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Research Article

Existence and Uniqueness Results for Fuzzy Fractional Differential Equation Upon Chinchole-Bhadane Intervals Fractional Calculus Method

Aziz El Ghazouani*, M'hamed El Omari and Said Melliani

Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Melal, Morocco
***Corresponding author:** aziz.elghazouani@usms.ac.ma

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Abstract. In this study, we derive certain features of a novel fractional derivative of two variables presented lately by Chinchole and Bhadane using a Mittag-Leffler kernel. Initially, It is given how to compute the fuzzy set value function's fractional derivative of the SABC. Hence, it is utilized to show that a solution to a fuzzy fractional differential equation involving the SABC fractional derivative exists and is unique. An example is presented and solved for further illustration.

Keywords. Fractional derivative of Chinchole-Bhadane, Fuzzy differential equation, Hukuhara generalized differentiability, Interval method

Mathematics Subject Classification (2020). 34A07, 34K36, 35R11, 35R13

1. Introduction

The use of fuzzy nonlinear systems in mathematical analysis of physical and engineering subjects has risen in recent years. Many major contributions, such as articles and books, have been produced on connected themes. The earliest book on the fuzzy notion is *Fuzzy Set Theory – and Its Applications* written by Zimmermann [10], while Kilbas *et al*. [8] wrote another handbook entitled *Theory and Applications of Fractional Differential Equations*. Furthermore, several research papers [1,3,9] have really been released to investigate fuzzy fractional differential equation solutions. Many studies have increasingly investigated on the idea of Hukuhara difference, which was presented by Bede and Gal [6]. As a result, in this work, we shall expand the gH-derivative into the fractional situation in the succeeding non-linear FFDE:

$$\begin{cases} {}^{SABC}D^{\alpha,\beta}x(s) = f(s,x(s)), \quad 0<\alpha,\ \beta<1,\ s\in J=[0,T], \\ x(0)=x_0\in E^1, \end{cases} \tag{1.1}$$

such that $x(s)\in X=\mathscr{C}_F(J)\cap\mathscr{L}_F(J)$, the set of all fuzzy continuous functions and the space of all fuzzy Lebesgue integrable functions on J correspondingly.

For that purpose, we will now provide the SABC fractional derivative [7] based on the gH-difference and use it to demonstrate existence and uniqueness solution to the equation (1.1).

The hereunder is how this article is framed: In Section 2, we examine several important concepts and terminology such as Generalized differentiability, Hausdorff distances, and fuzzy numbers. Section 3 explains the SABC fractional derivative for a fuzzy value function along with a few of its characteristics. As a consequence, the existence and uniqueness solution of FFDEs involving the SABC derivatives are shown, and many kinds of solutions are offered. Section 4 handles two basics examples to exemplify the approach. Section 5 concludes with a conclusion.

2. Fuzzy Fractional Arithmetic

We will discuss a number of essential ideas in this part that will be helpful for the length of this study.

Definition 1. A fuzzy number is a fuzzy set $x:\mathbb{R}\to[0,1]$ that satisfies the next conditions:

(1) x is normal, which means, there is a $t_0\in\mathbb{R}$ so that $x(t_0)=1$;

(2) x is a fuzzy convex set;

(3) x is an upper semi-continuous set;

(4) the closures of $\{s\in\mathbb{R}, x(s)>0\}$ is compact.

The E^1 represents the set of all fuzzy numbers on $\mathbb{R}$.

$$E^1=\{x:\mathbb{R}\to[0,1], x \text{ satisfies } (1)\text{-}(4)\}.$$

The ρ-cut of an element of E^1 is defined by

$$x^\rho=\{s\in\mathbb{R}, x(s)\ge\rho\}, \quad \rho\in(0,1].$$

A fuzzy number x's parameterized interval shape is represented as

$$x^\rho=[x_l(\rho),x_u(\rho)].$$

The distance (Hausdorff distance) between two element of E^1 is given by (see [2])

$$d(x,y)=\sup_{\rho\in(0,1]}\max\{|x_u(\rho)-y_u(\rho)|,|x_l(\rho)-y_l(\rho)|\}$$

and the following properties are valid:

(a) $d(\mu+\tau,\gamma+\tau)=d(\mu,\gamma)$,

(b) $d(\kappa\mu,\kappa\gamma)=|\kappa|d(\mu,\gamma)$,

(c) $d(\mu+\gamma,\rho+\varrho) \le d(\mu,\rho)+d(\gamma,\varrho)$.

