Journal of Informatics and Mathematical Sciences
Vol. 7, No. 1, pp. 49–67, 2015
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

b-Chromatic Number of Some Splitting Graphs

Research Article

S. Arockiaraj[1] and V. Premalatha[2]

[1] *Department of Mathematics, Mepco Schlenk Engineering College, Sivakasi 626 005, Tamilnadu, India*
[2] *Research Scholar of Karpagam University, Coimbatore, Tamilnadu, India;*
Department of Mathematics, Canara First Grade College, Kodialbail, Mangalore 575 003, Karnataka, India
***Corresponding author:** psarockiaraj@gmail.com

Abstract. A b-colouring of a graph G is a proper vertex colouring of G such that each colour class contains a vertex that has atleast one neighbour in every other colour class and b-chromatic number of a graph G is the largest integer $\phi(G)$ for which G has a b-colouring with $\phi(G)$ colours. In this paper, we have obtained the b-chromatic number of the splitting graphs of path P_n, cycle C_n, star $K_{1,n}$, fan graph F_n, triangular snake T_n, the H-graph H_n, the corona graph $P_n \circ K_1$ and $C_n \circ K_1$.

Keywords. b-colouring; b-chromatic number

MSC. 05C15; 05C38

Received: January 22, 2015 **Accepted:** June 19, 2015

1. Introduction

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A proper k-colouring of graph G is a function C defined from $V(G)$ onto a set of colours $\{1,2,\ldots,k\}$ such that any two adjacent vertices have different colours.

Path on n vertices is denoted by P_n and cycle on n vertices is denoted by C_n.

A triangular snake is obtained from a path by identifying each of the path with an edge of the cycle C_3. The graph $G \circ K_1$ is obtained from the graph G by attaching a new pendent vertex at each vertex of G.

The splitting graph $S(G)$ was introduced by Sampathkumar and Walikar [7]. For each vertex v of a graph G, take a new vertex v' and join v' to all the vertices of G adjacent to v.

The b-chromatic number of a graph was introduced by R. W. Irving and D. F. Manlove when considering minimal proper colouring with respect to a partial order defined on the set of all partition of vertices of graph. The b-chromatic number of a graph G, denoted by $\phi(G)$ is the largest positive integer t such that there exists a proper colouring for G with t colours in which every colour class contains at least one vertex adjacent to some vertex in all the other colour classes. Such a colouring is called a b-colouring.

So many authors have studied on b-chromatic number. Arockiaraj et al. [1] have studied an odd sum labeling of some splitting graphs. Motivated by these works, we have obtained the b-chromatic number of the splitting graphs of path P_n, cycle C_n, star $K_{1,n}$, fan graph F_n, triangular snake T_n, the H-graph H_n, the corona graph $P_n \circ K_1$ and $C_n \circ K_1$.

2. Main Results

Proposition 2.1. *The b-chromatic number of the splitting graph $SP(P_n)$ of path P_n is*

$$\phi(SP(P_n)) = \begin{cases} 5, & n \geq 9 \\ 4, & n = 6,7,8 \\ 3, & n = 5 \\ 2, & n = 2,3,4 \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on the path P_n and let v_i' be the duplicating vertex of v_i, $1 \leq i \leq n$. Assume that $n \geq 9$. Colour the vertices v_i, $1 \leq i \leq n$ by the colours $4,0,1,2,0,3,2,4,3,0,1,2,3,4,0,1,2,3,4,\ldots$ and the vertices v_i', $1 \leq i \leq n$ by the colours $2,3,3,4,4,1,1,4,0,1,2,3,4,0,1,2,3,4,\ldots$ respectively. Then the vertices v_2, v_3, v_4, v_6 and v_8 are the members of the colour classes $0,1,2,3$ and 4 respectively with their neighbours having all the remaining colours. Hence $\phi(SP(P_n)) = 5$, when $n \geq 9$.

When $n = 2$, $SP(P_2)$ is P_4 in which 2 vertices are of degree 2 and 2 vertices are of degree 1. Hence $\phi(SP(P_2)) \leq 2$ and Figure 1 shows that $\phi(SP(P_2)) = 2$.

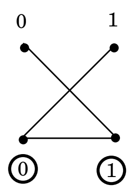


Figure 1. A b-colouring of $SP(P_2)$ with 2 colours

When $n = 3$, $SP(P_3)$ has only one vertex of degree 4 and 3 vertices of degree 2. Therefore $\phi(SP(P_3)) \leq 3$. Based on adjacency either v_2 and v_2' are of same colour or v_1 and v_3 are of same colour. Hence $\phi(SP(P_3)) \leq 2$. Figure 2 shows that $\phi(SP(P_3)) = 2$.

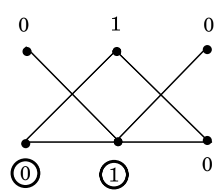


Figure 2. A b-colouring of $SP(P_3)$ with 2 colours

When $n = 4$, $SP(P_4)$ has only two vertices with the maximum degree 4 and the remaining vertices of degree atmost 2. Hence $\phi(SP(P_4)) \leq 3$. As in the case of $n = 3$, 3 colouring is not possible when $n = 4$. A b-colouring with 2 colours is shown in Figure 3.

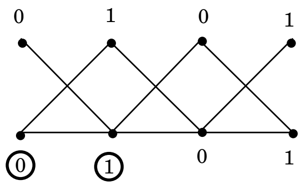


Figure 3. A b-colouring of $SP(P_4)$ with 2 colours

In $SP(P_5)$, only three vertices are of maximum degree 5 and the remaining vertices are of degree atmost 2. Hence $\phi(SP(P_5)) \leq 3$. A b-colouring with 3 colours is as shown in Figure 4.

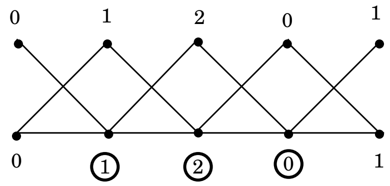


Figure 4. A b-colouring of $SP(P_5)$ with 3 colours

In $SP(P_6)$, four vertices are of maximum degree 4 and the remaining vertices are of degree atmost 2. Hence $\phi(SP(P_6)) \leq 4$. A b-colouring with 4 colours is shown in Figure 5.

