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On L^1-approximation of Trigonometric Series

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Abstract. In the paper [3] we defined three new classes of sequences motivated by the Logarithm Rest Bounded Variation Sequences defined by S.P. Zhou [4]. By means of these classes we extended Zhou's theorems pertaining to L^1-convergence of sine series. Very recently R.J. Le and S.P. Zhou [1] proved L^1-approximation theorems. Now we generalize their theorems to our wider classes.

1. **Introduction**

In a recent paper S.P. Zhou [4] defined the notion of Logarithm Rest Bounded Variation Sequences (LRBVS_N) which plays central role in his paper. He established, among others, necessary and sufficient condition for L^1-convergence of the series

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{1.1}$$

assuming that $\boldsymbol{a} := \{a_n\} \in \text{LRBVS}_\text{N}$, but without the prior condition that the sum function of (1.1) is integrable.

The notions and notations to be used in this paper are collected in Section 2.

Next, in a paper to be appearing in *Acta Math. Hungar.*, R.J. Le and S.P. Zhou [1] proved some theorems studying the order of approximation by the partial sums of series (1.1) also maintaining that $\boldsymbol{a} \in \text{LRBVS}_\text{N}$.

As one of the referees of the paper [1], we analized why the logarithm sequences play the crucial role in L^1-convergence of sine series. After collecting the cardinal properties of the sequence $\{\log n\}$, we could show that if a sequence has three essential properties of the sequence $\{\log n\}$, then all of the relevant results of Zhou hold for this sequence, too.

These sequences have been called Log-Type Sequences, in symbol LTS. By means of LTS two further classes of sequences have been defined, the Log-Type

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Rest Bounded Sequences (LTRBVS) and the $\boldsymbol{\gamma}$ Log-Type Rest Bounded Sequences ($\boldsymbol{\gamma}$LTRBVS), which satisfy the following embedding relations:

(1.2) $\quad \text{LRBVS}_\text{N} \subset \text{LTRBVS} \subset \boldsymbol{\gamma}\text{LTRBVS}.$

The embedding relations (1.2) have offered to extend Zhou's theorems. In [3] we established four theorems being analogies of Zhou's theorems.

The aim of the present paper is similar to that of [3], to extend the theorems of Le and Zhou from the class LRBVS_N to the classes LTRBVS or $\boldsymbol{\gamma}$LTRBVS.

2. Notions and Notations

Let $L_{2\pi}$ be the space of all real or complex integrable functions $f(x)$ of period 2π endowed with norm

$$\|f\| := \int_{-\pi}^{\pi} |f(x)| dx.$$

For those x where the trigonometric series converges, write

(2.1) $$f(x) := \sum_{n=1}^{\infty} a_n \sin nx,$$

(2.2) $$g(x) := \sum_{n=1}^{\infty} a_n \cos nx,$$

and

(2.3) $$h(x) := \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

As usual, let $s_n(f,x)$ and $s_n(g,x)$ be n-th partial sums of (2.1) and (2.2), respectively, furthermore denote

(2.4) $$s_n(h,x) := \sum_{k=-n}^{n} c_k e^{ikx}.$$

Next we recall some definitions of generalization of decreasing monotonicity related to our topic.

A sequence $\boldsymbol{a} := \{a_n\}$ of positive numbers will be called *Almost Monotone Sequence*, briefly $\boldsymbol{a} \in$ AMS, if $a_n \leqq K(\boldsymbol{a}) a_m$ for all $n \geqq m$, where $K(\boldsymbol{a})$ is a positive constant.

Let $\boldsymbol{\gamma} := \{\gamma_n\}$ be a given positive sequence. A null-sequence $\boldsymbol{a} := \{a_n\}$ ($a_n \to 0$) of real or complex numbers satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta a_n| \leqq K(\boldsymbol{a}) \gamma_m \quad (\Delta a_n := a_n - a_{n+1}),\ m = 1, 2, \ldots$$

is said to be a *sequence of* $\boldsymbol{\gamma}$ *rest bounded variation*, in symbol, $\boldsymbol{a} \in \boldsymbol{\gamma}$RBVS.

If $\gamma_n \equiv |a_n|$, then $\boldsymbol{\gamma}$RBVS reduces to RBVS, that is, to a *rest bounded variation sequence*.

We emphasize that if $\boldsymbol{a} \in \boldsymbol{\gamma}$RBVS it may have infinitely many zeros and negative terms, but this is not the case if $\boldsymbol{a} \in$ RBVS, see e.g. [2].