Addition and scalar combination operations of fuzzy numbers on $\mathbb{R}_{\mathscr{F}}$ have the form

$$[\Phi \oplus \Psi]^\rho = [\Phi]^\rho + [\Psi]^\rho \text{ and } [\lambda \odot \Phi]^\rho = \lambda[\Phi]^\rho, \quad \lambda \in \mathbb{R},$$

where

$$[\Phi]^\rho + [\Psi]^\rho = \{a+b : a \in [\Phi]^\rho, b \in [\Psi]^\rho\}$$

is the Minkowski sum of $[\Phi]^\rho$ and $[\Psi]^\rho$ and

$$\lambda[\Phi]^\rho = \{\lambda a : a \in [\Phi]^\rho\}.$$

For $\Phi, \Psi \in \mathbb{R}_{\mathscr{F}}$, the gH difference [6] of Φ and Ψ, denoted by $\Phi \ominus_{gH} \Psi$, is defined as the element $\Upsilon \in \mathbb{R}_{\mathscr{F}}$ such that

$$\Phi \ominus_{gH} \Psi = \Upsilon \iff \begin{cases} \text{(i) } \Phi = \Psi + \Upsilon, \text{ or} \\ \text{(ii) } \Psi = \Phi + (-1)\Upsilon. \end{cases} \tag{2.1}$$

In terms of r-levels we have

$$(\Phi \ominus_{gH} \Psi)^\rho = [\min\{\Phi_l(\rho)-\Psi_l(\rho), \Phi_u(\rho)-\Psi_u(\rho)\}, \max\{\Phi_l(\rho)-\Psi_l(\rho), \Phi_u(\rho)-\Psi_u(\rho)\}]$$

and the prerequisites for the existence of $\Upsilon = \Phi \ominus_g \Psi \in E^1$ are

$$\textit{Case (i)}: \begin{cases} \Upsilon_l(\rho) = \Phi_l(\rho) - \Psi_l(\rho) \text{ and } \Upsilon_u(\rho) = \Phi_u(\rho) - \Psi_u(\rho) \\ \text{with } \Upsilon_l(\rho) \text{ increasing, } \Upsilon_u(\rho) \text{ decreasing, } \Upsilon_l(\rho) \le \Upsilon_u(\rho) \end{cases}$$

$$\textit{Case (ii)}: \begin{cases} \Upsilon_l(\rho) = \Phi_u(\rho) - \Psi_u(\rho) \text{ and } \Upsilon_u(\rho) = \Phi_l(\rho) - \Psi_l(\rho) \\ \text{with } \Upsilon_l(\rho) \text{ increasing, } \Upsilon_u(\rho) \text{ decreasing, } \Upsilon_l(\rho) \le \Upsilon_u(\rho) \end{cases}$$

for all $\rho \in (0,1)$.

Definition 2 (Generalized derivative)**.** The gH-derivative of function x may defined in the following expression,

$$g_H x'(s) = \lim_{h\to 0^+} \frac{x(s+h) \ominus_{gH} x(s)}{h} = \lim_{\epsilon\to 0^+} \frac{x(s) \ominus_{gH} x(s+h)}{h}, \tag{2.2}$$

where

$$\left.\frac{dx(s)}{ds}\right|_{gH} \in X.$$

Taking a ρ-cut $(0 \le \rho \le 1)$ on both sides of the aforementioned derivative (2.2), and taking into account the concept of gH-difference (2.1), the derivatives in the interval shapes shown below

Case 1 (i-Diff) $_{i-gH}[x']_\rho = [x'_l(s;\rho), x'_u(s;\rho)]$,

Case 2 (ii-Diff) $_{ii-gH}[x']_\rho = [x'_u(s;\rho), x'_l(s;\rho)]$.

