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Research Article

Insights Into the Lucas *Q*-Matrix and Its Properties

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Abstract. This work proves that the equality

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^n = \begin{cases} 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, & \text{if } n \text{ is even,} \\ 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}, & \text{if } n \text{ is odd,} \end{cases}$$

holds for all integer n, where $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ is the Lucas Q-matrix introduced by Köken and Bozkurt [2]. While these authors established the case for natural n using induction, extending the result to integer exponents is considerably more challenging. One cannot simply assume that the formula holds for an arbitrary integer n, since, although Fibonacci and Lucas numbers are defined for negative indices, the proof by mathematical induction does not automatically extend to this setting. To achieve this, we developed several key prerequisites, including new relationships between the Fibonacci and Lucas Q-matrices, such as $\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = Q_L \cdot Q_F^{n-1}$, $n \in \mathbb{Z}$. Additionally, we demonstrate several previously known properties, but in a different way, using our properties. Because this text is also a survey article, we adopted a didactic, step-by-step approach, minimizing omissions whenever possible.

Keywords. Fibonacci numbers, Lucas numbers, Matrix method

Mathematics Subject Classification (2020). 11B39, 15A24, 40C05

1. Introduction

Consider (F_n) as the Fibonacci sequence given by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. In addition, consider (L_n) as the Lucas sequence given by $L_1 = 1$, $L_2 = 3$ and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. Such sequences are closely associated and have several identities and properties that relate them [3,4].

In 1960, in his master's thesis [1], Charles King introduced the Fibonacci Q-matrix, $Q_F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. This matrix holds significance because it satisfies the relation $Q_F^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, for $n \in \mathbb{Z}$, and numerous identities can be derived by manipulating it.

On the other hand, in 2010, Köken and Bozkurt [2] introduced the matrix $Q_L = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ as the Lucas Q-matrix. They demonstrate, via induction, that for $n \in \mathbb{N}$, the following holds:

$$Q_L^n = \begin{cases} 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, & \text{if } n \text{ is even,} \\ 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}, & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

Furthermore, in [2], the following well-known theorems are demonstrated with the aid of the new matrix Q_L:

Theorem 1.1. *For $n \in \mathbb{N}$,*

(1) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ (*Cassini's Identity*),

(2) $L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n-1}$.

Theorem 1.2. *Let n be any integer. So $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.*

Theorem 1.3. *For all integers m and n, the following equalities are valid:*

(1) $5F_{m+n} = L_n L_{m+1} + L_{n-1} L_m$,

(2) $F_{m+n} = F_n F_{m+1} + F_{n-1} F_m$,

(3) $L_{m+n} = L_m F_{n+1} + L_{m-1} F_n$,

(4) $5F_{m-n} = (-1)^{n-1}(L_m L_{n+1} - L_{m+1} L_n)$,

(5) $F_{m-n} = (-1)^n (F_m F_{n+1} - F_{m+1} F_n)$,

(6) $L_{m-n} = (-1)^{n-1}(F_m L_{n+1} - F_{m+1} L_n)$.

Although, Köken and Bozkurt [2] does not mention it, Cassini's identity holds for all integers n, not just natural ones. This can be easily proven, since it is known that $Q_F^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ holds for all integers n.

We find the introduction of the Lucas Q-matrix in [2] to be quite interesting. However, we have some observations that we hope to improve in the above-mentioned study. Additionally, interesting results were obtained that extend and further develop the research involving this matrix.

First, we list some observations. The n-th power computed in equation (1.1) was only considered for natural n. However, the theorems presented (1.2 and 1.3) derive the results under the assumption that the power holds for any integer n. This topic was not discussed in the article [2]. Nonetheless, we will demonstrate in this work that the n-th power computed in equation (1.1) indeed holds for all integers n. To do so, we prove that there is a direct relationship between the Fibonacci and Lucas Q-matrices, namely:

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = Q_L \cdot Q_F^{n-1}, \quad n \in \mathbb{Z}.$$

Furthermore, we observed that in the proof of Theorem 1.3, Köken and Bozkurt derived items (1) and (4) only for odd m and n; items (2) and (5) only for even m and n; and items (3) and (6) only for the case where m is odd and n is even (or vice versa). However, all these results are in fact independent of the parity of m and n because they hold for any integers. In this study, we present proofs that establish these properties without incurring this type of limitation. We will also demonstrate other properties similar to these, with the same technique, using Q-matrices.

Therefore, this paper provides insights into the Lucas Q_L-matrix and its properties.

2. Expanding Lucas and Fibonacci Numbers to Non-Positive Integer Indices

Before we present our contributions, it is pertinent to recall that extending the Fibonacci and Lucas numbers to non-positive indices is indeed possible, provided that the recurrence relations continue to hold.

Specifically, for the Fibonacci sequence, we can express $F_n = F_{n+2} - F_{n+1}$. In this manner, the sequence (F_n) is established for all $n \in \mathbb{Z}$. With this formula, we can assemble the Table 1, which displays a list of some values of F_n with integer indices as follows:

Table 1. Fibonacci numbers F_n, with $n \in \mathbb{Z}$

n	...	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	...
F_n	...	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	...

Similarly, we can extend the Lucas sequence to non-positive indices by rewriting the recurrence relation as $L_n = L_{n+2} - L_{n+1}$ and outlining the Table 2, which displays a list of some values of L_n with integer indices:

Table 2. Lucas numbers L_n, with $n \in \mathbb{Z}$

n	...	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	...
L_n	...	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	...

Observing the tables, it can be empirically noted that for $n \in \mathbb{N}$, we have $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$. We present a proof for this in the next sections, using the Q-matrices.

3. Main Content

3.1 Guiding Structure of This Work

To ensure that the reader clearly understands our approach, we outline the structure of our text in this brief section.

We begin by examining the matrix Q_L in Section 3.2. In this section, we establish an interesting relation between the Q-matrices of Fibonacci and Lucas, as given in Proposition 3.7. This result was identified by us and has not been presented elsewhere. Using this result, we can prove the subsequent proposition, which provides a version of Cassini's identity for the Lucas numbers.

In Section 3.3, we demonstrate that, for natural n, $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$. Although these results are well-known, our proof uses the matrices Q_F and Q_L. These equalities will be essential for establishing our main result in this article.

As mentioned previously, the main objective is to prove that

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^n = \begin{cases} 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, & \text{if } n \text{ is even,} \\ 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}, & \text{if } n \text{ is odd} \end{cases}$$

holds for any integer n. We do this in Section 3.4. To this end, we apply all the theories that we have developed in the previous sections. This result is particularly noteworthy because it was not established in the work [2], which introduced the matrix Q_L. One cannot simply assume that the formula holds for an arbitrary integer n, since, although Fibonacci and Lucas numbers are defined for negative indices, the proof by mathematical induction does not automatically extend to this setting.

In the last section, we provide proofs for several identities related to the Fibonacci and Lucas numbers. All demonstrations are based on the theoretical frameworks established in the previous sections. Moreover, we revisit and prove the results of Theorem 1.3, which appear in the work [2]; however, our proofs incorporate corrections based on the observations we made in the introduction regarding the proofs in [2].

3.2 Notes on the Matrix Q_L

As is well known, the n-th power of the matrix Q_F yields the matrix $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$. However, the matrix $\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}$ is not obtained from the n-th power of Q_L. This raises the question: how can we obtain $\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}$ from Q_L? We begin our exploration under the assumption that equation (1.1) has already been proven in [2] for natural n.