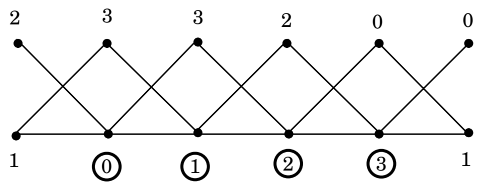


Figure 5. A b-colouring of $SP(P_6)$ with 4 colours

In $SP(P_7)$, five vertices are of maximum degree 5 and the remaining vertices are of degree atmost 2. Hence $\phi(SP(P_7)) \leq 5$. If $\phi = 5$, then the vertices v_2, v_3, v_4, v_5 and v_6 are the members of five colour classes. If v_2, v_3 and v_4 are members of different colour classes, then v_5 cannot be the member of the remaining colour classes. Hence $\phi(SP(P_7)) < 5$ and a b-colouring with 4 colours is given in Figure 6.

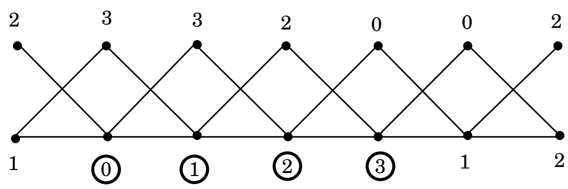


Figure 6. A b-colouring of $SP(P_7)$ with 4 colours

In $SP(P_8)$, six vertices are of the maximum degree 4 and all the remaining vertices are of degree atmost 2. Hence $\phi(SP(P_8)) \le 5$. If $\phi = 5$, the members of the colour classes with required property are to be in v_2, v_3, v_4, v_5, v_6 and v_7. By the same argument as in the case of $n = 7$, 5 colours are not possible. A b-colouring with 4 colours for $SP(P_8)$ is given in Figure 7.

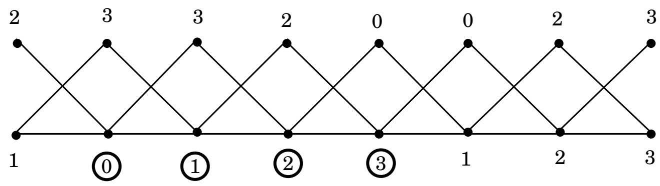


Figure 7. A b-colouring of $SP(P_8)$ with 4 colours

□

Proposition 2.2. *For the splitting graph $SP(C_n)$ of cycle C_n, the b-chromatic number is*

$$\phi(SP(C_n)) = \begin{cases} 5, & when\ n \ge 9 \\ 3, & when\ n = 3 \\ 2, & when\ n = 4 \\ 4, & when\ n = 5,6 \\ 5, & when\ n = 7 \\ 4, & when\ n = 8 \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices on the cycle C_n and $v'_1, v'_2, \dots, v'_n$ be the corresponding duplicated vertices in $SP(C_n)$. Since $\Delta(SP(C_n)) = 4$, $n \ge 3$, $\phi(SP(C_n)) \le 5$.

Assume that $n \ge 9$. Assign the colours $2,0,1,2,3,1,4,2$ to the vertices $v_1, v_2, \dots, v_8$ respectively, the colours $0,1,0,1,\dots$ to the vertices $v_9, v_{10}, \dots, v_n$ the colours $4,4,0,0$ to the vertices v'_3, v'_4, v'_5, v'_6 respectively and the colour 3 to all the remaining vertices $v'_1, v'_2, v'_7, v'_8, \dots, v'_n$. Then the vertices v_2, v_3, v_4, v_5 and v_6 are the members of the colour classes of $0,1,2,3$ and 4 in which they are adjacent to atleast one member of the remaining colour classes. Thus $\phi(SP(C_n)) = 5$ when $n \ge 9$.

When $n = 3$, only 3 vertices are of degree 4. Therefore $\phi(SP(C_3)) \le 3$. Figure 8 shows that $\phi(SP(C_3)) = 3$.

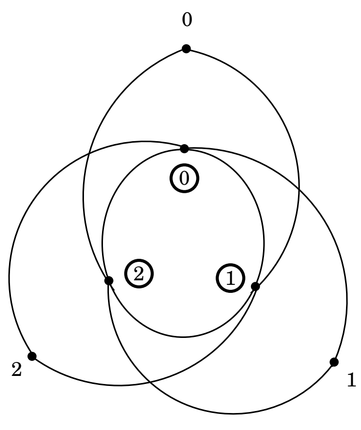


Figure 8. A b-colouring of $SP(C_3)$ with 3 colours

When $n = 4$, $SP(C_n)$ has only 4 vertices are of degree 4. So $\phi(SP(C_n)) \le 4$. If $\phi = 4$, then v_1 will be assigned by the colour 0(say) in which their neighbours having colours 1,2 and 3. So v_3 should be having the colour 0. If $\phi = 3$, by assigning the colour 0 to v_1, their neighbours will have the colours 1 and 2. By adjacency, v_3 should be coloured by 0. Similarly v_2 and v_4 are having same colours namely 1. Further if we assign the colour 2 to any one of $v_i, 1 \le i \le 4$, no member of the colour class 2 having the neighbours 0 and 1. Therefore $\phi(SP(C_4)) < 3$. Figure 9 shows that $\phi(SP(C_4)) = 2$.

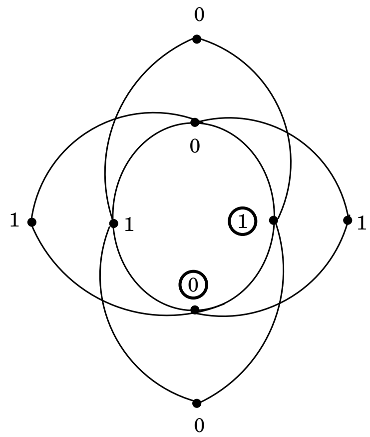


Figure 9. A b-colouring of $SP(C_4)$ with 2 colours

In $SP(C_5)$, since only 5 vertices are of degree 4, the colours of these vertices only are having the required property of b-colouring. By assigning the colours 0,1,2,3 and 4, for v_1, v_2, v_3, v_4 and v_5 respectively, v_2' and v_5' should have the colours 3 and 2 respectively. If it is so, then the neighbours of v_3 and v_4 have repeated colours and hence the colours 2 and 3 can't have all the colours in neighbours. Therefore $\phi(SP(C_5)) < 5$. Figure 10 shows that $\phi(SP(C_5)) = 4$.

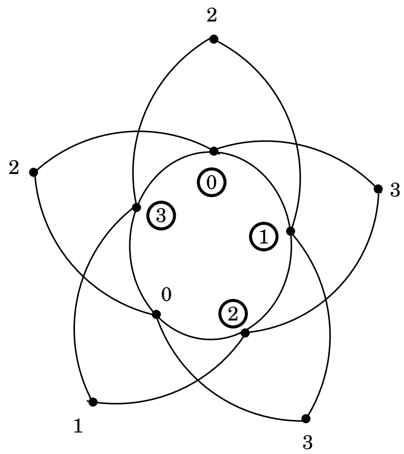


Figure 10. A b-colouring of $SP(C_5)$ with 4 colours

In $SP(C_6)$, since 6 vertices are of degree 4, other than a vertex all the remaining vertices such that the colours assigned to these are neighbours to the neighboring colours.