To define the *SABC* derivative in interval shape form, first define the Lebesgue integral of $x'(s)$ in interval shape as follows:

$$\left[\int_0^s x'(s)ds\right]_\rho = \int_0^s [x'(s)]_\rho ds$$
$$= \begin{cases} \left[\int_0^s x'_l(s;\rho)ds, \int_0^s x'_u(s;\rho)ds\right], & \text{in } \textit{Case } 1, \\ \left[\int_0^s x'_u(s;\rho)ds, \int_0^s x'_l(s;\rho)ds\right], & \text{in } \textit{Case } 2. \end{cases}$$

Definition 3. The *SABC* fractional derivative in the meaning of Caputo is described in two circumstances, as follows:

$$\left[{}_0^{SABC}D^{i,\alpha,\beta}x(s)\right]_\rho = \left[{}_0^{SABC}D^{i,\alpha,\beta}x_l(s;\rho), {}_0^{SABC}D^{i,\alpha,\beta}x_u(s;\rho)\right], \quad \textit{Case } 1,$$
$$\left[{}_0^{SABC}D^{ii,\alpha,\beta}x(s)\right]_\rho = \left[{}_0^{SABC}D^{ii,\alpha,\beta}x_u(s;\rho), {}_0^{SABC}D^{ii,\alpha,\beta}x_l(s;\rho)\right], \quad \textit{Case } 2,$$

where

$$\begin{aligned}
{}^{SABC}{}_0D^{i,\alpha,\beta}x(s) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s {}_{i-gH}x'(\tau)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)d\tau \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s {}_{i-gH}x'(\tau)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0} \\
&\quad + \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s {}_{ii-gH}x'(\tau)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}
\end{aligned}$$

and

$$\begin{aligned}
{}^{SABC}{}_0D^{ii,\alpha,\beta}x(s) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^t {}_{ii-gH}x'(\tau)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right]d\tau \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s {}_{ii-gH}x'(\tau)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0} \\
&\quad + \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s {}_{i-gH}x'(\tau)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}.
\end{aligned}$$

Since the sign of the extended Mittag-Leffler function $E_{\alpha,\beta}(s)$ is determined by α, β, s or $B(\alpha,\beta)>0$, $2-\alpha-\beta>0$ so,

$$\begin{aligned}
\left[{}^{SABC}{}_0D^{i,\alpha,\beta}x(s)\right]_\rho &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s {}_{i-gH}x'(\tau)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)d\tau \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s \left[{}_{i-gH}x'(\tau)\right]_\rho(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(t-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0} \\
&\quad + \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s \left[{}_{ii-gH}x'(\tau)\right]_\rho(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}, \\
\left[{}^{SABC}{}_0D_t^{ii,\alpha,\beta}x(s)\right]_\rho &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s \left[{}_{ii-gH}x'(\tau)\right]_\rho(s-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right]d\tau \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s \left[{}_{ii-gH}x'(\tau)\right]_\rho(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0} \\
&\quad + \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s \left[{}_{i-gH}x'(\tau)\right]_\rho(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}.
\end{aligned}$$

Then, we conclude that $\forall\ * \in \{i, ii\}$, the SABC fractional derivative are

$$\begin{aligned}
{}^{SABC}{}_0D^{*,\alpha,\beta}x_l(s;\rho) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_l'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0} \\
&\quad + \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_u'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}, \\
{}^{SABC}{}_0D^{*,\alpha,\beta}x_u(s;\rho) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_u'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^\alpha}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0}
\end{aligned}$$

$$+\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_l'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}.$$

Definition 4 ([7]). The Chinchole Bhadane fractional integral in two parameters $0<\alpha,\ \beta<1$ is defined as

$$^{SAB}{}_0I^{\alpha,\beta}x(s)=\begin{cases}\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^s x(\tau)(s-\tau)^{-\beta}d\tau+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^s x(\tau)(s-\tau)^{\alpha-\beta}d\tau, & \beta\neq 1,\\ \frac{1-\alpha}{N(\alpha)}x(s)+\frac{\alpha}{N(\alpha)\Gamma(\alpha)}\int_0^s x(\tau)(s-\tau)^{\alpha-1}d\tau, & \beta=1.\end{cases}\tag{2.3}$$

Lemma 1. *Let x be a fuzzy value function defined on X, the next expression is true,*

$$^{SAB}_{0}I^{\alpha,\beta}\left(^{SABC}_{0}D^{\alpha,\beta}x(s)\right)=\left(x(s)\ominus_{gH}x(0)\right)_{E_\alpha\geq 0}\oplus\left(x(s)\ominus_{gH}x(0)\right)_{E_\alpha<0}.\tag{2.4}$$

