Journal of Informatics and Mathematical Sciences
Vol. 6, No. 2, pp. 87–92, 2014
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Total Domination Polynomial of A Graph

Research Article

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Abstract. A total domination polynomial of a graph G of order n is the polynomial $D_{td}(G,x) = \sum_{t=\gamma_{td}(G)}^{n} d_{td}(G,t)x^t$, where $d_{td}(G,t)$ is the number of total dominating sets of G of cardinality t. In this paper, we present various properties of total domination polynomial of graph G. Also determine the total domination polynomial of some graph operations.

Keywords. Graph; Domination number; Sign domination number

MSC. 05C69; 05E15

Received: December 19, 2014 **Accepted:** December 27, 2014

1. Introduction

All the graphs $G = (V,E)$ considered here are simple, finite, nontrivial and undirected, where $|V| = n$ denotes number of vertices and $|E| = m$ denotes number of edges of G. Let $V = V_1 \cup V_2$, where V_1 and V_2 are two partitions of the vertex set of G. The number of distinct subsets with r vertices that can be selected from a set with n vertices is denoted by $\binom{n}{r}$ or $nC_r = \frac{n!}{(n-r)!r!}$. This number $\binom{n}{r}$ is called a binomial coefficient. For any undefined term in this paper, we refer Harary [6].

A set $D \subseteq V$ is a dominating set if every vertex not in D is adjacent to one or more vertices in D. The minimum cardinality taken over all dominating sets in G is called domination number and is denoted by $\gamma(G)$. The concept of domination has existed and studied for a long time. Book on domination [7] has stimulated sufficient inspiration leading to the expansive growth of this field.

A set S of vertices in a graph G is a total dominating set of a graph G if every vertex of G is adjacent to some vertex in S. The total domination number of a graph G, denoted by

$\gamma_{td}(G)$ is the minimum cardinality of total dominating set of G. Total domination in graphs was introduced by Cockayne et al. [4]. For more details on total domination, we refer [8], [9] and [10].

A domination polynomial of a graph G is the polynomial $D(G,x) = \sum_{t=\gamma(G)}^{n} d(G,t)x^t$, where $d(G,t)$ is number of dominating sets of G of cardinality t. Domination polynomial was initiated by Arocha et al. [3] and later developed by Alikhani et al. [1] and [2].

Analogously, we define total domination polynomial as follows: A total domination polynomial of a graph G of order n is the polynomial $D_{td}(G,x) = \sum_{t=\gamma_{td}(G)}^{n} d_{td}(G,t)x^t$, where $d_{td}(G,t)$ is the number of total dominating sets of G of cardinality t.

The nullity $\eta = \eta(D(G,x))$ of domination polynomial of a graph G is the multiplicity of the number zero. For further information on this parameter refer [5]. Let ξ denote number of roots of a graph polynomial.

2. Results

Theorem 2.1. *For any connected graph G with $n \geq 2$, the nullity of total domination polynomial of G, $\eta(D_t(G,x)) \geq 2$.*

Proof. Let S be total dominating set such that $|S| = \gamma_{td}(G)$. As $\langle S\rangle$ should not have isolated vertices, there should be at least two vertices in S adjacent to each other. As t ranges from $\gamma_{td}(G)$ to n, the minimum degree of x in G is greater than or equal to two. Also, every term of $D_{td}(G,x)$ has an x in it. If $D_{td}(G,x) = 0$, then $x = 0$ is of multiplicity greater than or equal to 2. □

Theorem 2.2. *For any graph G,*

$$\xi(D_{td}(G,x)) = \begin{cases} \xi(D(G,t)) & \text{if } G \text{ is connected graph with } n \geq 2, \\ \text{does not exist} & \text{if } G \text{ is totally disconnected graph.} \end{cases}$$

Proof. Let G be a nontrivial connected graph with n vertices. Degree of domination polynomial of G is n implies $\xi(D(G,x)) = n$. A set S with all n vertices of G forms a total dominating set. That is degree of total domination polynomial is n. Hence $\xi(D_{td}(G,x)) = n$.

If G is totally disconnected graph, then there is no total dominating set which implies total domination polynomial does not exist. Hence number of roots of total domination polynomial does not exist. □

To prove our next result, we use the following definition:

Let G be any graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Add n new vertices $\{u_1, u_2, \ldots, u_n\}$ and join u_i to v_i for $1 \leq i \leq n$. Let it be denoted as $G \circ K_1$.