If we assume that 0 and 1 are labels of adjacent vertices, to make these as the members so that they have all the remaining colours as neighbours, then among the remaining vertices atleast two vertices are having repeated colours in the neighbours. So $\phi(SP(C_6) < 5$. Figure 11 shows that $\phi(SP(C_6)) = 4$.

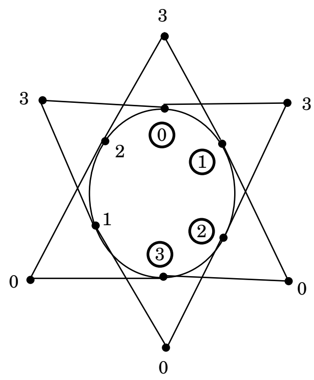


Figure 11. A b-colouring of $SP(C_6)$ with 4 colours

Figure 12 shows that $\phi(SP(C_7)) = 5$.

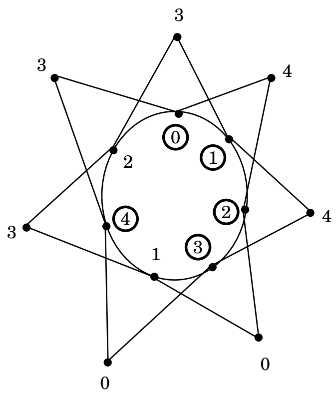


Figure 12. A b-colouring of $SP(C_7)$ with 5 colours

Repeating the arguments as in the case of $n = 4$ for $n = 8$, $\phi(SP(C_8)) < 5$. Figure 13 shows that $\phi(SP(C_8)) = 4$.

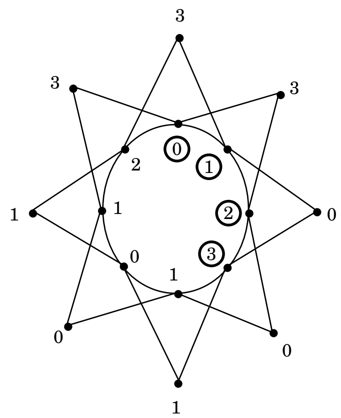


Figure 13. A b-colouring of $SP(C_8)$ with 4 colours

□

Proposition 2.3. *For any* $n \geq 2$, $\phi(SP(K_{1,n})) = 2$.

Proof. Let u be the central vertex and $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. Let u' and $y_1, y_2, \ldots, y_n$ be the duplicating vertices of u and $v_1, v_2, \ldots, v_n$ respectively.

In $SP(K_{1,n})$, the vertex u is of degree $2n$, the vertex u' is of degree n and the remaining vertices are of degree 2. Hence $\phi(SP(K_{1,n})) \leq 3$.

When we assign two distinct colours say 0 and 1 to u and u', and all v_i's are to be of same colour and y_i's may be assigned by the colours either 1 or 2. In this colouring, no member of colour classes 1 is adjacent to atleast one member of remaining colour classes. If we assign same colour say 0 to u and u', then v_i's and y_i's may be assigned by either 1 or 2. In this colouring, no member of 0 and 1 is adjacent to atleast one member of remaining colour classes. Hence $\phi(SP(K_{1,n})) < 3$. By assigning the colours 0 to u and u' and 1 to all v_i's and y_i's, $\phi(SP(K_{1,n})) = 2$. □

Proposition 2.4. *The b-chromatic number of the splitting graph $SP(F_n)$ of the Fan graph F_n is*

$$\phi(SP(F_n)) = \begin{cases} 6, & \text{for } n \geq 9 \\ 5, & \text{for } n = 6,7,8 \\ 4, & \text{for } n = 5 \\ 3, & \text{for } n = 2,3,4. \end{cases}$$

Proof. The graph $SP(F_n)$ is same as the graph obtained from $SP(P_n)$ by adding two new vertices say u and u' and join the vertices u and u' with all the vertices on the path P_n. Since u and u' cannot be the member of two different b-colour classes, the b-chromatic number of $SP(F_n)$ is $\phi(SP(P_n)) + 1$. Hence the result follows. □

Proposition 2.5. *The b-chromatic number of the splitting graph $SP(T_n)$ obtained from triangular snake T_n is*

$$\phi(SP(T_n)) = \begin{cases} 9, & \text{for } n \geq 11 \\ 8, & \text{for } n = 10 \\ n-1, & \text{for } 6 \leq n \leq 9 \\ 5, & \text{for } 3 \leq n \leq 5 \\ 4, & \text{for } n = 2 \\ 3, & \text{for } n = 1. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_{n+1}$ be the vertices on the path of length n and $u_i, 1 \leq i \leq n$ be the vertices so that $u_i v_i$ and $u_i v_{i+1}$ are edges of the triangular snake T_n with n triangles. Let x_i, y_j be the duplicating vertices of u_i, v_j respectively, $1 \leq i \leq n$ and $1 \leq j \leq n+1$. When $n \geq 2$, the maximum degree of $SP(T_n)$ is 8. Hence $\phi(SP(T_n)) \leq 9$ and the b-chromatic number will be varied according to the number of vertices with maximum degree and their adjacency. Assume that $n \geq 11$. By assigning the colours $2,3,4,5,6,0,1,2,3,4,5$ for the first eleven vertices of u_i, $4,5,6,0,1,2,3,4,5,6,0$ for the first eleven vertices of x_i, $6,8,1,2,3,4,5,6,0,1,7$ for the first eleven vertices of v_i, $0,0,7,7,8,8,7,7,8,8,2,2$ for the first twelve vertices of y_i and giving a proper colouring for the remaining vertices, the vertices $v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9$ and v_{11} are the members of the respective colour classes $8, 1, 2, 3, 4, 5, 6, 0$ and 7 in which they are having atleast one neighbour in all the remaining colour classes. So $\phi(SP(T_n)) = 9$ for $n \geq 11$. The b-colouring is shown in Figure 14 for $SP(T_{12})$.

