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η-Einstein (k,μ)-Space Forms

Research Article

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Abstract. In this paper we obtain the scalar curvatures of a (k,μ)-space form under h-projective, ϕ-projective semi symmetric and h-Weyl and ϕ-Weyl semisymmetry conditions.

Keywords. (k,μ)-space form; h-projective; ϕ-projective semi symmetric; h-Weyl; ϕ-Weyl semi symmetry

MSC. 53C40; 53C55; 53C25

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1. Introduction

A class of (k,μ) contact metric manifolds [3] is of interest as it contains both the classes of Sasakian and non-Sasakian cases. The contact metric manifolds for which the characteristic vector field ξ belongs to (k,μ)-nullity distribution for some real numbers k and μ are called (k,μ) contact metric manifolds. A full classification of (k,μ)-contact metric manifolds was given by Boeckx [4] and many authors [2] studied (k,μ)-contact metric manifolds. T. Koufogiorgos proved in [1] that if a (k,μ)-space M has constant ϕ-sectional curvature c and dimension greater than 3, the curvature tensor of this (k,μ)-space form is given by

$$\begin{aligned}4R(X,Y)Z = {}&[(c+3)\{g(Y,Z)X-g(X,Z)Y\}+(c+3-4k)\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\&+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}+(c-1)\{2g(X,\phi Y)\phi Z+g(X,\phi Z)\phi Y\\&-g(Y,\phi Z)\phi X\}-2\{g(hX,Z)hY-g(hY,Z)hX+2g(X,Z)hY-2g(Y,Z)hX\\&-2\eta(X)\eta(Z)hY+2\eta(Y)\eta(Z)hX+2g(hX,Z)Y-2g(hY,Z)X\\&+2g(hY,Z)\eta(X)\xi-2g(hX,Z)\eta(Y)\xi-g(\phi hX,Z)\phi hY+g(\phi hY,Z)\phi hX\}\\&+4\mu\{\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY+g(hY,Z)\eta(X)\xi-g(hX,Z)\eta(Y)\xi\}], \qquad (1.1)\end{aligned}$$

for any vector fields X, Y, Z, where $2h=L_\xi\phi$ and L is the usual Lie derivative.

The projective curvature tensor is an important tensor from the differential geometric point of view. An $(2n+1)$-dimensional Riemannian manifold M is locally projectively flat if their exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in Euclidean space. It is well-known that for $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor P vanishes. Here P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.2}$$

where S is the Ricci tensor of M.

In an $(2n+1)$-dimensional Riemannian manifold, the conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.3}$$

where r is a scalar curvature and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

The paper is organised as follows. In Section 2 we give some preliminary results of (k,μ)-space forms. Section 3 deals with h-projective and ϕ-projective semi-symmetric non-Sasakian (k,μ)-space forms. Section 4 is devoted to the study of h-Weyl and ϕ-Weyl semi-symmetric non-Sasakian (k,μ)-space forms. In all the cases the manifold becomes an η-Einstein manifold and we obtain scalar curvatures of (k,μ)-space forms.

2. Preliminaries

A $(2n+1)$-dimensional differential manifold M is said to admit an almost contact metric structure (ϕ,ξ,η,g) if it satisfies the following relations

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0, \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.5}$$

$$g(X, \phi Y) = d\eta(X,Y), \tag{2.6}$$

for all vector fields X, Y on M. In a contact metric manifold the $(1,1)$ tensor field h defined by $h = \frac{1}{2}L_\xi \phi$, where L denotes the Lie differentiation, is a symmetric operator anti-commutative with ϕ and satisfies $h\xi = 0$, $h\phi = -\phi h$, $Tr\,h = Tr\,\phi h = 0$. Moreover in any contact metric manifold, we have $\nabla_X \xi = -\phi X - \phi h X$. In [3] Blair et al. introduced a class of contact metric manifold M which satisfy

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{2.7}$$

where k and μ are real constants. This class of contact metric manifolds are called (k,μ) manifolds.

Also in a (k,μ)-contact metric manifold, the following relations hold ([3], [4]):

$$h^2 = (k-1)\phi^2, k \le 1, \tag{2.8}$$

$$S(X,Y) = \frac{1}{4}\Big[(c(2n+1)+6n+4k-5)g(X,Y) - (c(2n+1)+6n+4k-5-8nk)\eta(X)\eta(Y) + (8-8n+4\mu)g(Y,hX)\Big], \tag{2.9}$$

$$r = \frac{n}{2}[c(2n+1)+6n+4k-5]+2nk, \tag{2.10}$$

$$S(X,\xi) = 8nk\eta(X),\ S(\xi,\xi) = 8nk,$$

where S is the Ricci tensor of the type $(0,2)$ and r is the scalar curvature of the manifold.

If $\mu = 0$, the (k,μ)-nullity distribution $N(k,\mu)$ is reduced to the k-nullity distribution [5], where k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by

$$N(k): p \to N_p(k) = \{W \in T_p(M)/R(X,Y)W = k(g(Y,W)X - g(X,W)Y)\}.$$

If $\xi \in N(k)$, then we call M a $N(k)$-contact metric manifold.

