Some Results on Certain Symmetric Circulant Matrices

Research Article

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Abstract. A direct method for finding the inverse of a class of symmetric circulant matrices is given in [4]. In this paper a method of finding the Moore-Penrose inverse for a class of singular circulant matrices is presented and the spectral norm and spectral radius are calculated. Finally the spectral norm and spectral radius for symmetric circulant matrices with binomial coefficients are derived.

Keywords. Symmetric circulant matrix; Moore-Penrose inverse; Spectral radius

MSC. 05A19; 15A15

1. Introduction

In [4] the authors studied some properties including inversion and applications of a class of symmetric circulant matrices. When the matrix is singular, the Moore-Penrose inverse replaces the inverse. In Section 2 we calculate the Moore-Penrose inverse for matrices $\begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$. The concept of parallel sum of two hpsd (see ref 2.3) matrices was introduced by W.N. Anderson, Jr. and R.J. Duffin [1]. The parallel sum of matrices is useful in electrical networks. We calculate the parallel sum of two hpsd matrices in this section. In Section 3 the spectral norm and spectral radius for symmetric circulant matrices with binomial coefficients are derived using some Banach algebra results.

A circulant matrix $A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$ of order n is symmetric if

(i) $a_{\frac{n}{2}+j} = a_{\frac{n}{2}-(j-2)}$, $2 \le j \le \dfrac{n}{2}$, when n is even,

(ii) $a_{\frac{n}{2}+\left(j+\frac{1}{2}\right)} = a_{\frac{n}{2}+\left(\frac{3-2j}{2}\right)}$, $1 \le j \le \dfrac{n-1}{2}$, when n is odd.

In particular a matrix $A = \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$ of order n and the matrix

$$C_n = \begin{pmatrix} n_{C_0} & n_{C_1} & n_{C_2} & \cdots & n_{C_{n-1}} \end{pmatrix}$$

with Binomial coefficients are symmetric circulant.

We have [1], $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & 6 & 4 \\ 4 & 1 & 4 & 6 \\ 6 & 4 & 1 & 4 \\ 4 & 6 & 4 & 1 \end{bmatrix}$, $\cdots$ are symmetric circulant matrices with binomial coefficients.

Given an $n \times n$ matrix A with real (or complex) entries there exists a unique $n \times n$ matrix X satisfying the ***Moore-Penrose equations***:

(1) $AXA = A$,

(2) $XAX = X$,

(3) $(AX)^* = AX$,

(4) $(XA)^* = XA$,

where $*$ denotes the conjugate transpose. Such a unique X corresponding to A is called the Moore-Penrose inverse of A and is denoted by $A^{\dagger}$.

2. Moore-Penrose Inverse of Certain Singular Symmetric Circulant Matrix

Theorem 2.1. *For two distinct real numbers a, b with $a + (n-1)b = 0$, the Moore-Penrose $A^{\dagger}$ of $A = \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$ of order n is $A^{\dagger} = \dfrac{1}{(a-b)^2} \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$.*

Proof. Since $a + (n-1)b = 0$, clearly A is singular, we can find the Moore-Penrose inverse of A as follows:

When $A = \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$, $A^2 = \begin{pmatrix} c & d & d & \cdots & d \end{pmatrix}$, where c and d are given by $c = a^2 + (n-1)b^2$, $d = 2ab + (n-2)b^2$.

If $X = \begin{pmatrix} x_1 & x_2 & x_2 & \cdots & x_2 \end{pmatrix}$, since A, X are circulant, $AX = XA$.