(i) First, consider n to be odd. Since $Q_L^n = Q_L \cdot Q_L^{n-1}$, by equation (1.1), we have

$$\cancel{5^{\frac{n-1}{2}}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \cancel{5^{\frac{n-1}{2}}} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix},$$

that is,

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}.$$

(ii) Now, consider n to be even. Since $Q_L^{n+1} = Q_L \cdot Q_L^n$, by equation (1.1), we obtain

$$\cancel{5^{\frac{n}{2}}} \begin{bmatrix} L_{n+2} & L_{n+1} \\ L_{n+1} & L_n \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \cancel{5^{\frac{n}{2}}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix},$$

$$\begin{bmatrix} L_{n+2} & L_{n+1} \\ L_{n+1} & L_n \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot Q_F^n,$$

$$\begin{bmatrix} L_{n+2} & L_{n+1} \\ L_{n+1} & L_n \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot Q_F^{n-1} \cdot Q_F.$$

Multiplying both sides of the above equation by Q_F^{-1} on the right, we get

$$\begin{bmatrix} L_{n+2} & L_{n+1} \\ L_{n+1} & L_n \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot Q_F^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} L_{n+1} & L_{n+2} - L_{n+1} \\ L_n & L_{n+1} - L_n \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} L_{n+1} & (L_n + \cancel{L_{n+1}}) - \cancel{L_{n+1}} \\ L_n & (L_{n-1} + \cancel{L_n}) - \cancel{L_n} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix},$$

that is,

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}.$$

From (i) and (ii), we have established the following proposition:

Proposition 3.1.

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = Q_L \cdot Q_F^{n-1}, \quad n \in \mathbb{N}. \tag{3.1}$$

Note that Proposition 3.1 is valid only for natural n. However, it will be possible to extend it to all integer n at a later stage.

Based on Proposition 3.1, it can be easily demonstrated, through the use of determinants, an identity similar to Cassini's identity, but for the Lucas numbers.

Proposition 3.2. $L_{n+1}L_{n-1} - L_n^2 = -5(-1)^n$, $n \in \mathbb{N}$.

Proof. From Proposition 3.1,

$$\begin{aligned} \begin{vmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{vmatrix} &= \det(Q_L \cdot Q_F^{n-1}) \\ &= \det(Q_L \cdot Q_F^n \cdot Q_F^{-1}) \\ &= \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}. \end{aligned}$$

By Cassini's identity, we have:

$$L_{n+1}L_{n-1}-L_n^2=5(-1)^n(-1)=-5(-1)^n. \qquad \square$$

To generalize equation (1.1) for all $n\in\mathbb{Z}$, we need to establish that $L_{-n}=(-1)^nL_n$, as empirically observed in Section 2. In fact, this result is well-established in the literature; however, we will prove it differently, with the help of the Lucas Q-matrix. We first compute $(Q_L^n)^{-1}$.

Lemma 3.3. *For $n\in\mathbb{N}$, the inverse of the n-th power of Q_L is given by:*

$$(Q_L^n)^{-1}=\begin{cases}\frac{1}{5^{\frac{n}{2}}}\begin{bmatrix}F_{n-1} & -F_n\\ -F_n & F_{n+1}\end{bmatrix}, & \text{if } n \text{ is even},\\ \frac{1}{5^{\frac{n+1}{2}}}\begin{bmatrix}L_{n-1} & -L_n\\ -L_n & L_{n+1}\end{bmatrix}, & \text{if } n \text{ is odd}.\end{cases}$$

Proof. (i) Using equation (1.1), for even n:

$$(Q_L^n)^{-1}=\begin{bmatrix}5^{\frac{n}{2}}F_{n+1} & 5^{\frac{n}{2}}F_n\\ 5^{\frac{n}{2}}F_n & 5^{\frac{n}{2}}F_{n-1}\end{bmatrix}^{-1},$$

that is,

$$(Q_L^n)^{-1}=\frac{1}{5^{\frac{n}{2}}F_{n+1}5^{\frac{n}{2}}F_{n-1}-5^{\frac{n}{2}}F_n5^{\frac{n}{2}}F_n}\begin{bmatrix}5^{\frac{n}{2}}F_{n-1} & -5^{\frac{n}{2}}F_n\\ -5^{\frac{n}{2}}F_n & 5^{\frac{n}{2}}F_{n+1}\end{bmatrix}$$
$$=\frac{5^{\frac{n}{2}}}{5^n(F_{n+1}F_{n-1}-F_n^2)}\begin{bmatrix}F_{n-1} & -F_n\\ -F_n & F_{n+1}\end{bmatrix}$$
$$=\frac{1}{5^{\frac{n}{2}}(F_{n+1}F_{n-1}-F_n^2)}\begin{bmatrix}F_{n-1} & -F_n\\ -F_n & F_{n+1}\end{bmatrix}.$$

Cassini's identity implies

$$(Q_L^n)^{-1}=\frac{1}{5^{\frac{n}{2}}(-1)^n}\begin{bmatrix}F_{n-1} & -F_n\\ -F_n & F_{n+1}\end{bmatrix}.$$

Since n is even, we have

$$(Q_L^n)^{-1}=\frac{1}{5^{\frac{n}{2}}}\begin{bmatrix}F_{n-1} & -F_n\\ -F_n & F_{n+1}\end{bmatrix}.$$

(ii) Using equation (1.1), for odd n:

$$(Q_L^n)^{-1}=\begin{bmatrix}5^{\frac{n-1}{2}}L_{n+1} & 5^{\frac{n-1}{2}}L_n\\ 5^{\frac{n-1}{2}}L_n & 5^{\frac{n-1}{2}}L_{n-1}\end{bmatrix}^{-1},$$

that is,

$$(Q_L^n)^{-1}=\frac{1}{5^{\frac{n-1}{2}}L_{n+1}5^{\frac{n-1}{2}}L_{n-1}-5^{\frac{n-1}{2}}L_n5^{\frac{n-1}{2}}L_n}\begin{bmatrix}5^{\frac{n-1}{2}}L_{n-1} & -5^{\frac{n-1}{2}}L_n\\ -5^{\frac{n-1}{2}}L_n & 5^{\frac{n-1}{2}}L_{n+1}\end{bmatrix}$$
$$=\frac{5^{\frac{n-1}{2}}}{5^{n-1}(L_{n+1}L_{n-1}-L_n^2)}\begin{bmatrix}L_{n-1} & -L_n\\ -L_n & L_{n+1}\end{bmatrix}$$
$$=\frac{1}{5^{\frac{n-1}{2}}(L_{n+1}L_{n-1}-L_n^2)}\begin{bmatrix}L_{n-1} & -L_n\\ -L_n & L_{n+1}\end{bmatrix}.$$

Proposition 3.2 implies

$$(Q_L^n)^{-1} = \frac{1}{5^{\frac{n-1}{2}}(-5(-1)^n)} \begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix}.$$

Since n is odd, we have

$$\begin{aligned}(Q_L^n)^{-1} &= \frac{1}{5^{\frac{n-1}{2}}5} \begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix} \\ &= \frac{1}{5^{\frac{n+1}{2}}} \begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix}.\end{aligned}$$

From (i) and (ii), the result is demonstrated. □

On the other hand, note that

$$Q_L^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{6-1} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Based on this, we can compute $(Q_L^{-1})^n$.

Lemma 3.4. *For $n \in \mathbb{N}$, the n-th power of Q_L^{-1} is given by:*

$$(Q_L^{-1})^n = \begin{cases} \frac{1}{5^{\frac{n}{2}}} \begin{bmatrix} F_{-n+1} & F_{-n} \\ F_{-n} & F_{-n-1} \end{bmatrix}, & \text{if } n \text{ even}, \\ \frac{1}{5^{\frac{n+1}{2}}} \begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix}, & \text{if } n \text{ odd}. \end{cases}.$$

Proof. We will verify the lemma by induction.