A real or complex bounded sequence $\boldsymbol{c} := \{c_n\}$ is named *Logarithm Rest Bounded Variation Sequence*, $\boldsymbol{c} \in \mathrm{LRBVS_N}$, if N is a positive integer and the sequence $\{c_n \log^{-N} n\}$ belongs to $\boldsymbol{\gamma}$RBVS, where $\gamma_n := |c_n| \log^{-N} n$, see e.g. [1].

We shall also use the notation $L \ll R$ at inequalities if there exists a positive constant K such that $L \leqq KR$ holds, not necessarily the same at each occurance.

A positive nondecreasing sequence $\boldsymbol{\alpha} := \{\alpha_n\}$ will be called *Log-Type Sequence*, briefly LTS, if it satisfies the conditions:

(2.5) $\quad \alpha_n \to \infty,$

(2.6) $\quad \alpha_{n^2} \ll \alpha_n,$

and

(2.7) $\quad |\Delta \alpha_n| \ll \dfrac{\alpha_n}{n \log n}.$

By means of Log-Type Sequence we defined the following two classes of sequences, in [3] only for positive $\{a_n\}$.

Let $\boldsymbol{\gamma} := \{\gamma_n\}$ be a given positive sequence. If $\boldsymbol{\alpha} := \{\alpha_n\} \in$ LTS and $\left\{\frac{a_n}{\alpha_n}\right\} \in \boldsymbol{\gamma}$RBVS, then the sequence $\boldsymbol{a} := \{a_n\}$ will be called $\boldsymbol{\gamma}$ *Log-Type Rest Bounded Variation Sequence*, in symbol, $\boldsymbol{a} \in \boldsymbol{\gamma}$LTRBVS.

If $\gamma_n = \frac{|a_n|}{\alpha_n}$, then the sequence $\boldsymbol{a}$ will be said simply *Log-Type Rest Bounded Variation Sequence*, and denote by LTRBVS.

In other words, $\boldsymbol{a} \in$LTRBVS, if $\boldsymbol{\alpha} \in$LTS and $\left\{\frac{a_n}{\alpha_n}\right\} \in$RBVS.

3. Theorems

First we recall the main results of Le and Zhou [1], utilizing the notations of (2.i), $i = 1, 2, 3$.

Theorem A. *Let a nonnegative sequence* $\{a_n\} \in \mathrm{LRBVS_N}$, $\{\psi_n\}$ *a decreasing sequence tending to zero with*

(3.1) $\quad \psi_n \ll \psi_{2n}.$

Then

(3.2) $\quad \|f - s_n(f)\| \ll \psi_n$

if and only if

(3.3) $\quad a_n \log n \ll \psi_n \quad \textit{and} \quad \displaystyle\sum_{k=n}^{\infty} \frac{a_k}{k} \ll \psi_n.$

Theorem B. *Let* $\{c_n\} \in \mathrm{LRBVS_N}$ *and* $\{\psi_n\}$ *a decreasing null-sequence. If*

(3.4) $\quad |c_n| \log n \ll \psi_n \quad \textit{and} \quad \displaystyle\sum_{k=n}^{\infty} \frac{|c_k|}{k} \ll \psi_n$

and one of the following conditions

$$(3.5)\qquad \sum_{k=n+1}^{\infty} |\Delta c_k - \Delta c_{-k}| \log k \ll \psi_n$$

or

$$(3.6)\qquad \sum_{k=n+1}^{\infty} |\Delta c_k + \Delta c_{-k}| \log k \ll \psi_n$$

is satisfied, then

$$(3.7)\qquad \|h - s_n(h)\| \ll \psi_n$$

holds.

Corollary. *If a nonnegative sequence* $\{a_n\} \in \mathrm{LRBVS_N}$, *and* $\{\psi_n\}$ *is a decreasing null-sequence, then* (3.3) *implies that*

$$(3.8)\qquad \|f - s_n(f)\| + \|g - s_n(g)\| \ll \psi_n$$

holds.

As a sample result proved in [3] and being used in the proof of our first theorem reads as follows.

Theorem C. *Let* $\boldsymbol{a} \in \mathrm{LTRBVS}$, *then the assertions*

$$(3.9)\qquad \lim_{n\to\infty} \|f - s_n(f)\| = 0$$

and

$$(3.10)\qquad \sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$$

are equivalent.

We remark that if $\alpha_n = (\log n)^N$ then Theorem C includes Theorem 2 of [4].