Lemma 2.1. *For any graph G with n vertices,*

$$\gamma_{td}(G \circ K_1) = n.$$

Proof. The set $\{v_1, v_2, \dots, v_n\}$ is total dominating set as they dominate $\{u_1, u_2, \dots, u_n\}$. Hence $\gamma_{td}(G \circ K_1) = n$. □

Theorem 2.3. *For any graph G with n vertices,*

$$D_{td}(G \circ K_1, x) = x^n[(1+x)^n - 1].$$

Proof. Let G be a graph with vertex set $\{v_1, v_2, \dots, v_n\}$. Then, $G \circ K_1$ has $2n$ vertices. The set $S = \{v_1, v_2, \dots, v_n\}$ dominates vertices of G and vertices $\{u_1, u_2, \dots, u_n\}$, where as the set $S_1 = \{u_1, u_2, \dots, u_n\}$ is not a total dominating set as $\langle S_1 \rangle$ has isolates. For this set S, including vertices of S_1 one by one will be total dominating sets. Hence, select $(t-n)$ vertices out of n vertices of S_1, which can be done in nC_{t-n} ways. The total domination polynomial of $G \circ K_1$ is

$$\begin{aligned} D_{td}(G \circ K_1, x) &= \sum_{t=n}^{2n} d_{td}(G \circ K_1, t)x^t \\ &= x^n + nC_1 x^{n+1} + nC_2 x^{n+2} + \cdots + x^{2n} \\ &= x^n[(1+x)^n - 1]. \end{aligned}$$

□

Theorem 2.4. *For a complete graph K_n with $n \geq 2$ vertices,*

$$D_{td}(K_n, x) = \sum_{t=2}^{n} nC_t x^t.$$

Proof. Consider a complete graph K_n with $n \geq 2$ vertices, for which $\gamma_{td}(K_n) = 2$ and $t \in \{2, 3, \dots, n\}$. The number of total dominating set of cardinality t is nC_t. Hence the result follows. □

Theorem 2.5. *For a complete bipartite graph $G \cong K_{r,s}$ with $r, s \geq 2$ vertices,*

$$D_{td}(G, x) = D_{td}(K_{r+s}, x) - D_{td}(K_r, x) - D_{td}(K_s, x).$$

Proof. Let $|V_1| = r$ and $|V_2| = s$. A set consisting of one vertex from V_1 and another vertex from V_2 forms a total dominating set for G. Hence $\gamma_{td}(G) = 2$ and $t \in \{2, 3, \dots, r+s\}$. The number of ways of selecting t vertices from $(r+s)$ vertices of G is $(r+s)C_t$.

But all t vertices cannot be selected from the same set V_1 (or V_2) as vertices within sets V_1 and V_2 are not adjacent. Hence number of total dominating set of cardinality t is $(r+s)C_t - rC_t - sC_t$. From above Theorem,

$$\begin{aligned} D_{td}(G, x) &= \sum_{t=2}^{r+s} (r+s)C_t x^t - \sum_{t=2}^{r} rC_t x^t - \sum_{t=2}^{s} sC_t x^t \\ &= \sum_{t=2}^{r+s} d_{td}(K_{r+s}, t)x^t - \sum_{t=2}^{r} d_{td}(K_r, t)x^t - \sum_{t=2}^{s} d_{td}(K_s, t)x^t \\ &= D_{td}(K_{r+s}, x) - D_{td}(K_r, x) - D_{td}(K_s, x). \end{aligned}$$

□

Observation 2.1. *If G_1 and G_2 are nontrivial connected graphs, then*

$$\eta(D_{td}(G_1 + G_2, x)) = 2.$$

Theorem 2.6. *Let G_1 and G_2 be two connected graphs without isolated vertices. Then*

$$D_{td}(G_1+G_2,x)=D_{td}(K_{|V_1|,|V_2|},x)+D_{td}(G_1,x)+D_{td}(G_2,x),$$

where $V(G_1)=V_1$ and $V(G_2)=V_2$.

Proof. In a graph G_1+G_2, vertices of G_1 are adjacent to all vertices of G_2 and vice versa. Thus, one vertex of V_1 and another vertex of V_2 forms a total dominating set. So, $\gamma_{td}(G_1+G_2)=2$ and $t\in\{2,3,\dots,|V_1|+|V_2|\}$. A total dominating set of graph G_1+G_2 of cardinality t can be obtained by selecting j vertices from V_1 and $(t-j)$ vertices from V_2. Number of total dominating set is same as the number of total dominating set in complete graph $K_{|V_1|,|V_2|}$. Since G_1 and G_2 are connected, all t vertices can be selected from V_1, provided a set with t vertices in graph G_1 is total dominating set. Number of total dominating set is $d_{td}(G_1,t)$. Similarly, all t vertices can be selected from V_2, provided a set with t vertices in graph G_2 is total dominating set. Number of total dominating set is $d_{td}(G_2,t)$.

$$\begin{aligned}D_{td}(G_1+G_2,x)&=\sum_{t=2}^{|V_1|+|V_2|}d_{td}(K_{|V_1|,|V_2|},t)x^t+\sum_{t=\gamma_{td}(G_1)}^{|V_1|}d_{td}(G_1,t)x^t+\sum_{t=\gamma_{td}(G_2)}^{|V_2|}d_{td}(G_2,t)x^t\\&=D_{td}(K_{|V_1|,|V_2|},x)+D_{td}(G_1,x)+D_{td}(G_2,x).\end{aligned}$$ □