The class of (k,μ)-contact metric manifolds contain both the class of Sasakian ($k = 1$ and $h = 0$) and non-Sasakian ($k \ne 1$ and $h \ne 0$) manifolds. Throughout the paper we denote by M^{2n+1}, a $(2n+1)$-dimensional non-Sasakian (k,μ)-space form. A contact metric manifold is said to be η-Einstein if $Q = aId + b\eta\otimes\xi$, where a, b are smooth functions on M^{2n+1}.

3. h-Projectively and ϕ-Projectively Semi Symmetric Non-Sasakian (k,μ)-Space Form

Definition 3.1. *A (k,μ)-space form M is said to be h-projectively semi-symmetric if $P(X,Y)\cdot h = 0$ holds in M.*

We now prove the following theorem.

Theorem 3.1. *Let M be a non-Sasakian (k,μ)-space form. If M is h-projectively semi-symmetric, then M is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional h-projectively semi symmetric non-Sasakian (k,μ)-space form. The condition $P(X,Y)\cdot h = 0$ turns into

$$(P(X,Y)\cdot h)Z = P(X,Y)hZ - hP(X,Y)Z = 0. \tag{3.1}$$

From (1.1), we have

$$\begin{aligned}
&R(X,Y)hZ - hR(X,Y)Z \\
&= \frac{1}{4}[(c+3)\{g(Y,hZ)X - g(X,hZ)Y - g(Y,Z)hX + g(X,Z)hY\} \\
&\quad + (c+3-4k)\{g(X,hZ)\eta(Y)\xi - g(Y,hZ)\eta(X)\xi - \eta(X)\eta(Z)hY \\
&\quad + \eta(Y)\eta(Z)hX\} + (c-1)\{g(X,\phi hZ)\phi Y - g(Y,\phi hZ)\phi X \\
&\quad - g(X,\phi Z)h\phi Y + g(Y,\phi Z)h\phi X\} - 2\{g(hX,hZ)hY - g(hY,hZ)hX
\end{aligned}$$

$$
\begin{aligned}
&+2g(X,hZ)hY-2g(Y,hZ)hX+2g(hX,hZ)Y-2g(hY,hZ)X\\
&+2g(hY,hZ)\eta(X)\xi-2g(hX,hZ)\eta(Y)\xi-g(\phi hX,hZ)\phi hY\\
&+g(\phi hY,hZ)\phi hX+g(hX,Z)h^2Y-g(hY,Z)h^2X+2g(X,Z)h^2Y\\
&-2g(Y,Z)h^2X-2\eta(X)\eta(Z)h^2Y+2\eta(Y)\eta(Z)h^2X-g(\phi hX,Z)h\phi hY\\
&+g(\phi hY,Z)h\phi hX+2g(hX,Z)hY-2g(hY,Z)hX\}+4\mu\{g(hY,hZ)\eta(X)\xi\\
&-g(hX,hZ)\eta(Y)\xi+\eta(X)\eta(Z)h^2Y-\eta(Y)\eta(Z)h^2X\}].
\end{aligned}
\tag{3.2}
$$

for any vector fields X, Y, Z. Using (1.3), (3.1) and (3.2), we have

$$
\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,hZ)X-g(X,hZ)Y-g(Y,Z)hX+g(X,Z)hY\}\\
&+(c+3-4k)\{g(X,hZ)\eta(Y)\xi-g(Y,hZ)\eta(X)\xi-\eta(X)\eta(Z)hY\\
&+\eta(Y)\eta(Z)hX\}+(c-1)\{g(X,\phi hZ)\phi Y-g(Y,\phi hZ)\phi X-g(X,\phi Z)h\phi Y\\
&+g(Y,\phi Z)h\phi X\}-2\{g(hX,hZ)hY-g(hY,hZ)hX+2g(X,hZ)hY\\
&-2g(Y,hZ)hX+2g(hX,hZ)Y-2g(hY,hZ)X+2g(hY,hZ)\eta(X)\xi\\
&-2g(hX,hZ)\eta(Y)\xi g(\phi hX,hZ)\phi hY+g(\phi hY,hZ)\phi hX+g(hX,Z)h^2Y\\
&-g(hY,Z)h^2X+2g(X,Z)h^2Y-2g(Y,Z)h^2X-2\eta(X)\eta(Z)h^2Y+2\eta(Y)\eta(Z)h^2X\\
&-g(\phi hX,Z)h\phi hY+g(\phi hY,Z)h\phi hX+2g(hX,Z)hY-2g(hY,Z)hX\}\\
&+4\mu\{g(hY,hZ)\eta(X)\xi-g(hX,hZ)\eta(Y)\xi+\eta(X)\eta(Z)h^2Y-\eta(Y)\eta(Z)h^2X\}]\\
&-\frac{1}{2n}[S(Y,hZ)X-S(X,hZ)Y-S(Y,Z)hX+S(X,Z)hY]=0.
\end{aligned}
\tag{3.3}
$$