Proof. Using the ρ-cuts $(0<\rho<1)$, we get

$$\begin{aligned}&\left[^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{\alpha,\beta}x(s)\right)\right]_\rho\\
&={}^{SAB}_{0}I^{\alpha,\beta}\left(\left[^{SAB}_{0}D^{\alpha,\beta}x(s)\right]_\rho\right)\\
&=\begin{cases}^{SAB}_{0}I^{\alpha,\beta}\left([^{SAB}_{0}D^{i,\alpha,\beta}x_l(s;\rho),{}^{SAB}_{0}D^{i,\alpha,\beta}x_u(s;\rho)]\right), & \text{in } \mathit{Case}\ 1,\\ ^{SAB}_{0}I^{\alpha,\beta}\left([^{SAB}_{0}D^{ii,\alpha,\beta}x_u(s;\rho),{}^{SAB}_{0}D^{ii,\alpha,\beta}x_l(s;\rho)]\right), & \text{in } \mathit{Case}\ 2,\end{cases}\\
&=\begin{cases}\left[^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{i,\alpha,\beta}x_l(s;\rho)\right),{}^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{i,\alpha,\beta}x_u(s;r)\right)\right], & \text{in } \mathit{Case}\ 1,\\ \left[^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{ii,\alpha,\beta}x_u(s;\rho)\right),{}^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{ii,\alpha,\beta}x_l(s;\rho)\right)\right], & \text{in } \mathit{Case}\ 2,\end{cases}\end{aligned}$$

where

$$\begin{aligned}&^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{*,\alpha,\beta}x_l(s;\rho)\right)\\
&={}^{SAB}_{0}I^{\alpha,\beta}\left(\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^t x_l'(\tau;\rho)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0}\right.\\
&\quad\left.+\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_u'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}\right)\\
&={}^{SAB}_{0}I^{\alpha,\beta}\left(\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_l'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0}\right)\\
&\quad+{}^{SAB}_{0}I^{\alpha,\beta}\left(\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_u'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}\right)\\
&={}^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{*,\alpha,\beta}x_l(s;\rho)\right)|_{E_{\alpha,\beta}\geq 0}+{}^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{*,\alpha,\beta}x_u(s;\rho)\right)|_{E_{\alpha,\beta}<0}\\
&=\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_{\alpha,\beta}\geq 0}+\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_{\alpha,\beta}<0}\end{aligned}$$

and

$$\begin{aligned}&^{SAB}_{0}I^{\alpha,\beta}\left(^{SAB}_{0}D^{*,\alpha,\beta}x_u(s;r)\right)\\
&={}^{SAB}_{0}I^{\alpha,\beta}\left(\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_u'(\tau;r)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0}\right.\\
&\quad\left.+\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_l'(\tau;r)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}\right)\end{aligned}$$

$$= {}_0^{SAB}I^{\alpha,\beta}\left(\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_u'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}\geq 0}\right)$$
$$+ {}_0^{SAB}I^{\alpha,\beta}\left(\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^s x_l'(\tau;\rho)(s-\tau)^{\beta-1}E_{\alpha,\beta}\left(-\frac{(\alpha+\beta-1)(s-\tau)^{\alpha}}{2-\alpha-\beta}\right)\Big|_{E_{\alpha,\beta}<0}\right)$$
$$= {}_0^{SAB}I^{\alpha,\beta}\left({}_0^{SAB}D^{*,\alpha,\beta}x_u(s;\rho)\right)|_{E_{\alpha,\beta}\geq 0} + {}_0^{SAB}I^{\alpha,\beta}\left({}_0^{SAB}D_t^{*,\alpha,\beta}x_l(s;\rho)\right)\Big|_{E_{\alpha,\beta}<0}$$
$$= \left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_{\alpha,\beta}\geq 0} + \left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_{\alpha,\beta}<0}$$

for $* \in \{i, ii\}$.