(I) For even n:

(i) For $n = 2$, the result is true because

$$\begin{aligned}\frac{1}{5^{\frac{2}{2}}} \begin{bmatrix} F_{-1} & F_{-2} \\ F_{-2} & F_{-3} \end{bmatrix} &= \frac{1}{5} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix} \\ &= \frac{1}{25} \begin{bmatrix} 4+1 & -2-3 \\ -2-3 & 1+9 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = (Q_L^{-1})^2.\end{aligned}$$

(ii) Assume that the result holds for $n = k$, that is:

$$(Q_L^{-1})^k = \frac{1}{5^{\frac{k}{2}}} \begin{bmatrix} F_{-k+1} & F_{-k} \\ F_{-k} & F_{-k-1} \end{bmatrix}.$$

(iii) We show that it holds for $n = k+2$. In fact,

$$\begin{aligned}(Q_L^{-1})^{k+2} &= (Q_L^{-1})^k \cdot (Q_L^{-1})^2 \\ &= \frac{1}{5^{\frac{k}{2}}} \begin{bmatrix} F_{-k+1} & F_{-k} \\ F_{-k} & F_{-k-1} \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{5^{\frac{k+2}{2}}} \begin{bmatrix} F_{-k+1} - F_{-k} & -F_{-k+1} + 2F_{-k} \\ F_{-k} - F_{-k-1} & -F_{-k} + 2F_{-k-1} \end{bmatrix} \\ &= \frac{1}{5^{\frac{k+2}{2}}} \begin{bmatrix} (F_{-k-1} + F_{-k}) - F_{-k} & -(F_{-k-1} + F_{-k}) + 2F_{-k} \\ (F_{-k-2} + F_{-k-1}) - F_{-k-1} & -(F_{-k-2} + F_{-k-1}) + 2F_{-k-1} \end{bmatrix}\end{aligned}$$

$$= \frac{1}{5^{\frac{k+2}{2}}}\begin{bmatrix} F_{-k-1} & -F_{-k-1}+(F_{-k-2}+F_{-k-1}) \\ F_{-k-2} & -F_{-k-2}+(F_{-k-3}+F_{-k-2}) \end{bmatrix}$$

$$= \frac{1}{5^{\frac{k+2}{2}}}\begin{bmatrix} F_{-k-1} & F_{-k-2} \\ F_{-k-2} & F_{-k-3} \end{bmatrix}.$$

By (i), (ii), and (iii), we have proved the equality by mathematical induction for even n.

(II) For odd n:

(i) For $n=1$, the result is true because

$$\frac{1}{5^{\frac{1+1}{2}}}\begin{bmatrix} L_0 & L_{-1} \\ L_{-1} & L_{-2} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = Q_L^{-1}.$$

(ii) Assume that the result holds for $n=k$, i.e.,

$$(Q_L^{-1})^k = \frac{1}{5^{\frac{k+1}{2}}}\begin{bmatrix} L_{-k+1} & L_{-k} \\ L_{-k} & L_{-k-1} \end{bmatrix}.$$

(iii) We show that the result holds for $n=k+2$. In fact,

$$(Q_L^{-1})^{k+2} = (Q_L^{-1})^k \cdot (Q_L^{-1})^2$$

$$= \frac{1}{5^{\frac{k+1}{2}}}\begin{bmatrix} L_{-k+1} & L_{-k} \\ L_{-k} & L_{-k-1} \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{5^{\frac{k+3}{2}}}\begin{bmatrix} L_{-k+1}-L_{-k} & -L_{-k+1}+2L_{-k} \\ L_{-k}-L_{-k-1} & -L_{-k}+2L_{-k-1} \end{bmatrix}$$

$$= \frac{1}{5^{\frac{k+3}{2}}}\begin{bmatrix} (L_{-k-1}+L_{-k})-L_{-k} & -(L_{-k-1}-L_{-k})+2L_{-k} \\ (L_{-k-2}+L_{-k-1})-L_{-k-1} & -(L_{-k-2}+L_{-k-1})+2L_{-k-1} \end{bmatrix}$$

$$= \frac{1}{5^{\frac{k+3}{2}}}\begin{bmatrix} L_{-k-1} & -L_{-k-1}+(L_{-k-2}+L_{-k-1}) \\ L_{-k-2} & -L_{-k-2}+(L_{-k-3}+L_{-k-2}) \end{bmatrix}$$

$$= \frac{1}{5^{\frac{k+3}{2}}}\begin{bmatrix} L_{-k-1} & L_{-k-2} \\ L_{-k-2} & L_{-k-3} \end{bmatrix}.$$

By (i), (ii), and (iii), we have proved the equality by mathematical induction for odd n.

Thus, by (I) and (II), we have proved the lemma. □

We will now work on generalizing Proposition 3.1 to all integers n. To proceed, we will need the following lemmas.

Lemma 3.5.

$$\begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & -F_{n+1} \\ -F_{n+1} & F_{n+2} \end{bmatrix}, \quad n \in \mathbb{N}.$$

Proof. (i) First, consider the case where n is odd. Since $(Q_L^n)^{-1} = Q_L \cdot (Q_L^{n+1})^{-1}$, by Lemma 3.3, we have:

$$\cancel{\frac{1}{5^{\frac{n+1}{2}}}}\begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \cancel{\frac{1}{5^{\frac{n+1}{2}}}}\begin{bmatrix} F_{(n+1)-1} & -F_{(n+1)} \\ -F_{(n+1)} & F_{(n+1)+1} \end{bmatrix},$$

that is,

$$\begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & -F_{n+1} \\ -F_{n+1} & F_{n+2} \end{bmatrix}.$$

(ii) Now, consider the case where n is even. Since $(Q_L^{n-1})^{-1} = Q_L \cdot (Q_L^n)^{-1}$, by Lemma 3.4, we have:

$$\cancel{\frac{1}{5^{\frac{(n-1)+1}{2}}}}\begin{bmatrix} L_{(n-1)-1} & -L_{(n-1)} \\ -L_{(n-1)} & L_{(n-1)+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \cancel{\frac{1}{5^{\frac{n}{2}}}} \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix},$$

so,

$$\begin{bmatrix} L_{n-2} & -L_{n-1} \\ -L_{n-1} & L_n \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix}.$$

Multiplying both sides of the previous equality by $-Q_F^{-1}$, on the right, we get:

$$\begin{bmatrix} L_{n-2} & -L_{n-1} \\ -L_{n-1} & L_n \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} L_{n-1} & -L_{n-2} - L_{n-1} \\ -L_n & L_{n-1} + L_n \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & -F_{n-1} - F_n \\ -F_{n+1} & F_n + F_{n+1} \end{bmatrix},$$

that is,

$$\begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & -F_{n+1} \\ -F_{n+1} & F_{n+2} \end{bmatrix}.$$

By (i) and (ii), we have just established the lemma. □

Lemma 3.6.

$$\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_{-n} & F_{-n-1} \\ F_{-n-1} & F_{-n-2} \end{bmatrix}, \quad n \in \mathbb{N}.$$

Proof. (i) First, consider the case where n is odd. Since $(Q_L^{-1})^n = Q_L \cdot (Q_L^{-1})^{n+1}$, by Lemma 3.4, we have:

$$\cancel{\frac{1}{5^{\frac{n+1}{2}}}}\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \cancel{\frac{1}{5^{\frac{n+1}{2}}}} \begin{bmatrix} F_{-(n+1)+1} & F_{-(n+1)} \\ F_{-(n+1)} & F_{-(n+1)-1} \end{bmatrix},$$

that is,

$$\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_{-n} & F_{-n-1} \\ F_{-n-1} & F_{-n-2} \end{bmatrix}.$$