Replacing X by hX and contracting with W, by using (2.8) and symmetry property of h, we get

$$
\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,hZ)g(hX,W)-g(hX,hZ)g(Y,W)+(k-1)g(Y,Z)g(X,W)\\
&-(k-1)g(Y,Z)\eta(X)\eta(W)+g(hX,Z)g(hY,W)\}+(c+3-4k)\{g(hX,hZ)\eta(Y)\eta(W)\\
&-(k-1)\eta(Y)\eta(Z)g(X,W)+(k-1)\eta(X)\eta(Z)\eta(Y)\eta(W)\}+(c-1)\{g(hX,\phi hZ)g(\phi Y,W)\\
&-g(Y,\phi hZ)g(\phi hX,W)-g(hX,\phi Z)g(h\phi Y,W)+(k-1)g(Y,\phi Z)g(\phi X,W)\}\\
&-2\{-(k-1)g(X,hZ)g(hY,W)+(k-1)[g(hY,hZ)g(X,W)-g(hY,hZ)\eta(X)\eta(W)\\
&+2g(hX,hZ)g(hY,W)+2(k-1)[g(Y,hZ)g(X,W)-g(Y,hZ)\eta(X)\eta(W)]\\
&-2(k-1)g(X,hZ)g(Y,W)-2g(hY,hZ)g(hX,W)+2(k-1)g(X,hZ)\eta(Y)\eta(W)\\
&+(k-1)g(\phi X,hZ)g(\phi hY,W)-(k-1)g(\phi hY,hZ)g(\phi X,W)+(k-1)^2[g(X,Z)g(Y,W)\\
&+2(k-1)[-g(hX,Z)g(Y,W)+g(hX,Z)\eta(Y)\eta(W)]+2(k-1)g(Y,Z)g(hX,W)\\
&-2(k-1)\eta(Y)\eta(Z)g(hX,W)+(k-1)^2g(\phi X,Z)g(\phi Y,W)+(k-1)g(\phi hY,Z)g(\phi hX,W)\\
&+2(k-1)[-g(X,Z)g(hY,W)+\eta(X)\eta(Z)g(hY,W)]+2(k-1)[g(hY,Z)g(X,W)\\
&-g(hY,Z)\eta(X)\eta(W)]\}+4\mu\{(k-1)g(X,hZ)\eta(Y)\eta(W)+(k-1)\eta(Y)\eta(Z)g(hX,W)\}]\\
&-\frac{1}{2n}[S(Y,Z)g(hX,W)-S(hX,hZ)g(Y,W)+(k-1)[S(Y,Z)g(X,W)\\
&-S(Y,Z)\eta(X)\eta(W)]+S(hX,Z)g(hY,W)]=0.
\end{aligned}
\tag{3.4}
$$

Let e_i, $i = 1,2,3,\dots,2n+1$ be an orthonormal basis of vector fields in M. If we put $X = W = e_i$ in (3.4) and summing over i, then using (2.9), we obtain

$$\begin{aligned}&\frac{1}{4}\big[[(c+3)(k-1)2n+2(c-1)(k-1)-2(k-1)^2]g(Y,Z)\\&\quad+[-(c+3-4k)2n(k-1)-2(c-1)(k-1)+2(k-1)^2]\eta(Y)\eta(Z)\\&\quad+[(k-1)(-8+10n)]g(Y,hZ)\big]-(k-1)S(Y,Z)=0.\end{aligned} \tag{3.5}$$

Again using (2.9) in (3.5), we obtain

$$g(Y,hZ)=\left[\frac{6k-c-5}{18n-4\mu-16}\right]g(Y,Z)+\left[\frac{c-6k+5}{18n-4\mu-16}\right]\eta(Y)\eta(Z). \tag{3.6}$$

In view of (3.6), (3.5) takes the form

$$S(Y,Z)=A_1g(Y,Z)+B_1\eta(Y)\eta(Z), \tag{3.7}$$

where

$$\begin{aligned}A_1&=2n(c+3)+2(c-1)-2(k-1)+\frac{(8-10n)(c-6k+5)}{-16+18n-4\mu},\\B_1&=-2n(c+3-4k)-2(c-1)+2(k-1)+\frac{(-8+10n)(c-6k+5)}{-16+18n-4\mu}.\end{aligned}$$

Thus M is an η-Einstein manifold.

Taking $Y = Z = e_i$ in (3.7), we obtain

$$r=n\{(n+1)c+3n+k\}. \tag{3.8}$$

A h-projectively semi-symmetric non-Sasakian (k,μ)-space form is an η-Einstein manifold and the scalar curvature in this case is $r = n\{(n+1)c+3n+k\}$.