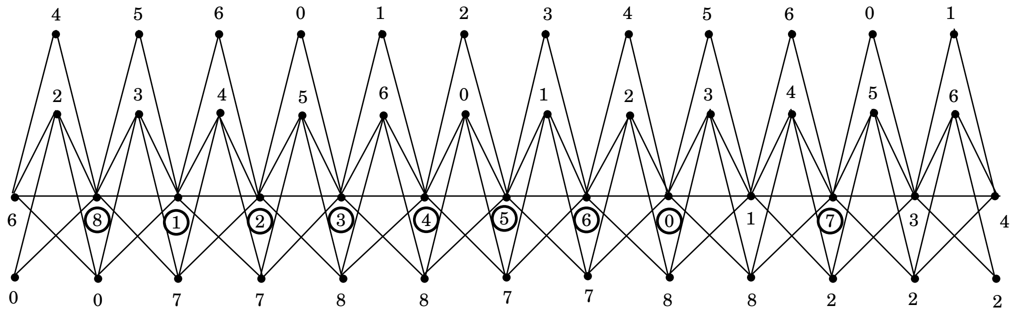


Figure 14. A b-colouring of $SP(T_{12})$ with 9 colours

Even through there are 9 vertices having the maximum degree 8 while $n = 10$, based on the adjacency, it is not possible to fill with 9 colours with the required property. So $\phi(SP(T_{10})) \leq 8$. The b-colouring shown in Figure 15 gives that $\phi(SP(T_{10})) = 8$.

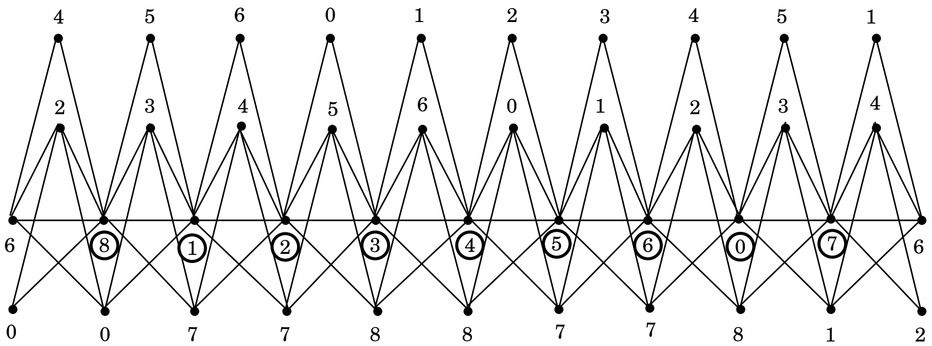
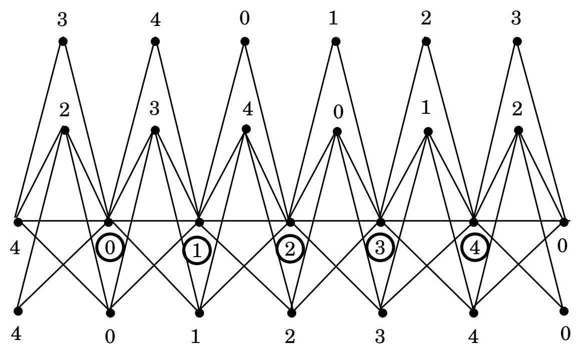


Figure 15. A b-colouring of $SP(T_{10})$ with 8 colours

While $6 \leq n \leq 9$, since $SP(T_n)$ has only $n-1$ vertices of degree 8 and the remaining vertices are of degree less than or equal to 4, $\phi(SP(T_n)) \leq n-1$ for $6 \leq n \leq 9$. The b-colouring shown in Figure 16 gives that $\phi(SP(T_n)) = n-1$ for $6 \leq n \leq 9$.



(Figure 16 *Contd.*)

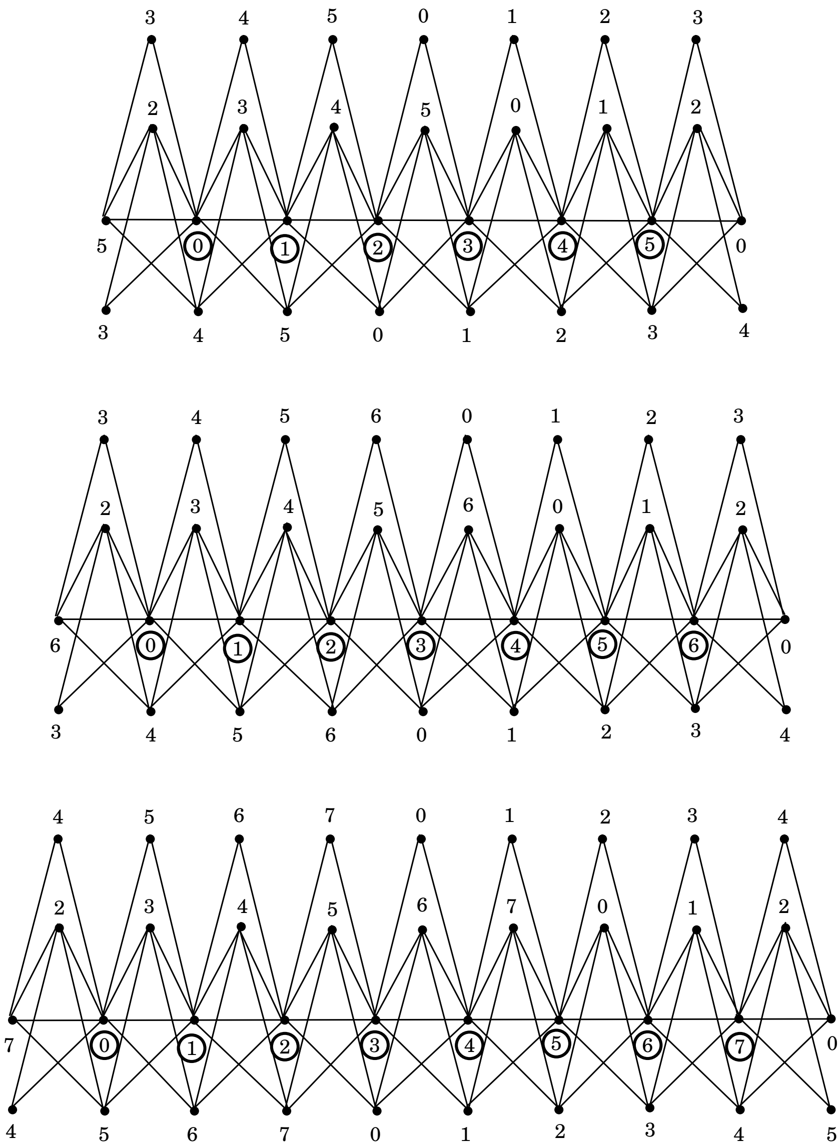
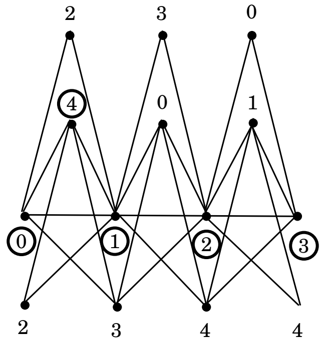


Figure 16. A b-colouring of $SP(T_n), 6 \le n \le 9$ with $n-1$ colours

When $3 \le n \le 5$, as only $n-2$ vertices are of degree 8 and remaining vertices are of degree 4, the b-chromatic number will be at most 5. The b-colouring shown in Figure 17 gives that $\phi(SP(T_n)) = 5$ for $3 \le n \le 5$.