Hence

$$AXA = A^2X = \begin{pmatrix} c & d & d & \cdots & d \end{pmatrix}\begin{pmatrix} x_1 & x_2 & x_2 & \cdots & x_2 \end{pmatrix}$$

$$= \begin{pmatrix} cx_1 + (n-1)dx_2 & dx_1 + (c+(n-1)d)x_2 & \cdots & dx_1 + (c+(n-1)d)x_2 \end{pmatrix}.$$

Hence

$$AXA = A$$

$$\implies \begin{pmatrix} cx_1 + (n-1)dx_2 & dx_1 + (c+(n-1)d)x_2 & \cdots & dx_1 + (c+(n-1)d)x_2 \end{pmatrix} = \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}.$$

$$\iff cx_1 + (n-1)dx_2 = a, \quad dx_1 + (c+(n-1)d)x_2 = b.$$

$$\iff (a-b)x_1 - (a-b)x_2 = 1$$

$$\iff x_1 - x_2 = \frac{1}{a-b}.$$

Thus

$$AXA = A \implies x_1 - x_2 = \frac{1}{a-b}, \qquad \text{(since } a+(n-1)b = 0).$$

When

$$x_1 = x_2 + \frac{1}{a-b},$$

$$X^2 = \begin{pmatrix} nx_2{}^2 + \frac{2}{a-b}x_2 + \frac{1}{(a-b)^2} & \cdots & \frac{2}{a-b}x_2 + nx_2{}^2 \end{pmatrix},$$

so that

$$X^2 A = \begin{pmatrix} nx_2{}^2 + \frac{2}{a-b}x_2 + \frac{1}{(a-b)^2} & \cdots & \frac{2}{a-b}x_2 + nx_2{}^2 \end{pmatrix} \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$$

$$= \begin{pmatrix} c & d & d & \cdots & d \end{pmatrix}$$

where

$$c = nax_2{}^2 + \frac{a}{(a-b)^2} + \frac{2ax_2}{a-b} + \frac{2b(n-1)x_2}{a-b} + n(n-1)bx_2{}^2,$$

$$d = nbx_2{}^2 + \frac{b}{(a-b)^2} + \frac{2bx_2}{a-b} + \frac{2ax_2}{a-b} + nax_2{}^2 + \frac{2(n-2)bx_2}{a-b} + n(n-2)bx_2{}^2$$

$$= \begin{pmatrix} \dfrac{a}{(a-b)^2} & \dfrac{b}{(a-b)^2} & \cdots & \dfrac{b}{(a-b)^2} \end{pmatrix}.$$

Hence $XAX = X$ iff

$$\begin{pmatrix} \dfrac{a}{(a-b)^2} & \dfrac{b}{(a-b)^2} & \cdots & \dfrac{b}{(a-b)^2} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_2 & \cdots & x_2 \end{pmatrix}.$$

Comparing

$$x_1 = \frac{a}{(a-b)^2}, \quad x_2 = \frac{b}{(a-b)^2}.$$

Thus

$$X = \begin{pmatrix} \dfrac{a}{(a-b)^2} & \dfrac{b}{(a-b)^2} & \cdots & \dfrac{b}{(a-b)^2} \end{pmatrix}.$$

Since A and X are symmetric circulant matrices, $(XA)^* = XA = AX$.

Hence the Moore-Penrose inverse $A^\dagger = \dfrac{1}{(a-b)^2} \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$. □

Corollary 2.2. *The Moore-Penrose inverse of a symmetric circulant matrix is symmetric circulant.*

Definition 2.3. A square matrix A is said to be Hermitian positive semi-definite(hpsd) if $A^* = A$ and all the eigenvalues of A are greater than or equal to zero and at least one eigenvalue is zero.

Lemma 2.4. *If $A = \begin{pmatrix} a_1 & a_2 & a_2 & \cdots & a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 & b_2 & \cdots & b_2 \end{pmatrix}$ are two symmetric circulant matrices of order n with $a_1 > a_2$, $a_1 + (n-1)a_2 = 0$ and $b_1 > b_2$, $b_1 + (n-1)b_2 = 0$ then $A + B$ is hpsd.*

Proof. Since $a_1 > a_2$, $a_1 + (n-1)a_2 = 0$, we have all the eigenvalues of A are greater than or equal to 0, so that A is hpsd. Similarly B is also hpsd.

Now $A + B = \begin{pmatrix} a_1 + & b_1 & a_2 + b_2 & \cdots & a_2 + b_2 \end{pmatrix}$, clearly $A + B$ is also hpsd. $\square$

Definition 2.5. The parallel sum of two $n \times n$ matrices A and B is defined as $A(A + B)^\dagger B$.