Comparing r of (3.8) with (2.10), we get $c = 6k-5$. □

Definition 3.2. *A (k,μ)-space form M is said to be ϕ-projectively semi-symmetric if $P(X,Y)\cdot\phi = 0$ holds in M.*

Theorem 3.2. *Let M be a non-Sasakian (k,μ)-space form-space form. If M is ϕ-projectively semi symmetric, then M is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional ϕ-projectively semi-symmetric non-Sasakian (k,μ)-space form-space form. The condition $P(X,Y)\cdot\phi = 0$ turns into

$$(P(X,Y)\cdot\phi)Z=P(X,Y)\phi Z-\phi P(X,Y)Z=0. \tag{3.9}$$

From (1.1), we have

$$\begin{aligned}&R(X,Y)\phi Z-\phi R(X,Y)Z\\&\quad=\frac{1}{4}[(c+3)\{g(Y,\phi Z)X-g(X,\phi Z)Y-g(Y,Z)\phi X+g(X,Z)\phi Y\}\\&\qquad+(c+3-4k)\{g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)\eta(X)\xi-\eta(X)\eta(Z)\phi Y\\&\qquad+\eta(Y)\eta(Z)\phi X\}+(c-1)\{g(X,\phi^2Z)\phi Y-g(Y,\phi^2Z)\phi X-g(X,\phi Z)\phi^2Y\end{aligned}$$

$$+g(Y,\phi Z)\phi^2X\}-2\{g(hX,\phi Z)hY-g(hY,\phi Z)hX+2g(X,\phi Z)hY$$
$$-2g(Y,\phi Z)hX+2g(hX,\phi Z)Y-2g(hY,\phi Z)X+2g(hY,\phi Z)\eta(X)\xi$$
$$-2g(hX,\phi Z)\eta(Y)\xi-g(\phi hX,\phi Z)\phi hY+g(\phi hY,\phi Z)\phi hX$$
$$+g(hX,Z)\phi hY-g(hY,Z)\phi hX+2g(X,Z)\phi hY-2g(Y,Z)\phi hX$$
$$-2\eta(X)\eta(Z)\phi hY+2\eta(Y)\eta(Z)\phi hX-g(\phi hX,Z)\phi^2hY g(\phi hY,Z)\phi^2hX$$
$$+2g(hX,Z)\phi Y-2g(hY,Z)\phi X\}+4\mu\{g(hY,\phi Z)\eta(X)\xi$$
$$-g(hX,\phi Z)\eta(Y)\xi+\eta(X)\eta(Z)\phi hY-\eta(Y)\eta(Z)\phi hX\}]. \tag{3.10}$$

for any vector fields X, Y, Z. Using (1.3) and (3.9) in (3.10), we have

$$\frac{1}{4}[(c+3)\{g(Y,\phi Z)X-g(X,\phi Z)Y-g(Y,Z)\phi X+g(X,Z)\phi Y\}$$
$$+(c+3-4k)\{g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)\eta(X)\xi-\eta(X)\eta(Z)\phi Y+\eta(Y)\eta(Z)\phi X\}$$
$$+(c-1)\{-g(X,Z)\phi Y+\eta(Z)\eta(X)\phi Y+g(Y,Z)\phi X-\eta(Z)\eta(Y)\phi X+g(X,\phi Z)\phi Y$$
$$-g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)X+g(Y,\phi Z)\eta(X)\xi\}-2\{g(hX,\phi Z)hY-g(hY,\phi Z)hX$$
$$+2g(X,\phi Z)hY-2g(Y,\phi Z)hX+2g(hX,\phi Z)Y-2g(hY,\phi Z)X+2g(hY,\phi Z)\eta(X)\xi$$
$$-2g(hX,\phi Z)\eta(Y)\xi-g(\phi hX,\phi Z)\phi hY+g(\phi hY,\phi Z)\phi hX+g(hX,Z)\phi hY-g(hY,Z)\phi hX$$
$$+2g(X,Z)\phi hY-2g(Y,Z)\phi hX-2\eta(X)\eta(Z)\phi hY+2\eta(Y)\eta(Z)\phi hX$$
$$-g(\phi hX,Z)\phi^2hY+g(\phi hY,Z)\phi^2hX+2g(hX,Z)\phi Y-2g(hY,Z)\phi X\}$$
$$+4\mu\{g(hY,\phi Z)\eta(X)\xi-g(hX,\phi Z)\eta(Y)\xi+\eta(X)\eta(Z)\phi hY-\eta(Y)\eta(Z)\phi hX\}]$$
$$-\frac{1}{2n}[S(Y,\phi Z)X-S(X,\phi Z)Y-S(Y,Z)\phi X+S(X,Z)\phi Y]=0. \tag{3.11}$$