(Figure 17 *Contd.*)

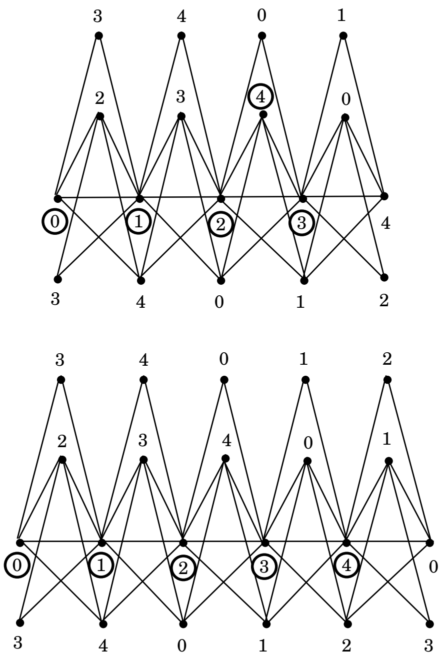


Figure 17. A b-colouring of $SP(T_n), 3 \le n \le 5$ with 5 colours

When $n = 2$, two adjacent edges will have to be received the same colour if we try to give b-colouring with 5 colours. The b-colouring shown in Figure 18 gives that $\phi(SP(T_2)) = 4$.

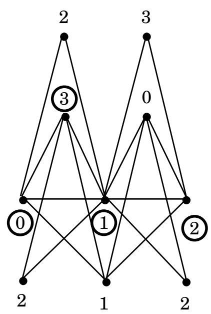


Figure 18. A b-colouring of $SP(T_2)$ with 4 colours

When $n = 1$, there are 3 vertices with maximum degree 4 and remaining vertices are of degree 2. The b-colouring shown in Figure 19 gives that $\phi(SP(T_1)) = 3$.

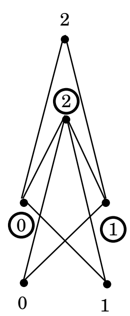


Figure 19. A b-colouring of $SP(T_1)$ with 3 colours

□

Proposition 2.6. *The b-chromatic number of the splitting graph of H-graph H_n is*

$$\phi(SP(H_n)) = \begin{cases} 5, & \text{for } n \geq 5 \\ 4, & \text{for } n = 4 \\ 2, & \text{for } n = 2,3. \end{cases}$$

Proof. Let u_i, v_i, $1 \leq i \leq n$ be the vertices on the paths of length $n-1$ in H-graph H_n. Let x_i and y_i be the duplicating vertices of u_i and $v_i, 1 \leq i \leq n$ respectively. In $SP(H_n)$, two vertices are having the maximum degree 5 and $2n-6$ vertices of degree 4. Hence $\phi(SP(H_n)) \leq 5$. Assume that $n \geq 7 (n \neq 8)$. Assign the colours $3,4,4,0,0$ for $x_i, 1 \leq i \leq 5$, $0,1,2,3,1$ for $u_i, 1 \leq i \leq 5$, $1,4,0,1$ for $v_i, 1 \leq i \leq 4$, $3,3,2,2$ for $y_i, 1 \leq i \leq 4$ and the proper 5-colouring for the remaining vertices. Then u_2, u_3, u_4, v_2 and v_3 are the members of the colour classes $1,2,3,4$ and 0 respectively in which they are having atleast one neighbour of the remaining colour classes. Hence $\phi(SP(H_n)) = 5$.

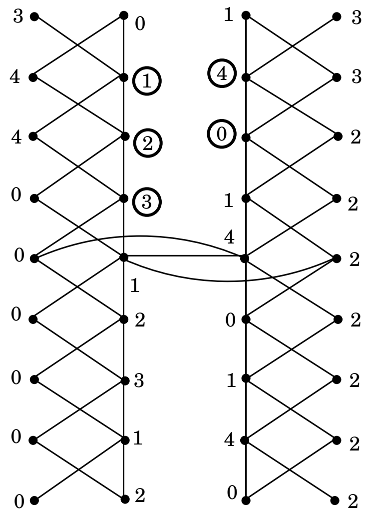


Figure 20. A b-colouring of $SP(H_9)$ with 5 colours

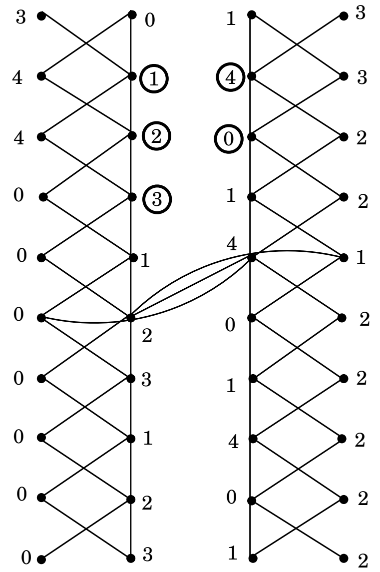


Figure 21. A b-colouring of $SP(H_{10})$ with 5 colours

A b-colouring with 5 colours for $n = 5$ is as shown in Figure 22.

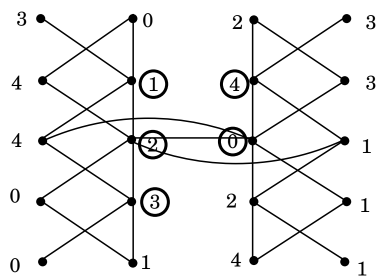


Figure 22. A b-colouring of $SP(H_5)$ with 5 colours

When $n = 3$, only two vertices are of degree 5, two vertices of degree 3 and remaining vertices are of degree less than 3. Based on the adjacency, 4 colours and 3 colours are not possible. A b-colouring with 2 colours for $n = 3$ is given in Figure 23.