(ii) Now, consider the case where n is even. Since $(Q_L^{-1})^{n-1} = Q_L \cdot (Q_L^{-1})^n$, by Lemma 3.4, we have:

$$\cancel{\frac{1}{5^{\frac{(n-1)+1}{2}}}}\begin{bmatrix} L_{-(n-1)+1} & L_{-(n-1)} \\ L_{-(n-1)} & L_{-(n-1)-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \cancel{\frac{1}{5^{\frac{n}{2}}}} \begin{bmatrix} F_{-n+1} & F_{-n} \\ F_{-n} & F_{-n-1} \end{bmatrix},$$

so,

$$\begin{bmatrix} L_{-n+2} & L_{-n+1} \\ L_{-n+1} & L_{-n} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_{-n+1} & F_{-n} \\ F_{-n} & F_{-n-1} \end{bmatrix}.$$

Multiplying both sides of the previous equality by Q_F^{-1}, on the right, we get:

$$\begin{bmatrix} L_{-n+2} & L_{-n+1} \\ L_{-n+1} & L_{-n} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_{-n+1} & F_{-n} \\ F_{-n} & F_{-n-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix},$$

that is,

$$\begin{bmatrix} L_{-n+1} & L_{-n+2} - L_{-n+1} \\ L_{-n} & L_{-n+1} - L_{-n} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_{-n} & F_{-n+1} - F_{-n} \\ F_{-n-1} & F_{-n} - F_{-n-1} \end{bmatrix},$$

in other terms,

$$\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_{-n} & F_{-n-1} \\ F_{-n-1} & F_{-n-2} \end{bmatrix}.$$

By (i) and (ii), we have just established the lemma. □

Now, we can establish that:

Proposition 3.7.

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = Q_L \cdot Q_F^{n-1}, \quad n \in \mathbb{Z}.$$

Proof. For $n = 0$, the result follows straightforwardly since

$$\begin{aligned} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_0 & F_{0-1} \\ F_{0-1} & F_{0-2} \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} L_{0+1} & L_0 \\ L_0 & L_{0-1} \end{bmatrix}. \end{aligned}$$

The case where n is a natural number has been established in Proposition 3.1. Furthermore, the case where $-n = k$, for some $k \in \mathbb{N}$, is demonstrated in Lemma 3.6. □

And as a consequence,

Proposition 3.8. *$L_{n+1}L_{n-1} - L_n^2 = -5(-1)^n$, $n \in \mathbb{Z}$.*

Proof. For $n = 0$, the result is straightforward, as we have

$$L_1 L_{-1} - L_0^2 = 1 \cdot (-1) - 2^2 = -5 = -5(-1)^0.$$

The case where n is a positive integer is established in Proposition 3.2. Now, consider when n is a negative integer. In this case, let $n = -k$, where $k \in \mathbb{N}$. Using the determinant from Lemma 3.6, we have

$$\begin{vmatrix} L_{-k+1} & L_{-k} \\ L_{-k} & L_{-k-1} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} F_{-k} & F_{-k-1} \\ F_{-k-1} & F_{-k-2} \end{vmatrix}.$$

This implies that

$$L_{-k+1}L_{-k-1} - L_{-k}^2 = (6-1)(F_{-k}F_{-k-2} - F_{-k-1}^2).$$

By Cassini's identity, it follows that

$$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n-1} = -5(-1)^n.$$ □

3.3 Proving $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$

To establish the equality in equation (1.1) for all integer n, it is necessary to show that $L_{-n} = (-1)^n L_n$, a result we will prove in this section. In addition, we first demonstrate the identity $F_{-n} = (-1)^{n+1}F_n$, as it will provide useful context.

To prove the desired equality, we will compare $(Q_F^n)^{-1}$ and $(Q_F^{-1})^n$.

Lemma 3.9. *For $n \in \mathbb{N}$, the inverse of the n-th power of Q_F is given by:*

$$(Q_F^n)^{-1} = (-1)^n \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix}.$$

Proof. By Cassini's identity, we have:

$$\begin{aligned}(Q_F^n)^{-1} &= \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}^{-1} \\ &= \frac{1}{(-1)^n} \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix} \\ &= (-1)^n \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} (-1)^n F_{n-1} & (-1)^{n+1} F_n \\ (-1)^{n+1} F_n & (-1)^n F_{n+1} \end{bmatrix}.\end{aligned}$$

□

Considering the powers of Q_F^{-1} the next lemma holds.

Lemma 3.10. *For $n \in \mathbb{N}$, the n-th power of Q_F^{-1} is given by:*

$$(Q_F^{-1})^n = \begin{bmatrix} F_{-n+1} & F_{-n} \\ F_{-n} & F_{-n-1} \end{bmatrix}.$$

Proof. Let us verify the lemma by induction.

(i) For $n = 1$, the result is true since

$$Q_F^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} F_0 & F_{-1} \\ F_{-1} & F_{-2} \end{bmatrix}.$$

(ii) We assume that the result is true for $n = k$, that is

$$(Q_F^{-1})^k = \begin{bmatrix} F_{-k+1} & F_{-k} \\ F_{-k} & F_{-k-1} \end{bmatrix}.$$

(iii) We show true for $n = k+1$,

$$\begin{aligned}(Q_F^{-1})^{k+1} &= (Q_F^{-1})^k \cdot (Q_F^{-1})^1 \\ &= \begin{bmatrix} F_{-k+1} & F_{-k} \\ F_{-k} & F_{-k-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} F_{-k} & F_{-k+1} - F_{-k} \\ F_{-k-1} & F_{-k} - F_{-k-1} \end{bmatrix} \\ &= \begin{bmatrix} F_{-k} & (F_{-k-1} + F_{-k}) - F_{-k} \\ F_{-k-1} & (F_{-k-2} + F_{-k-1}) - F_{-k-1} \end{bmatrix} \\ &= \begin{bmatrix} F_{-k} & F_{-k-1} \\ F_{-k-1} & F_{-k-2} \end{bmatrix}.\end{aligned}$$

By (i), (ii) and (iii) we have the equality for $n \in \mathbb{N}$. □

By comparing $(Q_F^n)^{-1}$ and $(Q_F^{-1})^n$ for $n \in \mathbb{N}$, as obtained in Lemmas 3.9 and 3.10, we will be able to prove, in the next result, that $F_{-n} = (-1)^{n+1} F_n$.

Theorem 3.11. $F_{-n} = (-1)^{n+1} F_n$, $n \in \mathbb{N}$.

Proof. To begin the proof, observe that

$$(Q_F^n)^{-1} = (Q_F^{-1})^n,$$

and from lemmas 3.9 and 3.10,

$$\begin{bmatrix} (-1)^n F_{n-1} & (-1)^{n+1} F_n \\ (-1)^{n+1} F_n & (-1)^n F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{-n+1} & F_{-n} \\ F_{-n} & F_{-n-1} \end{bmatrix}. \tag{3.2}$$

By comparing the entries in the first row and second column of the matrices in equation (3.2), we conclude that

$$F_{-n} = (-1)^{n+1} F_n.$$

□

Thus, we are motivated to extend the definition of Fibonacci numbers to include non-positive indices, setting $F_0 = 0$ and $F_{-n} = (-1)^{n+1} F_n$ for $n \in \mathbb{N}$.