Replacing X by ϕX and contracting with W in (3.11) from (2.4), we obtain

$$\frac{1}{4}[(c+3)\{g(Y,\phi Z)g(\phi X,W)-g(\phi X,\phi Z)g(Y,W)+g(Y,Z)g(X,W)-g(Y,Z)\eta(X)\eta(W)$$
$$+g(\phi X,Z)g(\phi Y,W)\}+(c+3-4k)\{g(\phi X,\phi Z)\eta(Y)\eta(W)-\eta(Y)\eta(Z)g(X,W)$$
$$+\eta(Y)\eta(Z)\eta(X)\eta(W)\}+(c-1)\{-g(\phi X,Z)g(\phi Y,W)-g(Y,Z)g(X,W)+g(Y,Z)\eta(X)\eta(W)$$
$$+\eta(Z)\eta(Y)g(X,W)+g(X,Z)g(Y,W)-\eta(Z)\eta(X)g(Y,W)-g(X,Z)\eta(Y)\eta(W)$$
$$-g(Y,\phi Z)g(\phi X,W)\}-2\{-g(X,Z)g(hY,W)-g(hY,\phi Z)g(h\phi X,W)+2[-g(X,Z)g(hY,W)$$
$$+\eta(Z)\eta(X)g(hY,W)]-2g(Y,\phi Z)g(h\phi X,W)-2g(hX,Z)g(Y,W)-2g(hY,\phi Z)g(\phi X,W)$$
$$+2g(hX,Z)\eta(Y)\eta(W)-g(h\phi X,Z)g(h\phi Y,W)+g(hY,Z)g(hX,W)+g(h\phi X,Z)g(\phi hY,W)$$
$$-g(hY,Z)g(hX,W)+2g(\phi X,Z)g(\phi hY,W)-2g(Y,Z)g(hX,W)+2\eta(Y)\eta(Z)g(hX,W)$$
$$-g(\phi hY,Z)g(h\phi X,W)\}+4\mu\{g(hX,Z)\eta(Y)\eta(W)-\eta(Y)\eta(Z)g(hX,W)\}]$$
$$-\frac{1}{2n}[S(Y,\phi Z)g(\phi X,W)-S(\phi X,\phi Z)g(Y,W)+S(Y,Z)g(X,W)-S(Y,Z)\eta(X)\eta(W)$$
$$-\frac{1}{2n}[S(Y,\phi Z)g(\phi X,W)-S(\phi X,\phi Z)g(Y,W)+S(Y,Z)g(X,W)-S(Y,Z)\eta(X)\eta(W)$$
$$+S(\phi X,Z)g(\phi Y,W)]=0. \tag{3.12}$$

Let $\{e_i\}$, $i = 1,2,\ldots,2n+1$ be an orthonormal basis of vector fields in M. If we put $X = W = e_i$ in (3.11) and summing over i, then using (2.2) and (2.8), we obtain

$$\frac{1}{4}\Big[\Big[8n+6k-14+\frac{c(2n+1)+6n+4k-5}{n}\Big]g(Y,Z) + \Big[c+8nk-2n-2k-6-\frac{c(2n+1)+6n+4k-5}{n}\Big]\eta(Y)\eta(Z) + 18g(Z,hY)\Big] - S(Y,Z) = 0. \tag{3.13}$$

Using (2.9) in (3.13), we obtain

$$S(Y,Z) = A_2 g(Y,Z) + B_2\eta(Y)\eta(Z), \tag{3.14}$$

where

$$A_2 = \Big[\Big(8n+6k-14+\frac{c(2n+1)+6n+4k-5}{n}\Big)(c-17)\Big] - (c(2n+1)+6n+4k-5),$$

$$B_2 = \Big[\Big(c+8nk-2n-2k-6-\frac{c(2n+1)+6n+4k-5}{n}\Big)(c-17)\Big] - (c(2n+1)+6n+4k-5).$$

Thus M is an η-Einstein manifold.

Taking $Y = Z = e_i$ in (3.13), we obtain

$$r = \frac{1}{4}[16n^2+20nk-16n+12k-14+4nc+2c]. \tag{3.15}$$

A ϕ-projectively semi symmetric non-Sasakian (k,μ)-space form is an η-Einstein manifold and the scalar curvature in this case is

$$r = \frac{1}{4}[16n^2+20nk-16n+12k-14+4nc+2c], \tag{3.16}$$

Comparing r of ϕ-projectively semi-symmetric non-Sasakian (k,μ)-contact metric space form with (2.10), we get

$$c = \frac{2n^2+2nk+3n+6k-7}{2n^2-n-1}. \qquad \square$$

4. h-Weyl and ϕ-Weyl Semi-symmetric Non-Sasakian (k,μ)-Space Form

Definition 4.1. *A (k,μ)-space form M is said to be h-Weyl semi-symmetric if $C(X,Y)\cdot h = 0$ holds on M.*

Theorem 4.1. *Let M be a non-Sasakian (k,μ)-space form-space form. If M is h-Weyl semi symmetric, then M is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional h-Weyl semi symmetric non-Sasakian (k,μ)-space form-space form. The condition $C(X,Y)\cdot h = 0$ turns into

