\ &= \sum_{i=1}^{m-n} s^i h^{i+n-1}|d(x_0,x_1)|.\end{aligned}$$

Therefore,

$$\begin{aligned}|d(x_n,x_m)| &\le \sum_{i=1}^{m-n} s^{i+n-1}h^{i+n-1}|d(x_0,x_1)| \\ &= \sum_{t=n}^{m-1} s^t h^t|d(x_0,x_1)| \\ &\le \frac{(sh)^n}{1-sh}|d(x_0,x_1)|\end{aligned}$$

and hence

$$|d(x_n,x_m)| \le \frac{(sh)^n}{1-sh}|d(x_0,x_1)| \to 0 \text{ as } m,n \to \infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Assume not, then there exists $z \in X$ such that

$$|d(u,Tu)| = |z| > 0. \tag{3.16}$$

So by using the triangular inequality and (3.14), we get

$$\begin{aligned}z &= d(u,Tu) \\ &\precsim sd(u,x_{n+1}) + sd(x_{n+1},Tu) \\ &= sd(u,x_{n+1}) + sd(Tx_n,Tu) \\ &\precsim sd(u,x_{n+1}) + \frac{s\alpha d(u,Tu)[1+d(x_n,Tx_n)]}{1+d(x_n,u)} + s\beta d(x_n,u) \\ &\quad + s\,\mathrm{Lmin}\{d(x_n,Tx_n),d(x_n,Tu),d(u,Tx_n)\} \\ &= sd(u,x_{n+1}) + \frac{s\alpha d(u,Tu)[1+d(x_n,x_{n+1})]}{1+d(x_n,u)} + s\beta d(x_n,u) \\ &\quad + s\,\mathrm{Lmin}\{d(x_n,x_{n+1}),d(x_n,Tu),d(u,x_{n+1})\}\end{aligned}$$

which implies that

$$\begin{aligned}|z| &= |d(u,Tu)| \\ &\le s|d(u,x_{n+1})| + \frac{s\alpha|d(u,Tu)[1+d(x_n,x_{n+1})]|}{|1+d(x_n,u)|} + s\beta|d(x_n,u)| \\ &\quad + s|\text{Lmin}\{d(x_n,x_{n+1}),d(x_n,Tu),d(u,x_{n+1})\}|. \end{aligned} \tag{3.17}$$

Taking the limit of (3.17) as $n \to \infty$, we get $|z| = |d(u,Tu)| \le 0$, a contradiction with (3.16). So $|z| = 0$. Hence $Tu = u$, u is a fixed point of T.

To prove the uniqueness of fixed point, let $u^\star$ in X be another fixed point of T that is $u^\star = Tu^\star$. Then

$$\begin{aligned} d(u,u^\star) &= d(Tu,Tu^\star) \\ &\precsim \frac{\alpha d(u^\star,Tu^\star)[1+d(u,Tu)]}{1+d(u,u^\star)} + \beta d(u,u^\star) + \text{Lmin}\{d(u,Tu),d(u,Tu^\star),d(u^\star,Tu)\} \\ &= \beta d(u,u^\star) \end{aligned}$$

so that

$$|d(u,u^\star)| \le \beta|d(u,u^\star)|,$$

which is a contradiction.
Hence $u = u^\star$. This completes the proof. □

Corollary 3.8. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$. Let $T : X \to X$ be a Dass-Gupta rational contraction satisfies the condition:*

$$d(Tx,Ty) \precsim \frac{\alpha d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y) \tag{3.18}$$

for all $x,y \in X$, where α, β are nonnegative reals such that $s(\alpha+\beta) < 1$. Then T has a unique fixed point in X.

Proof. We can prove this result by applying Theorem 3.7 with $L = 0$. □

Corollary 3.9. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$. Let $T : X \to X$ be a Dass-Gupta mapping satisfying (for some fixed n):*

$$d(T^n x,T^n y) \precsim \frac{\alpha d(y,T^n y)[1+d(x,T^n x)]}{1+d(x,y)} + \beta d(x,y) \tag{3.19}$$

for all $x,y \in X$, where α, β are nonnegative reals with $s(\alpha+\beta) < 1$. Then the map T has a unique fixed point in X.

Proof. From Corollary 3.8, we obtain that $u \in X$ such that $T^n u = u$. The uniqueness follows from

$$\begin{aligned} d(Tu,u) &= d(TT^n u, T^n u) \\ &= d(T^n Tu, T^n u) \\ &\precsim \frac{\alpha d(u,T^n u)[1+d(Tu,T^n Tu)]}{1+d(Tu,u)} + \beta d(Tu,u) \\ &= \frac{\alpha d(u,u)[1+d(Tu,Tu)]}{1+d(Tu,u)} + \beta d(Tu,u) \\ &\precsim \beta d(Tu,u). \end{aligned} \tag{3.20}$$

Taking modulus of above equation (3.20), we get $|d(Tu,u)| \le \beta |d(Tu,u)|$, a contradiction. So $Tu = u$. Hence $Tu = T^n u = u$. Therefore, the fixed point of T is unique. This completes the proof. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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