To prove the equality $L_{-n} = (-1)^n L_n$ for $n \in \mathbb{N}$, we will compare $(Q_L^n)^{-1}$ and $(Q_L^{-1})^n$, both of which were computed in the previous section.

Theorem 3.12. $L_{-n} = (-1)^n L_n$, $n \in \mathbb{N}$.

Proof. Lemma 3.5 implies that

$$\begin{aligned}
\begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix} &= \begin{bmatrix} 3F_n - F_{n+1} & -3F_{n+1} + F_{n+2} \\ F_n - 2F_{n+1} & -F_{n+1} + 2F_{n+2} \end{bmatrix} \\
&= \begin{bmatrix} 2F_n + (F_{n-2} + F_{n-1}) - F_{n+1} & -3F_{n+1} + (F_n + F_{n+1}) \\ \cancel{F_n} - F_{n+1} - (F_{n-1} + \cancel{F_n}) & -\cancel{F_{n+1}} + F_{n+2} + (F_n + \cancel{F_{n+1}}) \end{bmatrix} \\
&= \begin{bmatrix} F_n + F_{n-2} + \cancel{F_{n+1}} - \cancel{F_{n+1}} & -F_{n+1} - (F_{n-1} + \cancel{F_n}) + \cancel{F_n} \\ -F_{n-1} - F_{n+1} & F_n + F_{n+2} \end{bmatrix},
\end{aligned}$$

which simplifies to

$$\begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-2} & -F_{n-1} - F_{n+1} \\ -F_{n-1} - F_{n+1} & F_n + F_{n+2} \end{bmatrix}. \tag{3.3}$$

Lemma 3.6 implies that

$$\begin{aligned}
\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} &= \begin{bmatrix} 3F_{-n} + F_{-n-1} & 3F_{-n-1} + F_{-n-2} \\ F_{-n} + 2F_{-n-1} & F_{-n-1} + 2F_{-n-2} \end{bmatrix} \\
&= \begin{bmatrix} 2F_{-n} + F_{-n+1} & 2F_{-n-1} + F_{-n} \\ F_{-n-1} + F_{-n+1} & F_{-n-2} + F_{-n} \end{bmatrix},
\end{aligned}$$

which simplifies to

$$\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} = \begin{bmatrix} F_{-n} + F_{-n+2} & F_{-n-1} + F_{-n+1} \\ F_{-n-1} + F_{-n+1} & F_{-n-2} + F_{-n} \end{bmatrix}.$$

By Proposition 3.11, we have

$$\begin{aligned}
\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} &= \begin{bmatrix} (-1)^{n+1} F_n + (-1)^{n-2+1} F_{n-2} & (-1)^{n+1+1} F_{n+1} + (-1)^{n-1+1} F_{n-1} \\ (-1)^{n+1+1} F_{n+1} + (-1)^{n-1+1} F_{n-1} & (-1)^{n+2+1} F_{n+2} + (-1)^{n+1} F_n \end{bmatrix} \\
&= (-1)^{n+1} \begin{bmatrix} F_n + F_{n-2} & -F_{n+1} - F_{n-1} \\ -F_{n+1} - F_{n-1} & F_{n+2} + F_n \end{bmatrix}.
\end{aligned}$$

This implies that

$$(-1)^{n+1}\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-2} & -F_{n+1} - F_{n-1} \\ -F_{n+1} - F_{n-1} & F_{n+2} + F_n \end{bmatrix}.$$

Using equation (3.3), we obtain

$$(-1)^{n+1}\begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} = \begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix}.$$

By comparing the entries in the first row and second column of these matrices, it follows that

$$(-1)^{n+1}L_{-n} = -L_n$$

$$\implies \quad L_{-n} = (-1)^n L_n. \qquad \square$$

We will use these observations to complete the definition of Lucas numbers, in the same way we did for Fibonacci numbers, that is, we define $L_0 = 2$ and $L_{-n} = (-1)^n L_n$, for $n \in \mathbb{N}$.

3.3.1 On Binet's Theorem

The Binet formulas for Fibonacci and Lucas numbers can be trivially proved by induction for the case where n is a natural number.

Next, we prove that the Binet formula for Fibonacci numbers still holds, even with this more complete definition now including non-positive integer indices. In fact,

- For $n = 0$:
$$\frac{\alpha^0 - \beta^0}{\alpha - \beta} = \frac{1-1}{\alpha - \beta} = 0 = F_0.$$
- If $n \in \mathbb{N}$, for $-n$:
$$\frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{\frac{1}{\alpha^n} - \frac{1}{\beta^n}}{\alpha - \beta} = \frac{\frac{\beta^n - \alpha^n}{(\alpha\beta)^n}}{\alpha - \beta} = (-1)^n\left(\frac{\beta^n - \alpha^n}{\alpha - \beta}\right) = (-1)^{n+1}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) = (-1)^{n+1}F_n = F_{-n}.$$

We will now verify that the Binet formula for Lucas numbers also remains valid with non-positive integer indices. In fact,

- For $n = 0$:
$$\alpha^0 + \beta^0 = 1 + 1 = 2 = L_0.$$
- If $n \in \mathbb{N}$, for $-n$:
$$\alpha^{-n} + \beta^{-n} = \frac{1}{\alpha^n} + \frac{1}{\beta^n} = \frac{\beta^n + \alpha^n}{(\alpha\beta)^n} = (-1)^n(\alpha^n + \beta^n) = (-1)^n L_n = L_{-n}.$$

Thus, we have established the Binet Theorem for all integer n.

Theorem 3.13 (Binet formulas for Fibonacci and Lucas numbers)**.** *For $n \in \mathbb{Z}$, considering $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$,*

(a) *the n-th Fibonacci number is given by*
$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta};$$

(b) *the n-th Lucas number is given by*

$$L_n = \alpha^n + \beta^n.$$

3.4 Q_L^n for Non-positive n

In this section, our goal is to complete the proof of the n-th power of Q_L, as presented in equation (1.1), by extending it to non-positive integer values of n. Given that the case for $n \in \mathbb{N}$ was established in [2], the result holds for all integer values of n.

Theorem 3.14. *For $n \in \mathbb{Z}$, the n-th power of Q_L is given by:*

$$Q_L^n = \begin{cases} 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, & \text{if } n \text{ is even}, \\ 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}, & \text{if } n \text{ is odd}. \end{cases}$$

Proof. In the case where n is a positive integer, the proof has been established in [2]. We now examine the case when n is a negative integer. In this context, we can write $n = -k$, where $k \in \mathbb{N}$ and $Q_L^n = Q_L^{-k} = (Q_L^k)^{-1}$.

(I) Consider k even, that is, n even.