Thus,

$$\left[{}_0^{SAB}I^{\alpha,\beta}\left({}_0^{SAB}D^{\alpha,\beta}x(s)\right)\right]_\rho = \Big[\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_{\alpha,\beta}\geq 0} + \left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_{\alpha,\beta}<0},$$
$$\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_{\alpha,\beta}\geq 0} + \left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_{\alpha,\beta}<0}\Big].$$

Therefore, using the gH-difference (2.1),we obtain

$${}_0^{SAB}I^{\alpha,\beta}\left({}_0^{SABC}D^{\alpha,\beta}x(s)\right) = \left(x(s)\ominus_{gH}x(0)\right)_{E_{\alpha,\beta}\geq 0} \oplus \left(x(s)\ominus_{gH}x(0)\right)_{E_{\alpha,\beta}<0}.$$ □

3. Existence and Uniqueness Results using SABC Fractional Derivative

Theorem 1 (Existence)**.** *The solution to the equation* (1.1) *with the continuous fuzzified function* f *is as follows:*

$$\left(x(s)\Theta_{gH}x(0)\right)_{E_{\alpha,\beta}\geq 0} \oplus \left(x(s)\Theta_{gH}x(0)\right)_{E_{\alpha,\beta}<0} = \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x(\tau))\odot(s-\tau)^{-\beta}d\tau$$
$$\oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x(\tau))\odot(s-\tau)^{\alpha-\beta}d\tau. \tag{3.1}$$

Proof. The ρ-cuts of the equations (1.1) are as

$$\begin{cases} \left[{}^{SABC}D^{\alpha,\beta}x(s)\right]_\rho = [f(s,x(s))]_\rho, \\ [x(0)]_\rho = [x_0]_\rho. \end{cases}$$

So, in *Case* 1,

$${}_0^{SABC}D^{i,\alpha,\beta}x_l(s;\rho) = f_l(s,x(s);\rho) = f_1\left(s,x_l(s;\rho),x_u(s;\rho)\right),$$
$${}_0^{SABC}D^{i,\alpha,\beta}x_u(s;\rho) = f_u(s,y(s);\rho) = f_2\left(s,x_l(s;\rho),x_u(s;\rho)\right).$$

In *Case* 2,

$${}_0^{SABC}D^{ii,\alpha,\beta}x_l(s;\rho) = f_u(s,x(s);\rho) = f_2\left(s,x_l(s;\rho),x_u(s;\rho)\right),$$
$${}_0^{SABC}D^{i,\alpha,\beta}x_u(s;\rho) = f_l(s,x(s);\rho) = f_1\left(s,x_l(s;\rho),x_u(s;r)\right).$$

When we employ the *SAB* fractional integral of rank α,β to both halves of the above equations, we get:

In *Case* 1,

$$\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_{\alpha,\beta}\geq 0} + \left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_{\alpha,\beta}<0} = {}_0^{SAB}I^{\alpha,\beta}\left(f_l(s,x(s);\rho)\right),$$

$$\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_{\alpha,\beta}\geq 0}+\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_{\alpha,\beta}<0}={}_0^{SAB}I^{\alpha,\beta}\left(f_u(s,x(s);\rho)\right).$$

In *Case* 2,

$$\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_{\alpha,\beta}\geq 0}+\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_{\alpha,\beta}<0}={}_0^{SAB}I^{\alpha,\beta}\left(f_l(s,x(s);\rho)\right),$$

$$\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_{\alpha,\beta}\geq 0}+\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_{\alpha,\beta}<0}={}_0^{SAB}I^{\alpha,\beta}\left(f_u(s,x(s);\rho)\right)$$

such that

$$\begin{aligned}{}_0^{SAB}I^{\alpha,\beta}\left(f_l(s,x(s);\rho)\right)&=\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^s f_1(\tau,x_l(s),x_u(s);\rho)(s-\tau)^{-\beta}d\tau\\&\quad+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^s f_1(\tau,x_l(s),x_u(s);\rho)(\tau)(s-\tau)^{\alpha-\beta}d\tau,\\{}_0^{SAB}I^{\alpha,\beta}\left(f_u(s,x(s);\rho)\right)&=\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^s f_2(\tau,x_l(s),x_u(s);\rho)(s-\tau)^{-\beta}d\tau\\&\quad+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^s f_2(\tau,x_l(s),x_u(s);\rho)(\tau)(s-\tau)^{\alpha-\beta}d\tau.\end{aligned}$$