$$(C(X,Y)\cdot h)Z = C(X,Y)hZ - hC(X,Y)Z = 0, \tag{4.1}$$

for any vector fields X, Y, Z. Using (1.3)and (3.1) in (4.1), we have

$$\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,hZ)X - g(X,hZ)Y - g(Y,Z)hX + g(X,Z)hY\}\\
&+(c+3-4k)\{g(X,hZ)\eta(Y)\xi - g(Y,hZ)\eta(X)\xi - \eta(X)\eta(Z)hY\\
&+\eta(Y)\eta(Z)hX\} + (c-1)\{g(X,\phi hZ)\phi Y - g(Y,\phi hZ)\phi X - g(X,\phi Z)h\phi Y\\
&+g(Y,\phi Z)h\phi X\} - 2\{g(hX,hZ)hY - g(hY,hZ)hX + 2g(X,hZ)hY\\
&-2g(Y,hZ)hX + 2g(hX,hZ)Y - 2g(hY,hZ)X + 2g(hY,hZ)\eta(X)\xi\\
&-2g(hX,hZ)\eta(Y)\xi - g(\phi hX,hZ)\phi hY + g(\phi hY,hZ)\phi hX + g(hX,Z)h^2Y\\
&-g(hY,Z)h^2X + 2g(X,Z)h^2Y - 2g(Y,Z)h^2X - 2\eta(X)\eta(Z)h^2Y\\
&+2\eta(Y)\eta(Z)h^2X - g(\phi hX,Z)h\phi hY + g(\phi hY,Z)h\phi hX + 2g(hX,Z)hY\\
&-2g(hY,Z)hX\} + 4\mu\{g(hY,hZ)\eta(X)\xi - g(hX,hZ)\eta(Y)\xi + \eta(X)\eta(Z)h^2Y\\
&+\frac{r}{2n(2n-1)}\{g(Y,hZ)X - g(X,hZ)Y - g(Y,Z)hX + g(X,Z)hY\} = 0. \qquad (4.2)
\end{aligned}$$

Replacing X by hX, contracting with W and using (4.2) and symmetry property of h, we obtain,

$$\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,hZ)g(hX,W) - g(hX,hZ)g(Y,W) + (k-1)g(Y,Z)g(X,W)\\
&-(k-1)g(Y,Z)\eta(X)\eta(W) + g(hX,Z)g(hY,W)\} + (c+3-4k)\{g(hX,hZ)\eta(Y)\eta(W)\\
&-(k-1)\eta(Y)\eta(Z)g(X,W) + (k-1)\eta(X)\eta(Z)\eta(Y)\eta(W)\} + (c-1)\{g(hX,\phi hZ)g(\phi Y,W)\\
&-g(Y,\phi hZ)g(\phi hX,W) - g(hX,\phi Z)g(h\phi Y,W) + (k-1)g(Y,\phi Z)g(\phi X,W)\}\\
&-2\{-(k-1)g(X,hZ)g(hY,W) + (k-1)[g(hY,hZ)g(X,W) - g(hY,hZ)\eta(X)\eta(W)]\\
&+2g(hX,hZ)g(hY,W) + 2(k-1)[g(Y,hZ)g(X,W) - g(Y,hZ)\eta(X)\eta(W)]\\
&-2(k-1)g(X,hZ)g(Y,W) - 2g(hY,hZ)g(hX,W) + 2(k-1)g(X,hZ)\eta(Y)\eta(W)\\
&+(k-1)g(\phi X,hZ)g(\phi hY,W) - (k-1)g(\phi hY,hZ)g(\phi X,W) + (k-1)^2[g(X,Z)g(Y,W)\\
&-g(X,Z)\eta(Y)\eta(W) - \eta(X)\eta(Z)g(Y,W) + \eta(X)\eta(Y)\eta(Z)\eta(W)] + (k-1)g(hY,Z)g(hX,W)\\
&+2(k-1)[-g(hX,Z)g(Y,W) + g(hX,Z)\eta(Y)\eta(W)] + 2(k-1)g(Y,Z)g(hX,W)\\
&-g(hY,Z)\eta(X)\eta(W)]\} + 4\mu\{(k-1)g(X,hZ)\eta(Y)\eta(W) + (k-1)\eta(Y)\eta(Z)g(hX,W)\}]\\
&-\frac{1}{2n-1}[S(Y,hZ)g(hX,W) + (k-1)[S(X,Z)g(Y,W) - 2nk\eta(Z)\eta(X)g(Y,W)]\\
&+g(Y,hZ)g(QhX,W) + g(X,Z)g(QY,W) - \eta(Z)\eta(X)g(QY,W) + (k-1)[S(Y,Z)g(X,W)\\
&-S(Y,Z)\eta(X)\eta(W)] + S(hX,Z)g(hY,W) - g(Y,Z)g(hQhX,W) + g(Y,hZ)g(QhX,W)\\
&+g(X,Z)g(QY,W) - \eta(Z)\eta(X)g(QY,W) + (k-1)[S(Y,Z)g(X,W) - S(Y,Z)\eta(X)\eta(W)]\\
&+S(hX,Z)g(hY,W) - g(Y,Z)g(hQhX,W) + g(hX,Z)g(hQY,W)\}\\
&+\frac{r}{2n(2n-1)}\{g(Y,hZ)g(hX,W) + g(X,Z)g(Y,W) - \eta(X)\eta(Z)g(Y,W)\\
&+(k-1)[g(Y,Z)g(X,W) - g(Y,Z)\eta(X)\eta(W)] + g(hX,Z)g(hY,W)\} = 0. \qquad (4.3)
\end{aligned}$$

Taking $Y = W = \xi$ in (4.3), we obtain

$$S(X,Z) = \left[k(2n-1) - \frac{2nk}{k-1} + \frac{r}{2n(k-1)}\right] g(X,Z) + \left[-k(2n-1) + \frac{2nk^2}{k-1} - \frac{r}{2n(k-1)}\right]\eta(X)\eta(Z) + 4\mu(2n-1)g(hX,Z) \tag{4.4}$$