By Cassini's identity, we have:

$$\begin{aligned} Q_L^n = (Q_L^k)^{-1} &= \begin{bmatrix} 5^{\frac{k}{2}} F_{k+1} & 5^{\frac{k}{2}} F_k \\ 5^{\frac{k}{2}} F_k & 5^{\frac{k}{2}} F_{k-1} \end{bmatrix}^{-1} \\ &= \frac{1}{5^{2\frac{k}{2}}(-1)^k} \begin{bmatrix} 5^{\frac{k}{2}} F_{k-1} & -5^{\frac{k}{2}} F_k \\ -5^{\frac{k}{2}} F_k & 5^{\frac{k}{2}} F_{k+1} \end{bmatrix} \\ &= 5^{-k}(-1)^k 5^{\frac{k}{2}} \begin{bmatrix} F_{k-1} & -F_k \\ -F_k & F_{k+1} \end{bmatrix} \\ &= 5^{\frac{-k}{2}} \begin{bmatrix} (-1)^k F_{k-1} & (-1)^{k+1} F_k \\ (-1)^{k+1} F_k & (-1)^k F_{k+1} \end{bmatrix}. \end{aligned}$$

Nonetheless

$$F_n = F_{-k} = (-1)^{k+1} F_k,$$

$$F_{n+1} = F_{-k+1} = F_{-(k-1)} = (-1)^{(k-1)+1} F_{k-1} = (-1)^k F_{k-1}$$

and

$$F_{n-1} = F_{-k-1} = F_{-(k+1)} = (-1)^{(k+1)+1} F_{k+1} = (-1)^{k+2} F_{k+1} = (-1)^k F_{k+1}.$$

Thus, it follows that

$$Q_L^n = 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Before considering the case when n is odd, let us examine $n = 0$. Here, the theorem holds trivially, as

$$Q_F^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} F_1 & F_0 \\ F_0 & F_{-1} \end{bmatrix} = 5^{\frac{0}{2}} \begin{bmatrix} F_{0+1} & F_0 \\ F_0 & F_{0-1} \end{bmatrix}.$$

Thus, the theorem holds for even n.

(II) Consider k odd, that is, n odd.

Using Proposition 3.8, we obtain:

$$
\begin{aligned}
Q_L^n = (Q_L^k)^{-1} &= \begin{bmatrix} 5^{\frac{k-1}{2}} L_{k+1} & 5^{\frac{k-1}{2}} L_k \\ 5^{\frac{k-1}{2}} L_k & 5^{\frac{k-1}{2}} L_{k-1} \end{bmatrix}^{-1} \\
&= \frac{1}{5^{2\frac{(k-1)}{2}} 5(-1)^{k+1}} \begin{bmatrix} 5^{\frac{k-1}{2}} L_{k-1} & -5^{\frac{k-1}{2}} L_k \\ -5^{\frac{k-1}{2}} L_k & 5^{\frac{k-1}{2}} L_{k+1} \end{bmatrix} \\
&= \frac{5^{\frac{k-1}{2}}}{5^{k-1} 5(-1)^2(-1)^{k-1}} \begin{bmatrix} L_{k-1} & -L_k \\ -L_k & L_{k+1} \end{bmatrix} \\
&= 5^{\frac{-k-1}{2}} \begin{bmatrix} (-1)^{k-1} L_{k-1} & (-1)^k L_k \\ (-1)^k L_k & (-1)^{k-1} L_{k+1} \end{bmatrix}.
\end{aligned}
$$

Since

$$
\begin{aligned}
L_n &= L_{-k} = (-1)^k L_k, \\
L_{n+1} &= L_{-k+1} = L_{-(k-1)} = (-1)^{k-1} L_{k-1}
\end{aligned}
$$

and

$$
L_{n-1} = L_{-k-1} = L_{-(k+1)} = (-1)^{k+1} L_{k+1} = (-1)^{k-1} L_{k+1},
$$

we conclude that

$$
Q_L^n = 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}.
$$

Therefore, the theorem holds for odd n as well. □

3.5 Matrix-derived Identities

We will derive identities by combining the matrices Q_F and Q_L. Among the identities we present are those included in Theorem 1.3, as previously stated in [2]. However, these identities now have stronger justification, given that all our results are firmly established for all integer values of m and n.

Proposition 3.15. $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$, $m,n \in \mathbb{Z}$.

Proof. Observe that $Q_F^{m+n} = Q_F^m \cdot Q_F^n$. Therefore, we have

$$
\begin{bmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{bmatrix} = \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} \cdot \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix},
$$

which implies

$$
\begin{bmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{bmatrix} = \begin{bmatrix} F_{m+1}F_{n+1} + F_m F_n & F_{m+1}F_n + F_m F_{n-1} \\ F_m F_{n+1} + F_{m-1}F_n & F_m F_n + F_{m-1}F_{n-1} \end{bmatrix}. \tag{3.4}
$$

From the entry in the first row and second column of the matrices in equation (3.4), it follows that

$$
F_{m+n} = F_{m+1}F_n + F_m F_{n-1}.
$$

□

Proposition 3.16. $F_{m-n} = (-1)^n(F_mF_{n+1} - F_{m+1}F_n)$, $m,n \in \mathbb{Z}$.

Proof. Notice that $Q_F^{m-n} = Q_F^m \cdot (Q_F^n)^{-1}$. However,

$$(Q_F^n)^{-1} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}^{-1} = \frac{1}{(-1)^n}\begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix} = (-1)^n \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} F_{m-n+1} & F_{m-n} \\ F_{m-n} & F_{m-n-1} \end{bmatrix} = \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} \cdot (-1)^n \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix}.$$

Carrying out the matrix multiplication, we get:

$$\begin{bmatrix} F_{m-n+1} & F_{m-n} \\ F_{m-n} & F_{m-n-1} \end{bmatrix} = (-1)^n \begin{bmatrix} F_{m+1}F_{n-1} - F_mF_n & F_mF_{n+1} - F_{m+1}F_n \\ F_mF_{n-1} - F_{m-1}F_n & F_{m-1}F_{n+1} - F_mF_n \end{bmatrix}. \tag{3.5}$$

From the entry in the first row and second column of the matrices in equation (3.5), it follows that

$$F_{m-n} = (-1)^n(F_mF_{n+1} - F_{m+1}F_n).$$

□

Proposition 3.17. $L_n = F_{n-1} + F_{n+1}$, $n \in \mathbb{Z}$.

Proof. From Proposition 3.7, we have

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3F_n + F_{n-1} & 3F_{n-1} + F_{n-2} \\ F_n + 2F_{n-1} & F_{n-1} + 2F_{n-2} \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n+2} & F_{n-1} + F_{n+1} \\ F_{n-1} + F_{n+1} & F_{n-2} + F_n \end{bmatrix}. \tag{3.6}$$

By comparing the entries in the first row and second column of the matrices in the previous equality, the result follows. □

Proposition 3.18. $L_{n+1} + L_{n-1} = 5F_n$, $n \in \mathbb{Z}$.

Proof. Multiplying both sides of proposition 3.7 by Q_L on the left, we obtain:

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot Q_L \cdot Q_F^{n-1},$$

$$\begin{bmatrix} 3L_{n+1} + L_n & 3L_n + L_{n-1} \\ L_{n+1} + 2L_n & L_n + 2L_{n-1} \end{bmatrix} = Q_L^2 \cdot Q_F^{n-1}.$$

By Theorem 3.14, we know that $Q_L^2 = 5\begin{bmatrix} F_{2+1} & F_2 \\ F_2 & F_{2-1} \end{bmatrix} = 5Q_F^2$, so:

$$\begin{bmatrix} L_{n+1} + L_{n+3} & L_n + L_{n+2} \\ L_n + L_{n+2} & L_{n-1} + L_{n+1} \end{bmatrix} = 5^{\frac{2}{2}} Q_F^2 \cdot Q_F^{n-1},$$

$$\begin{bmatrix} L_{n+1} + L_{n+3} & L_n + L_{n+2} \\ L_n + L_{n+2} & L_{n-1} + L_{n+1} \end{bmatrix} = 5Q_F^{n+1},$$

which simplifies to:

$$\begin{bmatrix} L_{n+1} + L_{n+3} & L_n + L_{n+2} \\ L_n + L_{n+2} & L_{n-1} + L_{n+1} \end{bmatrix} = \begin{bmatrix} 5F_{n+2} & 5F_{n+1} \\ 5F_{n+1} & 5F_n \end{bmatrix}. \tag{3.7}$$

By comparing the entries in the second row and second column of the matrices in equation (3.7), the result follows. □

Proposition 3.19. $L_n + F_n = 2F_{n+1}$, $n \in \mathbb{Z}$.