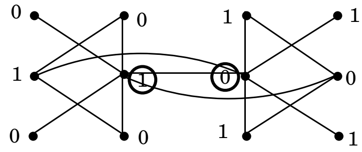


Figure 23. A b-colouring of $SP(H_3)$ with 2 colours

A b-colouring with 5-colours for $n = 6$ and 8 are shown in Figure 24 and Figure 25.

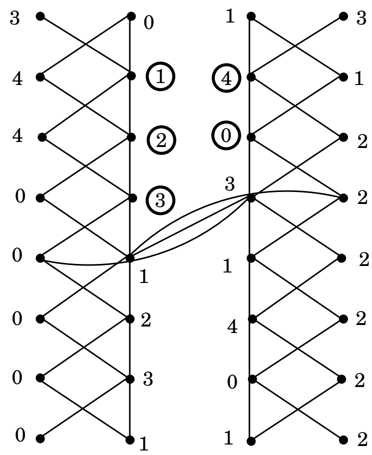


Figure 24. A b-colouring of $SP(H_8)$ with 5 colours

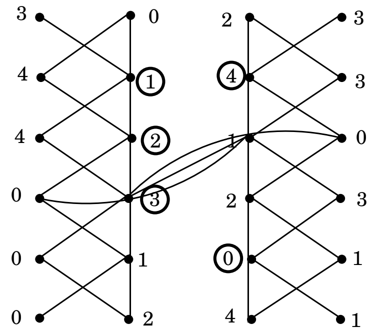


Figure 25. A b-colouring of $SP(H_6)$ with 5 colours

When $n = 4$, only 4 vertices are having degree atleast 4. Hence $\phi(SP(H_4)) \le 4$. A b-colouring with 4 colours for $n = 4$ given in Figure26.

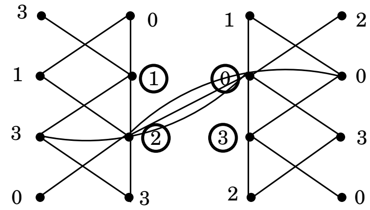


Figure 26. A b-colouring of $SP(H_4)$ with 4 colours

When $n = 2$, b-colouring with more than 2 colours is not possible. A b-colouring with two colours is given in Figure 27.

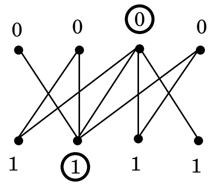


Figure 27. A b-colouring of $SP(H_2)$ with 2 colours

□

Proposition 2.7. *For splitting graph $SP(P_n \circ K_1)$ of the graph $P_n \circ K_1$, the b-chromatic number is*

$$\phi(SP(P_n \circ K_1)) = \begin{cases} 7, & when\ n \geq 9 \\ n-2, & when\ n = 7,8 \\ 5, & when\ n = 5,6 \\ 4, & when\ n = 3,4 \\ 2, & when\ n = 1,2 \end{cases}$$

Proof. In $P_n \circ K_1$, let $v_1, v_2, \ldots, v_n$ be the vertices on the path of length $n-1$ and u_i be the pendant vertex attached at $v_i, 1 \leq i \leq n$. Let v'_i and u'_i be the duplicating vertices of v_i and u_i respectively, $1 \leq i \leq n$.

In $SP(P_n \circ K_1), \Delta = 6$ for $n \geq 3$. Therefore $\phi(SP(P_n \circ K_1)) \leq 7$.

Case (1). $n \geq 9$.
Colour the vertices of $SP(P_n \circ K_1))$ as follows:

$$C(v_i) = i + 5 (\text{mod } 7)$$

$$C(u_i) = i (\text{mod } 7) \quad \text{and}$$

$$C(u'_i) = C(v'_i) = i + 2 (\text{mod } 7).$$

Then v_2, v_3, v_4, v_5, v_6 and v_7 are all the members of the colour classes $0, 1, 2, 3, 4, 5$ and 6 respectively in which their neighbouring colours are having all the remaining colours. Hence $\phi(SP(P_n \circ K_1) = 7$.

Case (2). $n = 8$.
Since 6 vertices are of the maximum degree 6 and two vertices are of the next maximum degree 4, $\phi(SP(P_8 \circ K_1)) \leq 6$. Figure 28 shows that $\phi(SP(P_8 \circ K_1)) = 6$.

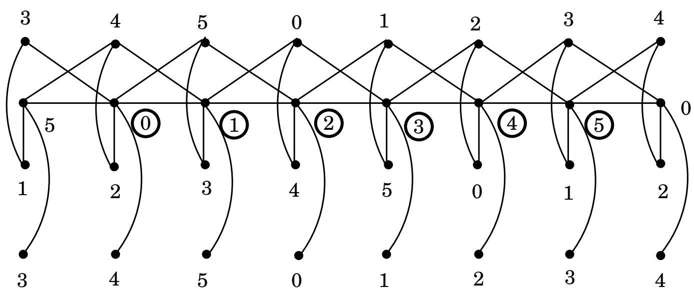


Figure 28. A b-colouring of $SP(P_8 \circ K_1)$ with 5 colours

Case (3). $n = 7$ (or 6, 5).
Since five (or four, three) vertices are of the maximum degree 6 and two vertices are of the next maximum degree 4, $\phi(SP(P_n \circ K_1)) \leq 5$. The b-colouring with 5 colours is given in Figure 29 and hence $\phi(SP(P_n \circ K_1)) = 5$.

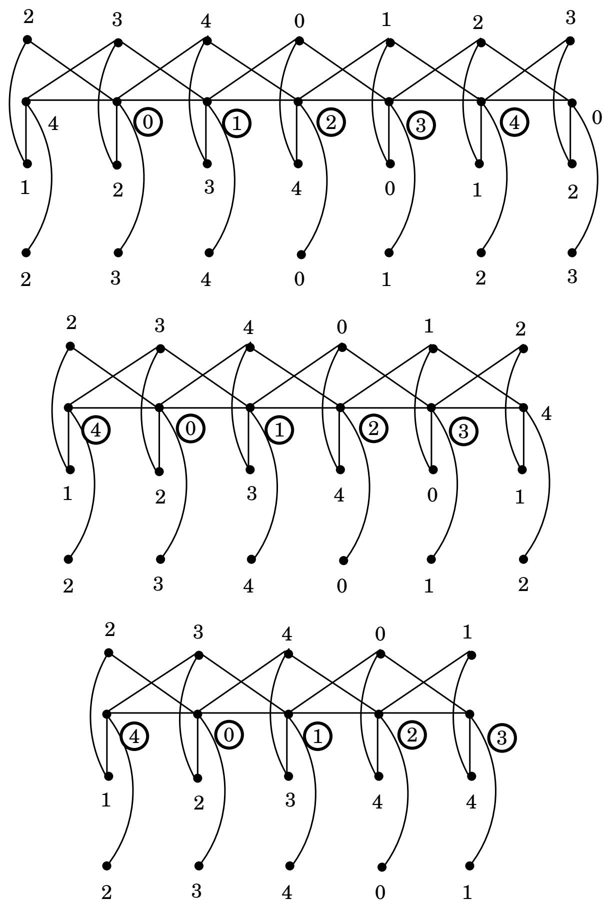


Figure 29. A b-colouring of $SP(P_n \circ K_1)$ with 5 colours for $n = 7, 6$ and 5

Case (4). $n = 4$.