Now using (2.9) in (4.4), we get

$$g(X,hZ) = \frac{1}{2-2n-8n\mu+5\mu}\left[\frac{8nk(2n-1)(k-1)-4n^2k+r}{2n(k-1)} - \frac{c(2n+1)+6n+4k-5}{4}\right] g(X,Z) + \left[\frac{-8nk(2n-1)(k-1)+4n^2k^2-r}{2n(k-1)} + \frac{c(2n+1)+6n+4k-5-8nk}{4}\right]\eta(X)\eta(Z) \tag{4.5}$$

Using (4.5) in (4.4), we obtain

$$S(X,Z) = A_1' g(X,Z) + B_1' \eta(X)\eta(Z), \tag{4.6}$$

where

$$A_1' = \frac{4\mu(2n-1)}{2-2n-8n\mu+5\mu}\left[\frac{8nk(2n-1)(k-1)-4n^2k+r}{2n(k-1)} - \frac{c(2n+1)+6n+4k-5}{4}\right] + \frac{2nk(2n-1)(k-1)+ +4n^2k^2-r}{2n(k-1)},$$

$$B_1' = \frac{4\mu(2n-1)}{2-2n-8n\mu+5\mu}\left[\frac{-8nk(2n-1)(k-1)+4n^2k^2-r}{2n(k-1)} + \frac{c(2n+1)+6n+4k-5-8nk}{4}\right] + \frac{-2nk(2n-1)(k-1)+ +4n^2k^2-r}{2n(k-1)}.$$

Thus M is an η-Einstein manifold.

Taking $X = Z = e_i$ in (4.4), we obtain

$$r = 4n^2. \tag{4.7}$$

An h-Weyl semi-symmetric non-Sasakian (k,μ)-space form is an η-Einstein manifold and the scalar curvature in this case is $4n^2$.

Comparing r of (4.7) with (2.10), we obtain

$$c = \frac{5+2n-8k}{2n+1}.$$

□

Definition 4.2. *A (k,μ)-space form M is said to be ϕ-Weyl semi-symmetric if $C(X,Y)\cdot\phi = 0$ holds on M.*

Theorem 4.2. *Let M be a non-Sasakian (k,μ)-space form. If M is ϕ-Weyl semi-symmetric, then M is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional ϕ-Weyl semi-symmetric non-Sasakian (k,μ)-space form. The condition $C(X,Y)\cdot\phi=0$ turns into

$$(C(X,Y)\cdot\phi)Z = C(X,Y)\phi Z - \phi C(X,Y)Z = 0, \tag{4.8}$$

for any vector fields X, Y, Z. Using (1.3)and (3.8) in (4.7), we have

$$\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y\} \\
&\quad + (c+3-4k)\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X\} \\
&\quad + (c-1)\{g(X,\phi^2 Z)\phi Y - g(Y,\phi^2 Z)\phi X - g(X,\phi Z)\phi^2 Y + g(y,\phi Z)\phi^2 X\} - 2\{g(hX,\phi Z)hY \\
&\quad - g(hY,\phi Z)hX + 2g(X,\phi Z)hY - 2g(Y,\phi Z)hX + 2g(hX,\phi Z)Y - 2g(hY,\phi Z)X \\
&\quad + 2g(hY,\phi Z)\eta(X)\xi - 2g(hX,\phi Z)\eta(Y)\xi - g(\phi hX,\phi Z)\phi hY + g(\phi hY,\phi Z)\phi hX + g(hX,Z)\phi hY \\
&\quad - g(hY,Z)\phi hX + 2g(X,Z)\phi hY - 2g(Y,Z)\phi hX - 2\eta(X)\eta(Z)\phi hY + 2\eta(Y)\eta(Z)\phi hX \\
&\quad - g(\phi hX,Z)\phi^2 hY + g(\phi hY,Z)\phi^2 hX + 2g(hX,Z)\phi Y - 2g(hY,Z)\phi X\} + 4\mu\{g(hY,\phi Z)\eta(X)\xi \\
&\quad - g(hX,\phi Z)\eta(Y)\xi + \eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX\}] - \frac{1}{2n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y \\
&\quad + g(Y,\phi Z)QX - g(X,\phi Z)QY - S(Y,Z)\phi X + S(X,Z)\phi Y - g(Y,Z)\phi QX + g(X,Z)\phi QY\} \\
&\quad + \frac{r}{2n(2n-1)}[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] = 0.
\end{aligned} \tag{4.9}$$