Theorem 2.7. *Let T be a tree with $n\ge 3$ vertices out of which l are leaves. Then*

$$D_{td}(T,x)=\begin{cases}x^{n-l}(1+x)^l & \text{if } n-l\ge 2,\\ x^{n-l}[(1+x)^l-1] & \text{if } n-l=1.\end{cases}$$

Proof. Consider a tree T with n vertices out of which l are leaves. Let S be a total dominating set. As any two leaves are not adjacent to each other, a set S with all leaf vertices is not a total dominating set. Since parent vertices dominates leaf vertices of T, a set with all parent vertices forms a total dominating set. Number of parent vertices in T is $n-l$.

(i) If $\gamma_{td}(T)=n-l\ge 2$, inclusion of leaf vertices in this set will still be total dominating set. Hence t ranges from $n-l$ to n. The total domination polynomial of T implies that

$$D_{td}(T,x)=\sum_{t=n-l}^{n}d_{td}(T,t)x^t=x^{n-l}+lC_1x^{n-l+1}+lC_2x^{n-l+2}+\ldots+x^n.$$

(ii) If $n-l=1$, then the set with only one parent vertex cannot be total dominating set. Hence t ranges from $n-l+1$ to n. This implies that

$$D_{td}(T,x)=lC_1x^{n-l+1}+lC_2x^{n-l+2}+\ldots+x^l.$$

Thus, the results follow. □

Observation 2.2. *If G_1, G_2,$\dots$,G_k are nontrivial connected graphs, then*

$$\eta(D_{td}(G_1\cup G_2\cup\ldots\cup G_k,x))=\sum_{i=1}^{k}\gamma_{td}(G_i).$$

Theorem 2.8. *Let G_1, G_2,...,G_k be nontrivial connected graphs. Then*

$$D_{td}(G_1 \cup G_2 \cup \ldots \cup G_k, x) = \prod_{i=1}^{k} D_{td}(G_i, x).$$

Proof. We shall prove this by mathematical induction. For $k = 1$, it is vacuously true. For $k = 2$, let $\gamma_{td}(G_1)$ and $\gamma_{td}(G_2)$ be a total domination number of graphs G_1 and G_2 respectively. Then for graph G_1, $t \in \{\gamma_{td}(G_1), \gamma_{td}(G_1)+1, \ldots, |V(G_1)|\}$ and for graph G_2, $t \in \{\gamma_{td}(G_2), \gamma_{td}(G_2)+1, \ldots, |V(G_2)|\}$. The total domination number for graph $G_1 \cup G_2$ is $\gamma_{td}(G_1)+\gamma_{td}(G_2)$ and $t \in \{\gamma_{td}(G_1)+\gamma_{td}(G_2), \gamma_{td}(G_1)+\gamma_{td}(G_2)+1, \ldots, |V(G_1)|+|V(G_2)|\}$. To select t vertices from vertex set of graph $G_1 \cup G_2$, select j vertices from $V(G_1)$, where $j \in \{\gamma_{td}(G_1)+1, \ldots, |V(G_1)|\}$ and $(t-j)$ vertices from $V(G_2)$, where $(t-j) \in \{\gamma_{td}(G_2), \gamma_{td}(G_2)+1, \ldots, |V(G_2)|\}$. The number of total dominating sets in $G_1 \cup G_2$ is equal to the coefficient of x^t in $D_{td}(G_1,x)D_{td}(G_2,x)$. Hence the coefficient of x^t in $D_{td}(G_1 \cup G_2)$ and $D_{td}(G_1)D_{td}(G_2)$ are equal. Thus result is true for $k = 2$.

Now assume the result to be true of $k-1$ nontrivial connected graphs, that is $D_{td}(G_1 \cup G_2 \cup \ldots \cup G_{k-1}, x) = \prod_{i=1}^{k-1} D_{td}(G_i, t)$.

We shall prove the result for k nontrivial connected graphs.

$$\begin{aligned} D_{td}(G_1 \cup G_2 \cup \ldots \cup G_k, x) &= D_{td}(G_1 \cup G_2 \cup \ldots \cup G_{k-1}, x) D_{td}(G_k, x) \\ &= \prod_{i=1}^{k-1} D_{td}(G_i, x) D_{td}(G_k, x) \\ &= \prod_{i=1}^{k} D_{td}(G_i, x). \end{aligned}$$ □

3. Conclusion

The two fundamental parameters among all domination related parameters are the domination number and the total domination number. To dominate a graph, every vertex not in the dominating set is adjacent to at least one vertex in the set, while to totally dominate a graph, every vertex is adjacent to a vertex in the set. The main aim of this article is to initialize the study of the total domination polynomial, which gives algebraic information about the graphs.

Acknowledgement

The authors would like to express their gratitude to referee for his/her careful reading and helpful comments.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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