Theorem 2.6. *If $A = \begin{pmatrix} a_1 & a_2 & a_2 & \cdots & a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 & b_2 & \cdots & b_2 \end{pmatrix}$ satisfying the conditions of Lemma 2.4 then the parallel sum of A and B is $\left(\dfrac{1}{((a_1 - a_2) + (b_1 - b_2))^2} A(A + B) \right) B$.*

Proof. Since A and B are hpsd, $A + B$ is also hpsd.

Now $A + B = \begin{pmatrix} a_1 + & b_1 & a_2 + b_2 & \cdots & a_2 + b_2 \end{pmatrix}$.

By Theorem 2.1, $(A + B)^\dagger = \dfrac{1}{((a_1 - a_2) + (b_1 - b_2))^2} \begin{pmatrix} a_1 + & b_1 & a_2 + b_2 & \cdots & a_2 + b_2 \end{pmatrix}$.

Therefore the parallel sum is $\dfrac{1}{((a_1 - a_2) + (b_1 - b_2))^2} A(A + B)B$. $\square$

Theorem 2.7. *For two distinct elements a, b in the complex field, the spectral norm of the symmetric circulant matrix $A = \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$ of order n is $\rho(A) = \max\{|a + (n-1)b|, |a - b|\}$.*

Proof. From [2] the eigenvalues of A are $a + (n-1)b$, $a - b$.

Thus the spectrum of A is $\sigma(A) = \{a + (n-1)b, a - b\}$.

Hence the spectral norm of A is $\rho(A) = \max\{|a + (n-1)b|, |a - b|\}$.

More over if A is singular then $a + (n-1)b = 0$ and hence spectral radius of A is $|a - b|$. $\square$

Remark 2.8. Perron and Frobenius theorem [3] states that if $A = (a_{ij})$ is a matrix with non-negative elements $a_{ij} \geq 0$, there exists an eigenvector of A with non-negative coordinates and with eigenvalue ρ, such that all other eigenvalues satisfy $|\lambda| \leq \rho$. In particular if $A = \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$ (of order n) and if a, b are nonnegative then $a + (n-1)b$ is the maximum eigenvalue of A and by Perron and Frobenius theorem an eigenvector corresponding to $a + (n-1)b$ contains non-negative coordinates. We can choose this eigenvector as $(1, 1, 1, \cdots, 1)$.

Theorem 2.9 ([2])**.** *For any element of a Banach Algebra X of the form $e - z$ with $\|z\| < 1$ there exists a unique $y = e - x$ in X with $\|x\| < 1$ such that $y^2 = e - z$ i.e., $(e - x)^2 = e - z$. In other words $e - z$ has a square root.*

This could be used to establish the following theorem:

Theorem 2.10. *For two distinct elements a,b in a Banach algebra X of all $n \times n$ complex matrices with $|a| < \dfrac{1}{2}$ and $|b| < \dfrac{1}{2}$ the of symmetric circulant matrix $I - A$, where $A = \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$ of order n has a square root $I - B$ in X.*

Proof. Since the eigenvalues of $A = \begin{pmatrix} a & b & b & \cdots & b \end{pmatrix}$ are $\lambda = a + (n-1)b$, $a - b$, $a - b, \cdots, a - b$ ($n - 1$ times). For $|a| < \dfrac{1}{2}$ and $|b| < \dfrac{1}{2}$, we have each eigen value λ of A satisfies $|\lambda| < 1$.

Thus the spectral radius $\rho(A) = \max\{|\lambda_i| : i = 1, \cdots, n\}$ is less than 1.

Therefore by Theorem 2.9 there exists a unique $I - B$ such that $(I - B)^2 = I - A$ in X.

Hence $I - B$ is the square root of $I - A$. □

3. Symmetric Circulant Matrices with Binomial Coefficients

Theorem 3.1. *The eigenvalues of the symmetric circulant matrix*

$$C_n = \begin{pmatrix} {}_nC_0 & {}_nC_1 & {}_nC_2 & \cdots & {}_nC_{n-1} \end{pmatrix}$$

are given by $2^{n-1}, \left(2^n \cos^{2n} \dfrac{\pi m}{n} - 1\right), m = 1, 2, 3, \cdots, n - 1$.