Since two vertices are of the maximum degree 6, two vertices are of the next maximum degree 4 and the remaining vertices are having degree less then 4, $\phi(SP(P_4 \circ K_1)) \le 4$. Figure 30 shows that $\phi(SP(P_4 \circ K_1)) = 4$.

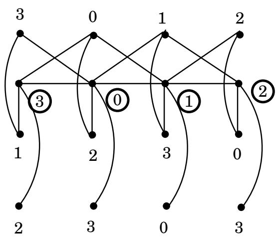


Figure 30. A b-colouring of $SP(P_4 \circ K_1)$ with 4 colours

Case (5). $n = 3$.

Since one vertex say v_2 is of the maximum degree 6, two vertices (say v_1, v_3) are of the next maximum degree 4 and one vertex say v_2' is of the next maximum degree 3, $\phi(SP(P_3 \circ K_1)) \le 4$.

If $\phi = 4$, then the vertices v_1, v_2, v_3 and v_2' only should be the members of the colour classes with the required property. If it is so, then the adjacent vertices v_2' and u_2 are to be of same colour. So $\phi(SP(P_3 \circ K_1)) < 4$. The b-colouring with 3 colours for $SP(P_3 \circ K_1)$ is given in Figure 31.

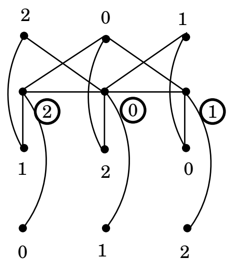


Figure 31. A b-colouring of $SP(P_3 \circ K_1)$ with 3 colours

Case (6). $n = 1, 2$.
By the same argument as in Case (5), $\phi(SP(P_n \circ K_1)) = 2$.

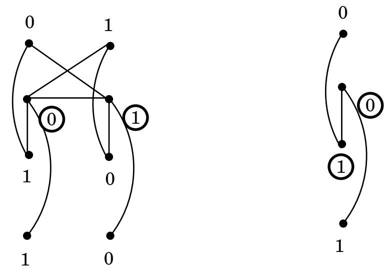


Figure 32. a b-colouring of $SP(P_n \circ K_1)$ with 2 colours for $n = 2$ and 1

□

Proposition 2.8. *For any* $n \geq 3$,

$$\phi(SP(C_n \circ K_1)) = \begin{cases} n, & \text{if } 3 \leq n \leq 6 \\ 7, & \text{if } n \geq 7 \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices on the cycle C_n and $u_i, 1 \leq i \leq n$ be the pendant vertex attached at v_i in $C_n \circ K_1$. Let x_i and y_i be the duplicating vertices of u_i and $v_i, 1 \leq i \leq n$ respectively.

In $SP(C_n \circ K_1)$, the maximum degree is 6 and the number of vertices with maximum degree 6 is n. So $\phi(SP(C_n \circ K_1)) \leq n$ when $3 \leq n \leq 6$ and $\phi(SP(C_n \circ K_1)) \leq 7$ when $n \geq 7$.

Assume that $n \geq 9$. Assign the colours for the vertices of $SP(C_n \circ K_1)$ as follows.

For $1 \leq i \leq n$,

$$C(u_i) = i + 1 (\text{mod } 7)$$

$$C(v_i) = i - 1 (\text{mod } 7)$$

$$C(x_i) = \begin{cases} i + 2, & 1 \leq i \leq 3 \\ i - 4 (\text{mod } 7), & 4 \leq i \leq n \end{cases}$$

and

$$C(y_i) = \begin{cases} 5, & \text{for } i = 1 \\ 6, & \text{for } i = 2, 3 \\ i - 4 (\text{mod } 7), & \text{for } 4 \leq i \leq n - 1 \\ 5, & \text{for } i = n. \end{cases}$$

Then the colouring is a proper colouring and the vertices $v_2, v_3, v_4, v_5, v_6, v_7$ and v_8 are the members of the respective colour classes of the colours 1,2,3,4,5,6 and 0 so that they are having all the remaining colours in their neighboring vertices. Thus $\phi(SP(C_n \circ K_1)) = 7$ for $n \geq 9$.

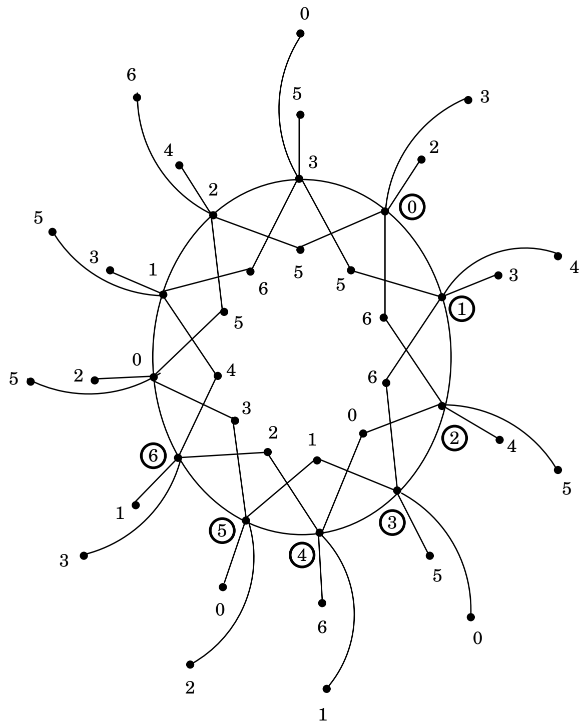
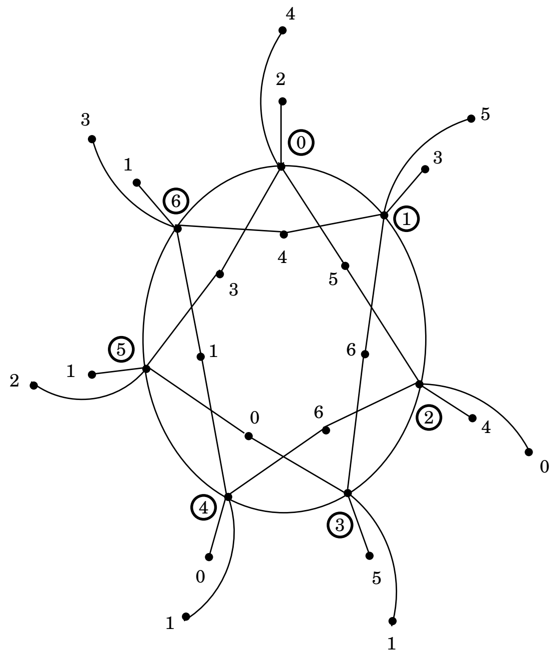


Figure 33. A b-colouring of $SP(C_{11} \circ K_1)$ with 7 colours

A b-colouring for $SP(C_n \circ K_1)$ when $n = 7$ and 8 are given in Figure 34.