We intend to prove the following theorems:

Theorem 1. *Let a nonnegative sequence* $\boldsymbol{a} \in \mathrm{LTRBVS}$ *and* $\{\psi_n\}$ *be a decreasing null-sequence with* (3.1). *Then the assertions* (3.2) *and* (3.3) *are equivalent.*

It is plain that if $\alpha_n = (\log n)^N$, then Theorem 1 reduces to Theorem A.

The implication (3.3)⇒(3.2) has a further generalization.

Theorem 2. *Let* $\boldsymbol{\gamma} := \{\gamma_n\} \in \mathrm{AMS}$ *and a nonnegative sequence* $\boldsymbol{a} \in \boldsymbol{\gamma}\mathrm{LTRBVS}$, *furthermore* $\{\psi_n\}$ *be a decreasing null-sequence. If*

$$(3.11)\qquad \alpha_n \gamma_n \log n \ll \psi_n \quad \textit{and} \quad \sum_{k=n}^{\infty} \frac{\alpha_k \gamma_k}{k} \ll \psi_n,$$

then (3.2) *holds.*

Theorem 3. *Both Theorem* B *and Corollary can be improved such that the condition* $\{c_n\}(\{a_n\}) \in \text{LRBVS}_{\text{N}}$ *is replaced by the assumption* $\{c_n\}(\{a_n\}) \in \boldsymbol{\gamma}\text{LTRBVS}$*, where* $\gamma_n := \frac{|c_n|}{\alpha_n}\left(\frac{a_n}{\alpha_n}\right)$*, respectively.*

4. **Proofs of the Theorems**

***Proof of Theorem* 1.** Principally our proof follows the proof of Theorem A. First we prove the *sufficiency* of the assumptions of (3.3). By Theorem C, condition (3.10) implies that $\|f - s_n(f)\|$ tends to zero, consequently we only have to verify that (3.2) also holds.

By Abel's transformation

$$
\begin{aligned}
(4.1) \qquad f(x) - s_n(f,x) &= \sum_{k=n+1}^{\infty} a_k \sin kx \\
&= \sum_{k=n+1}^{\infty} \frac{a_k}{\alpha_k} \alpha_k \sin kx \\
&= -\frac{a_{n+1}}{\alpha_{n+1}} \sum_{k=1}^{n} \alpha_k \sin kx + \sum_{k=n+1}^{\infty} \Delta \frac{a_k}{\alpha_k} \sum_{\nu=1}^{k} \alpha_\nu \sin \nu x \\
&=: I_1(x) + I_2(x).
\end{aligned}
$$

Since

$$
\sum_{k=1}^{n} \alpha_k \sin kx = \sum_{k=1}^{n-1} \Delta\alpha_k \sum_{\nu=1}^{k} \sin \nu x + \alpha_n \sum_{k=1}^{n} \sin kx,
$$

thus

$$
\begin{aligned}
\int_0^\pi \left|\sum_{k=1}^{n} \alpha_k \sin kx\right| dx &\ll \sum_{k=1}^{n-1} |\Delta\alpha_k| \int_0^\pi \left|\sum_{\nu=1}^{k} \sin \nu x\right| dx \\
&\quad + \alpha_n \int_0^\pi \left|\sum_{k=1}^{n} \sin kx\right| dx \\
&\ll \left(\sum_{k=1}^{n-1} |\Delta\alpha_k| \log k + \alpha_n \log n\right) \\
&\ll \alpha_n \log n.
\end{aligned}
$$

Hence

$$
(4.2) \qquad I_1 := \int_0^\pi |I_1(x)| dx \ll \frac{a_{n+1}}{\alpha_{n+1}} \alpha_n \log n \ll a_{n+1} \log n
$$

and

$$I_2 := \int_0^\pi |I_2(x)|dx$$

$$\ll \sum_{k=n+1}^{\infty} \left|\Delta \frac{a_k}{\alpha_k}\right| \int_0^\pi \left|\sum_{\nu=1}^{k} \alpha_\nu \sin \nu x\right| dx$$

$$\ll \sum_{k=n+1}^{\infty} \left|\Delta \frac{a_k}{\alpha_k}\right| \alpha_k \log k.$$

Denote

$$R_n := \sum_{k=n}^{\infty} \left|\Delta \frac{a_k}{\alpha_k}\right|, \quad n \geqq 1.$$

Then

$$I_2 \ll \sum_{k=n+1}^{\infty} (R_k - R_{k+1})\alpha_k \log k \tag{4.3}$$

$$\ll \sum_{k=n+1}^{\infty} R_{k+1}(\alpha_{k+1} \log(k+1) - \alpha_k \log k) - R_{n+1}\alpha_{n+1} \log(n+1).$$