Proof. By adding Q_F^n to both sides of the matrix equation (3.6), we have:

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} + \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n+2} & F_{n-1} + F_{n+1} \\ F_{n-1} + F_{n+1} & F_{n-2} + F_n \end{bmatrix} + \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} F_n + F_{n+2} + F_{n+1} & F_{n-1} + F_{n+1} + F_n \\ F_{n-1} + F_{n+1} + F_n & F_{n-2} + F_n + F_{n-1} \end{bmatrix}.$$

This implies:

$$\begin{bmatrix} L_{n+1} + F_{n+1} & L_n + F_n \\ L_n + F_n & L_{n-1} + F_{n-1} \end{bmatrix} = \begin{bmatrix} 2F_{n+2} & 2F_{n+1} \\ 2F_{n+1} & 2F_n \end{bmatrix}. \tag{3.8}$$

By comparing the entries in the first row and second column of the matrix equation (3.8), we obtain the desired identity. □

Proposition 3.20. $L_n - F_n = 2F_{n-1}$, $n \in \mathbb{Z}$.

Proof. By subtracting Q_F^n from both sides of the matrix equation (3.6), we have:

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} - \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n+2} & F_{n-1} + F_{n+1} \\ F_{n-1} + F_{n+1} & F_{n-2} + F_n \end{bmatrix} - \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} F_n + F_{n+2} - F_{n+1} & F_{n-1} + F_{n+1} - F_n \\ F_{n-1} + F_{n+1} - F_n & F_{n-2} + F_n - F_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} 2F_n + \cancel{F_{n+1}} - \cancel{F_{n+1}} & 2F_{n-1} + \cancel{F_n} - \cancel{F_n} \\ 2F_{n-1} + \cancel{F_n} - \cancel{F_n} & 2F_{n-2} + \cancel{F_{n-1}} - \cancel{F_{n-1}} \end{bmatrix}.$$

This implies:

$$\begin{bmatrix} L_{n+1} - F_{n+1} & L_n - F_n \\ L_n - F_n & L_{n-1} - F_{n-1} \end{bmatrix} = \begin{bmatrix} 2F_n & 2F_{n-1} \\ 2F_{n-1} & 2F_{n-2} \end{bmatrix}. \tag{3.9}$$

By comparing the entries in the first row and second column of the matrix equation (3.9), we obtain the desired identity. □

Proposition 3.21. $L_n^2 - F_n^2 = 4F_{n+1}F_{n-1}$, $n \in \mathbb{Z}$.

Proof. From Propositions 3.19 and 3.20, we have:

$$(L_n + F_n)(L_n - F_n) = 2F_{n+1}2F_{n-1},$$

that is,

$$L_n^2 - F_n^2 = 4F_{n+1}F_{n-1}.$$ □

Proposition 3.22. $L_n^2 - 4(-1)^n = 5F_n^2$, $n \in \mathbb{Z}$.

Proof. By adding the equalities produced by the Cassini's identity and Proposition 3.8, we get:

$$-5(-1)^n + (-1)^n = L_{n+1}L_{n-1} - L_n^2 + F_{n+1}F_{n-1} - F_n^2,$$

that is,

$$L_n^2 - 4(-1)^n = L_{n+1}L_{n-1} + F_{n+1}F_{n-1} - F_n^2.$$

Using Proposition 3.17, we compute:

$$\begin{aligned}
L_n^2 - 4(-1)^n &= (F_n + F_{n+2})(F_{n-2} + F_n) + F_{n+1}F_{n-1} - F_n^2 \\
&= F_nF_{n-2} + \cancel{F_nF_n} + F_{n+2}F_{n-2} + F_{n+2}F_n + F_{n+1}F_{n-1} - \cancel{F_n^2} \\
&= F_nF_{n-2} + (F_n + F_{n+1})F_{n-2} + (F_n + F_{n+1})F_n + F_{n+1}F_{n-1} \\
&= F_nF_{n-2} + F_nF_{n-2} + F_{n+1}F_{n-2} + F_nF_n + F_{n+1}F_n + F_{n+1}F_{n-1} \\
&= F_n^2 + F_nF_{n-2} + F_nF_{n-2} + F_{n+1}F_{n-2} + F_{n+1}F_n + F_{n+1}F_{n-1} \\
&= F_n^2 + F_nF_{n-2} + F_nF_{n-2} + F_{n+1}(F_{n-2} + F_{n-1}) + F_{n+1}F_n \\
&= F_n^2 + F_nF_{n-2} + F_nF_{n-2} + F_{n+1}F_n + F_{n+1}F_n \\
&= F_n^2 + 2F_nF_{n-2} + 2F_{n+1}F_n \\
&= F_n^2 + 2F_nF_{n-2} + 2(F_{n-1} + F_n)F_n \\
&= F_n^2 + 2F_nF_{n-2} + 2F_nF_{n-1} + 2F_nF_n \\
&= 3F_n^2 + 2F_n(F_{n-2} + F_{n-1}) \\
&= 3F_n^2 + 2F_nF_n \\
&= 5F_n^2.
\end{aligned}$$

□

Proposition 3.23. $L_{m+n} = L_{m+1}F_n + L_mF_{n-1},\ n, m \in \mathbb{Z}.$

Proof. From Proposition 3.7, we have:

$$\begin{aligned}
\begin{bmatrix} L_{m+n+1} & L_{m+n} \\ L_{m+n} & L_{m+n-1} \end{bmatrix} &= Q_L \cdot Q_F^{m+n-1} \\
&= Q_L \cdot Q_F^{m-1} \cdot Q_F^n \\
&= \begin{bmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{bmatrix} \cdot Q_F^n,
\end{aligned}$$

that is,

$$\begin{bmatrix} L_{m+n+1} & L_{m+n} \\ L_{m+n} & L_{m+n-1} \end{bmatrix} = \begin{bmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{bmatrix} \cdot \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Implying

$$\begin{bmatrix} L_{m+n+1} & L_{m+n} \\ L_{m+n} & L_{m+n-1} \end{bmatrix} = \begin{bmatrix} L_{m+1}F_{n+1} + L_mF_n & L_{m+1}F_n + L_mF_{n-1} \\ L_mF_{n+1} + L_{m-1}F_n & L_mF_n + L_{m-1}F_{n-1} \end{bmatrix}. \tag{3.10}$$

Comparing the entries in the first row and second column in the equality (3.10), we obtain the desired identity. □

Proposition 3.24. $2L_{m+n} = L_mL_n + 5F_mF_n,\ n \in \mathbb{Z}.$

Proof. By summing the terms of the secondary diagonal in equation (3.10), we have:

$$\begin{aligned}
2L_{m+n} &= L_{m+1}F_n + L_mF_{n-1} + L_mF_{n+1} + L_{m-1}F_n \\
&= L_m(F_{n-1} + F_{n+1}) + (L_{m-1} + L_{m+1})F_n.
\end{aligned}$$

□

Substituting $F_{n-1}+F_{n+1}$ according to Proposition 3.17 and $L_{m-1}+L_{m+1}$ according to Proposition 3.18, we obtain the desired proposition.

Proposition 3.25. $2F_{m+n} = F_m L_n + F_n L_m$, $n \in \mathbb{Z}$.