Thus, in *Case* 1

$$\begin{aligned}&\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_\alpha\geq 0}+\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_\alpha<0}\\&\quad=\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^s f_1(\tau,x_l(s),x_u(s);\rho)(s-\tau)^{-\beta}d\tau\\&\qquad+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^s f_1(\tau,x_l(s),x_u(s);\rho)(\tau)(s-\tau)^{\alpha-\beta}d\tau,\\&\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_\alpha\geq 0}+\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_\alpha<0}\\&\quad=\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^s f_2(\tau,x_l(s),x_u(s);\rho)(s-\tau)^{-\beta}d\tau\\&\qquad+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^s f_2(\tau,x_l(s),x_u(s);\rho)(s-\tau)^{\alpha-\beta}d\tau.\end{aligned}$$

In *Case* 2

$$\begin{aligned}&\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_\alpha\geq 0}+\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_\alpha<0}\\&\quad=\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^s f_1(\tau,x_l(s),x_u(s);\rho)(s-\tau)^{-\beta}d\tau\\&\qquad+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^s f_1(\tau,x_l(s),x_u(s);\rho)(\tau)(s-\tau)^{\alpha-\beta}d\tau,\\&\left(x_l(s;\rho)-x_l(0;\rho)\right)_{E_\alpha\geq 0}+\left(x_u(s;\rho)-x_u(0;\rho)\right)_{E_\alpha<0}\\&\quad=\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\int_0^s f_2(\tau,x_l(s),x_u(s);\rho)(s-\tau)^{-\beta}d\tau\\&\qquad+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^s f_2(\tau,x_l(s),x_u(s);\rho)(s-\tau)^{\alpha-\beta}d\tau.\end{aligned}$$

We obtain by employing fuzzy arithmetic:

In *Case* 1

$$(x(t)\ominus x(0))_{E_{\alpha,\beta}\geq 0}\ominus(-1)(x(s)\ominus x(0))_{E_{\alpha,\beta}<0}=\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\odot\int_0^s f(\tau,x(s))\odot(s-\tau)^{-\beta}d\tau$$

$$\oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{\alpha-\beta} d\tau.$$

In *Case* 2

$$\ominus(-1)(x(s)\ominus x(0))_{E_{\alpha,\beta}\geq 0} \oplus (x(s)\ominus x(0))_{E_{\alpha,\beta}<0} = \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{-\beta} d\tau$$
$$\oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{\alpha-\beta} d\tau.$$

Thus, in general, we have

$$\left(x(s)\Theta_{gH}x(0)\right)_{E_{\alpha,\beta}\geq 0} \oplus \left(x(s)\Theta_{gH}x(0)\right)_{E_{\alpha,\beta}<0} = \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{-\beta} d\tau$$
$$\oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{\alpha-\beta} d\tau. \quad \square$$

Corollary 1. (a) *If $E_{\alpha,\beta} > 0\ \forall\ t,\alpha,\beta$, well there's:*

In Case 1

$$x(s) = x(0) \oplus \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{-\beta} d\tau$$
$$\oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{\alpha-\beta} d\tau.$$

In Case 2

$$x(s) = x(0) \ominus (-1)\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{-\beta} d\tau$$
$$\ominus (-1)\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{\alpha-\beta} d\tau$$

(b) *If $E_{\alpha,\beta} < 0\ \forall\ s,\alpha,\beta$, thus, the above equation is also valid in the opposite way.*

Theorem 2 (Uniqueness)**.** *The function f is presumed to be a continuous fuzzy valued function defined on $\mathscr{C}(J\times X,X)$, as well as fulfilling the Lipschitz criteria in $\mathscr{C}(J\times X,X)$ with regards to x,*

$$d(f(s,x_1(s)), f(s,x_2(s))) \leq K d(x_1(s),x_2(s)), \quad \forall\ x_1,x_2 \in X.$$

Once the very next condition is true, the problem (1.1) *possesses a one-of-a-kind fuzzy number solution $x(s)$.*

$$\left(\frac{(2-\alpha-\beta)T^{-\beta+1}}{B(\alpha,\beta)\Gamma(-\beta+2)}K + \frac{(\alpha+\beta-1)T^{\alpha-\beta+1}}{B(\alpha,\beta)\Gamma(\alpha-\beta+2)}K\right) \leq 1.$$

Proof. We consider the operator $H: X \to X$ defined as follow:

In *Case* 1

$$H(x(s)) = x(0) \oplus \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{-\beta} d\tau$$
$$\oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{\alpha-\beta} d\tau.$$

In *Case* 2

$$H(x(s)) = x(0) \ominus (-1)\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{-\beta} d\tau \\ \ominus (-1)\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x(s)) \odot (s-\tau)^{\alpha-\beta} d\tau.$$