Replacing X by ϕX, contracting with W and using (4.9) and symmetry property of h, we obtain,

$$\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,\phi Z)g(\phi X,W) - g(\phi X,\phi Z)g(Y,W) + g(Y,Z)g(X,W) - g(Y,Z)\eta(X)\eta(W) \\
&\quad + g(\phi X,Z)g(\phi Y,W)\} + (c+3-4k)\{g(\phi X,\phi Z)\eta(Y)\eta(W) - \eta(Y)\eta(Z)g(X,W) \\
&\quad + \eta(X)\eta(Z)\eta(Y)\eta(W)\} + (c-1)\{-g(\phi X,Z)g(\phi Y,W) - g(Y,Z)g(X,W) + g(Y,Z)\eta(X)\eta(W) \\
&\quad + \eta(Y)\eta(Z)g(X,W) + g(X,Z)g(Y,W) - \eta(X)\eta(Z)g(Y,W) - g(X,Z)\eta(Y)\eta(W) \\
&\quad - g(Y,\phi Z)g(\phi X,W)\} - 2\{-g(X,Z)g(hY,W) - g(hy,\phi Z)g(h\phi X,W) + 2[-g(X,Z)g(hY,W) \\
&\quad + \eta(X)\eta(Z)g(hY,W)] - 2g(Y,\phi Z)g(h\phi X,W) - 2g(hX,Z)g(Y,W) - 2g(hY,\phi Z)g(\phi X,W) \\
&\quad + 2g(hX,Z)\eta(Y)\eta(W) - g(h\phi X,Z)g(h\phi Y,W) + g(hY,Z)g(hX,W) + g(h\phi X,Z)g(h\phi Y,W) \\
&\quad - g(hY,Z)g(hX,W) + 2g(h\phi X,Z)g(\phi hY,W) - 2g(Y,Z)g(hX,W) + 2\eta(Y)\eta(Z)g(hX,W) \\
&\quad + 2g(h\phi X,Z)g(\phi Y,W) + 2[g(hY,Z)g(X,W) - g(hY,Z)\eta(X)\eta(W)] + g(hX,Z)g(hY,W) \\
&\quad - g(\phi hY,Z)g(hX,W)\} + 4\mu\{g(hX,Z)\eta(Y)\eta(W) - \eta(Y)\eta(Z)g(hX,W)\}] \\
&\quad - \frac{1}{2n-1}[S(Y,\phi Z)g(\phi X,W) - S(\phi X,\phi Z)g(Y,W) + g(Y,\phi Z)S(\phi X,W) - g(\phi X,\phi Z)S(Y,W) \\
&\quad - S(Y,Z)g(\phi^2 X,W) + S(\phi X,Z)g(\phi Y,W) - g(Y,Z)g(\phi Q\phi X,W) + g(\phi X,Z)g(\phi QY,W)\} \\
&\quad + \frac{r}{2n(2n-1)}\{g(Y,\phi Z)g(\phi X,W) - g(\phi X,\phi Z)g(Y,W) - g(Y,Z)g(\phi^2 X,W) \\
&\quad + g(\phi X,Z)g(\phi Y,W)\} = 0.
\end{aligned} \tag{4.10}$$

Taking $Y = W = \xi$ in (4.10), we obtain

$$\left[\frac{16nk+2n^2c+nc+6n^2-5n-2r}{4n(2n-1)}\right]g(X,Z) + \left[\frac{-16nk-2n^2c-nc-6n^2+5n+2r}{4n(2n-1)}\right]\eta(X)\eta(Z)+\frac{2\mu n-8n+3\mu+8}{2n-1}g(hX,Z)=0. \quad (4.11)$$

Using (2.9) in (4.11), we get

$$S(X,Z)=A'_2 g(X,Z)+B'_2\eta(X)\eta(Z), \quad (4.12)$$

where

$$A'_2=\frac{-(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n-2r)}{16n(2\mu n-8n+3\mu+8)}+\frac{c(2n+1)+6n+4k-5}{4},$$

$$B'_2=\frac{(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n-2r)}{16n(2\mu n-8n+3\mu+8)}-\frac{c(2n+1)+6n+4k-5}{4}.$$

Thus M is an η-Einstein manifold.

Taking $X = Z = e_i$ in (4.12), we obtain

$$r=\frac{(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n)}{48n-16\mu n-22\mu-48}.$$

A ϕ-Weyl non-Sasakian (k,μ)-space form is an η-Einstein manifold and the scalar curvature in this case is given by

$$r=\frac{(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n)}{48n-16\mu n-22\mu-48}. \quad (4.13)$$

Comparing r of (4.13) with (2.10), we obtain

$$c=\frac{-12\mu n^2+24n^2-5\mu n-64n-16k\mu n-80nk-26\mu k-16k+15\mu+40}{4\mu n^2-16n^2+7\mu n+4n+3\mu+8}. \qquad \square$$

Table 1. Scalar curvature of a non-Sasakian (k,μ)-space form M

S. No.	Curvature tensor	Condition	Scalar curvature
1	Projective curvature tensor	$P(X,Y)\cdot h=0$	$r=n\{(n+1)c+3n+k\}$
2	Projective curvature tensor	$P(X,Y)\cdot\phi=0$	$r=\frac{1}{4}[16n(n-1)+4k(5n+3)]+\frac{1}{4}[2c(2n+1)-14]$
3	Conformal curvature tensor	$C(X,Y)\cdot h=0$	$r=4n^2$
4	Conformal curvature tensor	$C(X,Y)\cdot\phi=0$	$r=\frac{(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n)}{48n-16\mu n-22\mu-48}$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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