Proof. By summing the terms of the secondary diagonal in equation (3.4), we have:

$$\begin{aligned} 2F_{m+n} &= F_{m+1}F_n + F_m F_{n-1} + F_m F_{n+1} + F_{m-1}F_n \\ &= F_n(F_{m-1}+F_{m+1}) + F_m(F_{n-1}+F_{n+1}). \end{aligned}$$

Substituting $F_{m-1}+F_{m+1}$ and $F_{n-1}+F_{n+1}$ according to Proposition 3.17, we obtain the desired proposition. □

Proposition 3.26. $5F_{m+n} = L_n L_{m+1} + L_{n-1}L_m$, $n \in \mathbb{Z}$.

Proof. Multiplying $\begin{bmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{bmatrix}$ by $\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}$, according to equation (3.6), we obtain:

$$\begin{bmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{bmatrix} \cdot \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} F_m + F_{m+2} & F_{m-1}+F_{m+1} \\ F_{m-1}+F_{m+1} & F_{m-2}+F_m \end{bmatrix} \cdot \begin{bmatrix} F_n + F_{n+2} & F_{n-1}+F_{n+1} \\ F_{n-1}+F_{n+1} & F_{n-2}+F_n \end{bmatrix}.$$

This implies, in the entries of the first row and second column:

$$L_{m+1}L_n + L_m L_{n-1} = (F_m + F_{m+2})(F_{n-1}+F_{n+1}) + (F_{m-1}+F_{m+1})(F_{n-2}+F_n),$$

that is,

$$\begin{aligned} L_n L_{m+1} + L_{n-1}L_m &= (F_m + F_{m+2})(F_{n-1}+F_{n+1}) + (F_{m-1}+F_{m+1})(F_{n-2}+F_n) \\ &= F_m F_{n-1} + F_{m+2}F_{n-1} + F_m F_{n+1} + F_{m+2}F_{n+1} \\ &\quad + F_{m-1}F_{n-2} + F_{m+1}F_{n-2} + F_{m-1}F_n + F_{m+1}F_n \\ &= (F_{m+1}F_n + F_m F_{n-1}) + (F_{m+1}+F_m)F_{n-1} + F_{m-1}F_n \\ &\quad + (F_{m+1}+F_m)F_{n+1} + F_{m+1}F_{n-2} + F_m F_{n+1} + F_{m-1}F_{n-2} \\ &= (F_{m+1}F_n + F_m F_{n-1}) + (F_{m+1}(F_{n-2}+F_{n-1}) + F_m F_{n-1}) \\ &\quad + F_{m+1}(F_{n-1}+F_n) + 2F_m(F_{n-1}+F_n) + F_{m-1}F_{n-2} + F_{m-1}F_n \\ &= 2(F_{m+1}F_n + F_m F_{n-1}) + 2F_m F_{n-1} + F_{m+1}F_n \\ &\quad + F_{m+1}F_{n-1} + 2F_m F_n + F_{m-1}F_{n-2} + F_{m-1}F_n \\ &= 3(F_{m+1}F_n + F_m F_{n-1}) + F_m F_{n-1} + F_{m+1}F_{n-1} \\ &\quad + F_m F_n + F_m(F_{n-2}+F_{n-1}) + F_{m-1}F_{n-2} + F_{m-1}F_n \\ &= 3(F_{m+1}F_n + F_m F_{n-1}) + 2F_m F_{n-1} + F_m F_n \\ &\quad + F_{m+1}F_{n-1} + (F_{m-1}+F_m)F_{n-2} + F_{m-1}F_n \\ &= 3(F_{m+1}F_n + F_m F_{n-1}) + 2F_m F_{n-1} \\ &\quad + F_{m+1}(F_{n-2}+F_{n-1}) + (F_{m-1}+F_m)F_n \\ &= 5(F_{m+1}F_n + F_m F_{n-1}). \end{aligned}$$

Substituting $F_{m+1}F_n + F_m F_{n-1}$ according to Proposition 3.15, we obtain the desired proposition. □

Proposition 3.27. $L_{m-n} = (-1)^{n-1}(F_m L_{n+1} - F_{m+1} L_n)$, $n, m \in \mathbb{Z}$.

Proof. By Proposition 3.7, we have:

$$\begin{aligned}\begin{bmatrix} L_{m-n+1} & L_{m-n} \\ L_{m-n} & L_{m-n-1} \end{bmatrix} &= Q_L \cdot Q_F^{m-n-1} \\ &= Q_L \cdot Q_F^{-n-1} \cdot Q_F^m \\ &= \begin{bmatrix} L_{-n+1} & L_{-n} \\ L_{-n} & L_{-n-1} \end{bmatrix} \cdot Q_F^m.\end{aligned}$$

And by Theorem 3.12, we have:

$$\begin{aligned}\begin{bmatrix} L_{m-n+1} & L_{m-n} \\ L_{m-n} & L_{m-n-1} \end{bmatrix} &= (-1)^n \begin{bmatrix} -L_{n-1} & L_n \\ L_n & -L_{n+1} \end{bmatrix} \cdot Q_F^m \\ &= (-1)^{n-1} \begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix} \cdot \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix}.\end{aligned}$$

This implies:

$$\begin{bmatrix} L_{m-n+1} & L_{m-n} \\ L_{m-n} & L_{m-n-1} \end{bmatrix} = (-1)^{n-1} \begin{bmatrix} L_{n-1}F_{m+1} - L_n F_m & L_{n-1}F_m - L_n F_{m-1} \\ L_{n+1}F_m - L_n F_{m+1} & L_{n+1}F_{m-1} - L_n F_m \end{bmatrix}. \tag{3.11}$$

Comparing the entries in the second row and first column in the equality (3.27), we obtain the desired identity. □

Proposition 3.28. $5F_{m-n} = (-1)^{n-1}(L_m L_{n+1} - L_{m+1} L_n)$, $n, m \in \mathbb{Z}$.

Proof. Equating the trace of the matrices in equation (3.11), we have:

$$\begin{aligned}L_{m-n+1} + L_{m-n-1} &= (-1)^{n-1}(L_{n-1}F_{m+1} - L_n F_m) + (-1)^{n-1}(L_{n+1}F_{m-1} - L_n F_m) \\ &= (-1)^{n-1}(L_{n-1}F_{m+1} + L_{n+1}F_{m-1} + L_n F_{m+1} \\ &\quad - L_n F_{m+1} - L_n F_m - L_n F_m) \\ &= (-1)^{n-1}((L_{n-1} + L_n)F_{m+1} + L_{n+1}F_{m-1} - L_n F_m - L_n(F_m + F_{m+1})) \\ &= (-1)^{n-1}(L_{n+1}(F_{m-1} + F_{m+1}) - L_n(F_m + F_{m+2})).\end{aligned}$$

Replacing $L_{m-n+1} + L_{m-n-1}$ according to Proposition 3.18 and $F_{m-1} + F_{m+1}$ and $F_m + F_{m+2}$ according to Proposition 3.17, we have the desired proposition. □

4. Final Remarks

In this paper, we were able to generalize the equality

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^n = \begin{cases} 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, & \text{if } n \text{ is even,} \\ 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}, & \text{if } n \text{ is odd,} \end{cases}$$

for all integers n.

We also introduce the identity

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = Q_L \cdot Q_F^{n-1}, \quad n \in \mathbb{Z}.$$

Additionally, we prove that, for natural n, $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$, using the matrices Q_F and Q_L.

To conclude, we provide proofs for various identities involving the Lucas and Fibonacci numbers while also refining the proofs given in [2] for Theorem 1.3 by consistently accounting for integer indices regardless of parity. Specifically, item (1) of Theorem 1.3 corresponds to our Proposition 3.26, item (2) to our Proposition 3.15, item (3) to our Proposition 3.23, item (4) to our Proposition 3.28, item (5) to our Proposition 3.16, and item (6) to our Proposition 3.27.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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