In *Case* 1

$$d(H(x_1(s)),H(x_2(s))) = d\Bigg(x_1(0) \oplus \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x_1(s)) \odot (s-\tau)^{-\beta} d\tau \\ \oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x_1(s)) \odot (s-\tau)^{\alpha-\beta} d\tau, x_2(0) \\ \oplus \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x_2(s)) \odot (s-\tau)^{-\beta} d\tau \\ \oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x_2(\tau)) \odot (s-\tau)^{\alpha-\beta} d\tau\Bigg).$$

Since $x_1(0) = x_2(0)$ and by using Hausdorff distance properties mentioned before, we obtain

$$d(H(x_1(s)),H(x_2(s))) \\ = d\Bigg(\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x_1(s)) \odot (s-\tau)^{-\beta} d\tau \\ \oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x_1(s)) \odot (s-\tau)^{\alpha-\beta} d\tau, \\ \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x_2(s)) \odot (s-\tau)^{-\beta} d\tau \\ \oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x_2(s)) \odot (s-\tau)^{\alpha-\beta} d\tau\Bigg) \\ \le d\Bigg(\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x_1(s)) \odot (s-\tau)^{-\beta} d\tau, \\ \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s f(\tau,x_2(s)) \odot (s-\tau)^{-\beta} d\tau\Bigg) \\ + d\Bigg(\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x_1(s)) \odot (s-\tau)^{\alpha-\beta} d\tau, \\ \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s f(\tau,x_2(s)) \odot (s-\tau)^{\alpha-\beta} d\tau\Bigg) \\ \le \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} d\left(\int_0^s f(\tau,x_1(s)) \odot (s-\tau)^{-\beta} d\tau, \int_0^s f(\tau,x_2(s)) \odot (s-\tau)^{-\beta} d\tau\right) \\ + \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} d\left(\int_0^s f(\tau,x_1(s)) \odot (s-\tau)^{\alpha-\beta} d\tau, \int_0^s f(\tau,x_2(s)) \odot (s-\tau)^{\alpha-\beta} d\tau\right) \\ \le \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \int_0^s |(s-\tau)^{-\beta}| \, d(f(\tau,x_1(s)), f(\tau,x_2(s))) d\tau \\ + \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \int_0^s |(s-\tau)^{\alpha-\beta}| \, d(f(\tau,x_1(s)), f(\tau,x_2(s))) d\tau \\ \le \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \int_0^s |(s-\tau)^{-\beta}| \, K d(x_1(s),x_2(s)) d\tau$$

$$
\begin{aligned}
&+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^s |(s-\tau)^{\alpha-\beta}|\,Kd(x_1(s),x_2(s))\,d\tau\\
&\le \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\frac{t^{-\beta+1}}{-\beta+1}Kd(x_1(s),x_2(s))+\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\frac{s^{\alpha-\beta+1}}{\alpha-\beta+1}Kd(x_1(s),x_2(s))\\
&\le \left(\frac{(2-\alpha-\beta)s^{-\beta+1}}{B(\alpha,\beta)\Gamma(-\beta+2)}K+\frac{(\alpha+\beta-1)s^{\alpha-\beta+1}}{B(\alpha,\beta)\Gamma(\alpha-\beta+2)}K\right)d(x_1(s),x_2(s))\\
&\le \left(\frac{(2-\alpha-\beta)T^{-\beta+1}}{B(\alpha,\beta)\Gamma(-\beta+2)}K+\frac{(\alpha+\beta-1)T^{\alpha-\beta+1}}{B(\alpha,\beta)\Gamma(\alpha-\beta+2)}K\right)d(x_1(s),x_2(s)).
\end{aligned}
$$

Thus

$$d(F(x_1(s)),F(x_2(s)))\le Md(x_1(s),x_2(t)),$$

where $M=\left(\frac{(2-\alpha-\beta)T^{-\beta+1}}{B(\alpha,\beta)\Gamma(-\beta+2)}K+\frac{(\alpha+\beta-1)T^{\alpha-\beta+1}}{B(\alpha,\beta)\Gamma(\alpha-\beta+2)}K\right)\le 1$.