Next, using the conditions $\{\frac{a_n}{\alpha_n}\} \in$ RBVS and (2.7), we get

$$I_2 \ll \sum_{k=n+1}^{\infty} \frac{a_{k+1}}{\alpha_{k+1}} \left(|\Delta \alpha_k| \log(k+1) + \frac{\alpha_\nu}{\nu} \right) + a_{n+1} \log(n+1) \tag{4.4}$$

$$\ll \sum_{k=n+1}^{\infty} \frac{a_{k+1}}{k} + a_{n+1} \log(n+1).$$

Collecting the estimations (4.1)–(4.4), and using the assumptions (3.3), the implication (3.3)⇒(3.2) is proved.

In order to prove the *necessity* of (3.3) we define the following function:

$$\phi_n(x) := \sum_{k=1}^{n} \left(\frac{\sin(n+k)x}{k} - \frac{\sin(n-k)x}{k} \right) = 2\cos nx \sum_{k=1}^{n} \frac{\sin kx}{k},$$

and utilize the well-known inequality

$$\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \ll 1.$$

Since, by (3.2), we have that

$$\sum_{k=1}^{n} \frac{a_{n+k}}{k} = \left| \int_0^{2\pi} (f(x) - s_n(f,x))\phi_n(x)dx \right| \ll \|f - s_n(f)\| \ll \psi_n. \tag{4.5}$$

Furthermore, by $\{\frac{a_n}{\alpha_n}\} \in$ RBVS, for all $1 \leqq k \leqq n$

$$\frac{a_{2n+1}}{\alpha_{2n+1}} \ll \frac{a_{2n}}{\alpha_{2n}} \ll \frac{a_{n+k}}{\alpha_{n+k}}, \tag{4.6}$$

and, by (2.5) and (2.6), $\alpha_{2n+1} \ll \alpha_{n+k}$, thus (4.6) implies that

$$(4.7) \qquad a_{2n+1} \ll a_{2n} \ll a_{n+k}, \quad 1 \leqq k \leqq n.$$

Now, using (4.5) and (4.7), we get

$$(4.8) \qquad a_{2n+1} \log(2n+1) \ll a_{2n} \log 2n \ll a_{2n} \sum_{k=1}^{n} \frac{1}{k} \ll \sum_{k=1}^{n} \frac{a_{n+k}}{k} \ll \psi_n,$$

whence, by (3.1),

$$(4.9) \qquad a_n \log n \ll \psi_n$$

also holds.

Finally we show that

$$(4.10) \qquad \sum_{k=n}^{\infty} \frac{a_k}{k} \ll \psi_n$$

also comes from (3.2).

Since

$$(4.11) \qquad 2 \sum_{k=[(n+1)/2]}^{\infty} \frac{a_{2k+1}}{2k+1} = \int_0^{\pi} (f(x) - s_n(f,x))dx \ll \|f - s_n(f)\| \ll \psi_n,$$

thus, by virtue of (3.1), (4.7) and (4.11), we also verified (4.10).

This completes the proof. □

***Proof of Theorem* 2.** The proof proceeds on the line of Theorem 1 up to the estimation given in (4.3). Next, we utilize the new assumption $\{\frac{a_n}{\alpha_n}\} \in \gamma$RBVS instead of $\{\frac{a_n}{\alpha_n}\} \in$ RBVS, which implies that $\frac{a_n}{\alpha_n} \leqq R_n \ll \gamma_n$, whence

$$(4.12) \qquad a_n \ll \alpha_n \gamma_n, \quad n = 1, 2, \ldots$$

follows. Using these estimation at the end of (4.4), we obtain that

$$(4.13) \qquad I_2 \ll \sum_{k=n+1}^{\infty} \frac{\alpha_{k+1}\gamma_{k+1}}{k} + \alpha_{n+1}\gamma_{n+1} \log(n+1).$$

Hence, by (3.11),

$$(4.14) \qquad I_2 \ll \psi_n$$

follows.

If we put the estimations (4.12) into (4.2), too, then, by (3.11), we get that

$$(4.15) \qquad I_1 \ll \psi_n$$

also holds.

The last two estimations and (4.1) convey the assertion of Theorem 2, thus the proof is complete. □

***Proof of Theorem* 3.** The proof is a simple repetition of the proof of Theorem B, putting everywhere α_n in place of $(\log n)^N$, and using Theorem 1 instead of Theorem A.

We omit the details. □

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*Obviously the author has read only the English translation of [4] as referee of [1].