(Figure 34 *Contd.*)

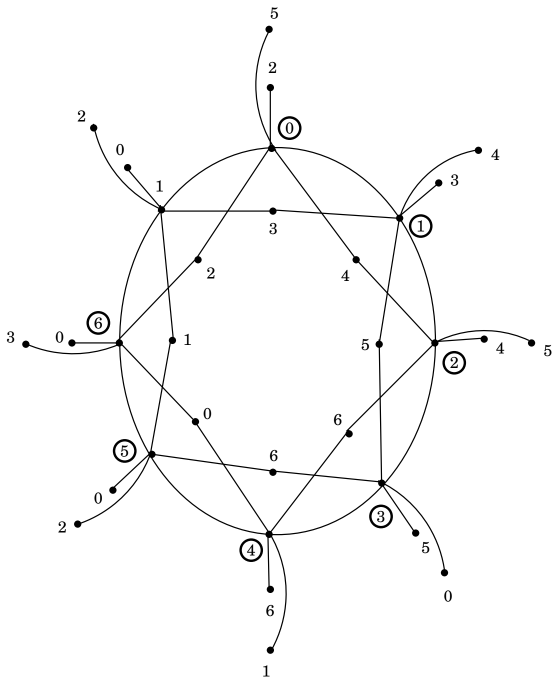
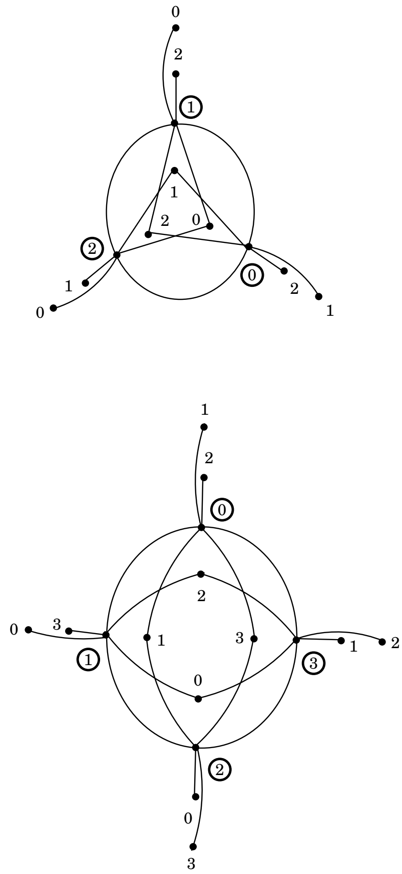


Figure 34. A b-colouring of $SP(C_n \circ K_1)$ with 7 colours for $n = 7$ and 8

Hence $SP(C_n \circ K_1) = 7$ for $n \geq 7$. A b-colouring of $C_n \circ K_1$ with n colours when $3 \leq n \leq 6$ are given in Figure 35.



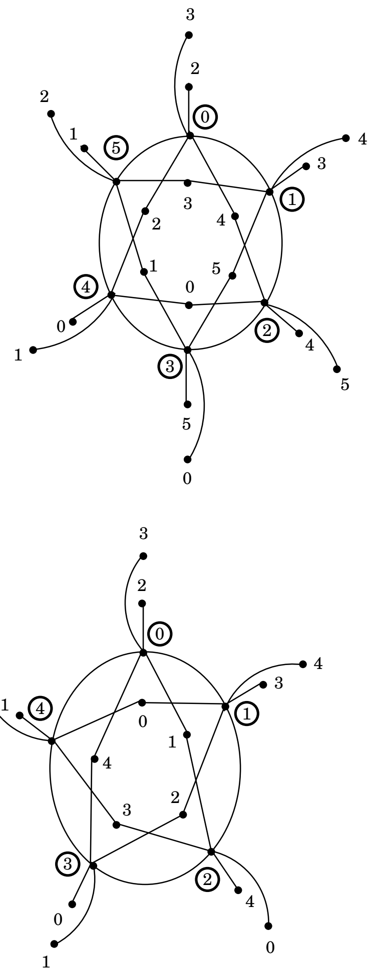


Figure 35. A b-colouring of $SP(C_n \circ K_1)$ with n colours for $3 \le n \le 6$

for reading manuscript critically. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

References

[1] S. Arockiaraj, P. Mahalakshmi and P. Namasivayam, Odd sum labeling on splitting graphs, *Util. Math.*, (accepted)

[2] N. Alon and B. Mohar, The chromatic number of graph power, *Combinatorics Probability and Computing* **11** (1993), 1–10.

[3] I.W. Irving and D.F. Manlove, The b-chromatic number of a graph, *Discrete Appl. Math.* **91** (1999), 127–141.

[4] J. Kratochvil, Zs. Tuza and M. Voigt, On the b-chromatic number of graphs, *Lecturer Notes in Computer Science*, Springer Berlin **2573** (2002), 310–320.

[5] M. Kouider, *b-chromatic number of a graph, subgraphs, degrees*, Res. Rep. 1392, LRI, Univ. Orsay, France, 2004.

[6] R. Javedi and B. Omoomi, On b-coloring of Cartesian product of graphs, *Ars Combin.* **107** (2012), 521–536.

[7] E. Sampathkumar and H.B. Walikar, On splitting graph of a graph, *J. Karnatak Univ. Sci.* **25** (13) (1980), 13–16.

[8] K. Thilagavathi, D. Vijayalakshmi and N. Roopesh, b-coloring of central graphs, *International Journal of Computer Applications* **3** (11) (2010), 27–29.

[9] Vivin. J. Vernold and M. Venkatachalam, The b-chromatic number of corona graphs, *Util. Math.* **88** (2012), 299–307.