As a result, F is a contraction operator on X. So, the equation (1.1)) has a unique solution premised by the Banach fixed point theorem. The same technique may be applied in *Case* 2. □

Lemma 2. *Assume* $0<\alpha,\ \beta<1$. x *is a solution of the following FFDE,*

$${}^{SABC}{}_0D^{\alpha,\beta}x(s)=0,\quad t\in J.$$

Then, x is a constant fuzzy function.

Proof. The interval parametric form of the above equation are

$$
\begin{aligned}
&{}^{SABC}D^{i,\alpha,\beta}x_l(s;\rho)=x_l(s;\rho)=x_l(0;\rho)=x_{0,l}(\rho),\qquad 0<s<1,\ 0<\alpha,\ \beta<1,\\
&{}^{SABC}D^{i,\alpha,\beta}x_u(s;\rho)=x_u(s;\rho)=x_u(0;\rho)=x_{0,u}(\rho),\quad 0<s<1,\ 0<\alpha,\ \beta<1,\quad 0\le\rho\le1.
\end{aligned}
$$

Thus, in general

$$x(s)=x_0.$$

This proves that x is a constant fuzzy function. □

4. Fundamentals Examples

Example 1. Say we have the next fractional differential problem with a fuzzy set as the start condition.

$$
\begin{cases}
{}^{SABC}D_s^{\alpha,\beta}x(s)=x(s), & 0<s<1,\ 0<\alpha,\ \beta<1,\\
x(0)=x_0
\end{cases}
$$

with

$$x_0[\rho]=[x_{0,l}(\rho),x_{0,u}(\rho)]=[\rho-1,2\rho],\quad 0\le\rho\le1.$$

The one-of-a-kind approach is as follows:

$$
\begin{aligned}
x(s)=x(0)\oplus\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)}\odot\int_0^s x(\tau)\odot(s-\tau)^{-\beta}d\tau\oplus\frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)}\\
\odot\int_0^s x(\tau)\odot(s-\tau)^{\alpha-\beta}d\tau.
\end{aligned}
$$

The interval parameterized shape is comprised by

$$\begin{cases} {}^{SABC}D_s^{i,\alpha,\beta} x_l(s;\rho) = x_l(s;\rho), & 0<s<1, \quad 0<\alpha,\ \beta<1, \\ x_l(0;\rho) = x_{0,l}(\rho) = \rho - 1, \end{cases}$$

$$\begin{cases} {}^{SABC}D_s^{i,\alpha,\beta} x_u(s;\rho) = x_u(s;\rho), & 0<s<1,\ 0<\alpha,\ \beta<1, \\ x_u(0;\rho) = x_{0,u}(\rho) = 2\rho. \end{cases}$$

Example 2.

$$\begin{cases} {}^{SABC}D_t^{\alpha,\beta} x(s) = s, & 0<s<1,\ 0<\alpha,\ \beta<1 \\ x(0) = x_0 \end{cases}$$

depending on initial condition

$$x_0[\rho] = [x_{0,l}(\rho), x_{0,u}(\rho)] = [\rho, 3\rho], \quad 0 \le \rho \le 1.$$

The one-of-a-kind solution is as follows:

$$x(s) = x(0) \oplus \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(-\beta+1)} \odot \int_0^s \tau \odot (s-\tau)^{-\beta} d\tau \oplus \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha-\beta+1)} \odot \int_0^s \tau \odot (s-\tau)^{\alpha-\beta} d\tau.$$

For $\alpha = \beta = \frac{1}{2}$, we get

$$x(s) = x(0) \oplus 1.470553 \odot s^{3/2}.$$

The interval parametric form are

$$x_l(s) = \rho - 1 + 1.470553 \times s^{3/2},$$

$$x_u(s) = 2\rho + 1.470553 \times s^{3/2},$$

for $\rho = \frac{1}{2}$, we get

$$x_l(s) = -\frac{1}{2} + 1.470553 \times s^{3/2},$$

$$x_u(t) = 1 + 1.470553 \times s^{3/2}.$$

5. Conclusion

Using a Mittag-Leffler kernel, we deduce several properties of a new fractional derivative of two variables recently published by Chinchole and Bhadane. Initially, we establish the SABC fractional derivative on fuzzy set value functions in parameterized interval representation. It is then utilized to show that a solution to a FFDE with SABC fractional derivative exists and also is unique. For additional clarification, two examples are shown and solved.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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