Proof. We have the eigenvalues of a circulant matrix $A = (a_0, a_1, a_2, \cdots, a_{n-1})$ are

$$\lambda_j = \sum_{j=0}^{n-1} a_j \left(e^{\frac{2\pi i k}{n}}\right)^j, \quad k = 0, 1, 2, \cdots, n - 1.$$

i.e.,

$$\lambda_j = \sum_{j=0}^{n-1} a_j \cos \frac{2\pi k j}{n} + i \sum_{j=0}^{n-1} a_j \sin \frac{2\pi k j}{n}, \quad k = 0, 1, 2, \cdots, n - 1.$$

Since C_n is symmetric,

$$\lambda_k = \sum_{j=0}^{n-1} {}_nC_j \cos \frac{2\pi k j}{n}, \quad k = 0, 1, 2, \cdots, n - 1.$$

Thus the eigen values of C_{n-1} are

$$\lambda_0 = {}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{n-1} = 2^n - 1,$$

$$\lambda_1 = {}_nC_0 + {}_nC_1 \cos \frac{2\pi}{n} + {}_nC_2 \cos \frac{4\pi}{n} + \cdots + {}_nC_{n-1} \cos \frac{2(n-1)\pi}{n} = \left(1 + \cos \frac{2\pi}{n}\right)^n - 1,$$

$$\lambda_2 = {}_nC_0 + {}_nC_1 \cos \frac{4\pi}{n} + {}_nC_2 \cos \frac{8\pi}{n} + {}_nC_3 \cos \frac{12\pi}{n} + \cdots = \left(1 + \cos \frac{4\pi}{n}\right)^n - 1,$$

$$\lambda_n = \left(1 + \cos \frac{2(n-1)\pi}{n}\right)^n - 1.$$

Also $\det(C_n) = \displaystyle\prod_{m=0}^{n-1} \left(2^n \cos^{2n} \frac{\pi m}{n} - 1\right)$. □

Theorem 3.2. *The spectral norm of C_n is $\rho(C_n) = \max\{2^{n-1}, |(2\cos^2\frac{\pi}{n})^n - 1|\}$.*

Proof. From Theorem 3.1 we have its spectrum is

$$\sigma(C_n) = \left\{ 2^n - 1, \left(1 + \cos\frac{2\pi}{n}\right)^n - 1, \left(1 + \cos\frac{4\pi}{n}\right)^n - 1, \cdots, \left(1 + \cos\frac{2(n-1)\pi}{n}\right)^n - 1 \right\}$$

$$= \left\{ 2^n - 1, 2^n\cos^{2n}\frac{\pi}{n}, 2^n\cos^{2n}\frac{2\pi}{n} - 1, \cdots, 2^n\cos^{2n}\frac{(n-1)\pi}{n} - 1 \right\}.$$

Now

$$\rho(C_n) = \max\left\{ 2^{n-1}, \left|\left(2\cos^2\frac{\pi}{n}\right)^n - 1\right|, \left|\left(2\cos^2\frac{2\pi}{n}\right)^n - 1\right|, \cdots, \left|\left(2\cos^2\frac{(n-1)\pi}{n}\right)^n - 1\right| \right\}.$$

Since $\cos^{2n}\frac{\pi m}{n}$ is maximum for $m = 1$ we get $\rho(C_n) = \max\left\{ 2^{n-1}, \left|\left(2\cos^2\frac{\pi}{n}\right)^n - 1\right| \right\}$.   □

Remark 3.3. In C_n all the elements are positive and 2^{n-1} is the maximum eigenvalue. So by Perron and Frobenius theorem an eigenvector corresponding to 2^{n-1} contains non-negative coordinates. We can choose this eigenvector as $(1, 1, 1, \cdots, 1)$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both the authors contributed equally and significantly in writing this article as well as read and approved the final manuscript.

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