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A Note on k-Gamma Function and Pochhammer k-Symbol

Research Article

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Abstract. In this note, we discuss some extended results involving the Pochhammer's symbol and express the multiple factorials in terms of the said symbol. We prove the k-analogue of Vandermonde's theorem which contains the binomial theorem as a limiting case. Also, we introduce some limit formulae involving the k-symbol and prove the k-analogue Gauss multiplication and Legendere's duplication theorems by using these formulae.

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1. Introduction

The factorial notation (!) was introduced by Christian Kramp in 1808 for positive integers and is frequently used to compute the binomial coefficients. When x is any positive real number, the problem was solved in 1729 by Euler, who defined the generalized factorial function which is now called the gamma function. The relationship between Euler gamma function and ordinary factorial function is $\Gamma(n) = (n-1)!$, n is a positive integer. On the other hand, the gamma function is defined for all real numbers except $n = 0, -1, -2, \dots$. Here, we begin with a simpler generalization of $n!$ called a shifted factorial and named as Appells symbol (see [1])

$$(\alpha, n) = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1). \tag{1.1}$$

This product of n factors, beginning with any complex number α and increasing by unit steps, as a special case $(\alpha, 0) = 1$ and $(1, n) = n!$. The product was studied by James Stirling (1730). Afterwords, the German mathematician Leo Pochhammer defined shifted (rising) factorial, which was named as Pochhammer's symbol and is denoted by $(\alpha)_n$ used more widely for the same quantity. The Pochhammer's symbol can be expressed in terms of Euler gamma function by the following relation (1.2) which has more fundamental importance (see [2]).

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=(\alpha)_n. \tag{1.2}$$

The Pochhammer's symbol $\left(\frac{1}{2}\right)_n$ is a rational number for all integers n, but in limiting case for large n, it has remarkable connections with irrational numbers π and e. The first of the connections was formed by Johan Wallies at Oxford in 1656 given by

$$\frac{\pi}{2}=\frac{2.2.4.4.6.6\ldots}{1.3.3.5.5.7\ldots}, \tag{1.3}$$

which can be written in the form of Pochhammer's symbol as

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{(n!)^2}{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n}, \tag{1.4}$$

and also $\frac{\pi}{2}$ is the first positive root of the trigonometric equation $\cos\theta=0$, so

$$\lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)_n\sqrt{n}}{(1)_n}=\frac{1}{\pi^{\frac{1}{2}}}. \tag{1.5}$$

If n is very large positive integer, then computation of $n!$ is tedious. An easy technique of computing an approximate value was introduced by Stirling (1730) and modified by De Moiver, which is given as

$$n!=(1)_n\sim(2\pi n)^{\frac{1}{2}}\left(\frac{n}{e}\right)^n, \tag{1.6}$$

where e is the irrational number and symbol $\sim$ shows the ratio of the two sides approaches to unity as $n\to\infty$. The connection between e and Pochhammer's symbol for large values of n is given by

$$\left(\frac{1}{2}\right)_n\sim(2)^{\frac{1}{2}}\left(\frac{n}{e}\right)^n. \tag{1.7}$$

2. Pochhammer's symbol and gamma function

Definition 2.1. For $\alpha\in\mathbb{C}$ and a non-negative integer n, the Pochhammer's symbol is defined by

$$(\alpha)_n=\begin{cases}\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), & n\in\mathbb{N}\\ 1, & n=0,\alpha\neq 0.\end{cases} \tag{2.1}$$

Remarks. From the above definition, we conclude $(\alpha)_n=(\alpha+n-1)(\alpha)_{n-1}$. For $\alpha\neq 1,2,\ldots,n$, the above definition becomes $(\alpha)_{-n}=\frac{1}{(\alpha-1)(\alpha-2)\ldots(\alpha-n)}$. Also, we see that$(-n)_m=0$ if n, m are integers and $0 \le n < m$. For example, $(-3)_5 = (-3)(-2)(-1)(0)(1) = 0$ and consequently a series sometimes terminates after a finite number of terms. So, consider the binomial series

$$(1-x)^{-a} = 1 + ax + a(a+1)\frac{x^2}{2!} + \ldots = \sum_{m=0}^{\infty} (a)_m \frac{x^m}{m!}, \quad |x| < 1. \tag{2.2}$$

If $a = -n$ is a negative integer, all the coefficients $(-n)_m$ with $n < m$ become zero and the series (2.2) terminates reducing to the binomial theorem.

Proposition 2.2. *Let the complex number α and the integers m and n be such that both sides of the following equations are satisfied, then we have addition formula, reflection formula and the duplication formula respectively as*

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n, \tag{2.3}$$

$$(\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha)_n}, \tag{2.4}$$

$$(2\alpha)_{2n} = 2^{2n}(\alpha)_n\left(\alpha+\frac{1}{2}\right)_n. \tag{2.5}$$

Remarks. Above three results are proved in [3] in the form of Appell's symbol. The use of (2.3) and (2.4) occurs in the sums like $\sum_{m=0}^{n} f(m,n)$. A factor of the form $\frac{1}{(a)_m}$ or $\frac{1}{(a)_{n-m}}$ can be changed into $(-1)^m(b)_{n-m}$ or $(-1)^m(b)_m$ respectively after multiplied by a quantity which does not depend upon m. The more explicit case is

$$\frac{1}{(a)_{n-m}} = \frac{1}{(a)_n(a+n)_{-m}} = \frac{(-1)^m(1-a-n)_m}{(a)_n}, \tag{2.6}$$

where $b = 1-a-n$ does not involve the summation index. In the second case, it is useful way to add and subtract n i.e.,

$$\frac{1}{(a)_m} = \frac{1}{(a)_{n-n+m}} = \frac{1}{(a)_n(a+n)_{-n+m}} = \frac{(-1)^{n-m}(1-a-n)_{n-m}}{(a)_n}. \tag{2.7}$$

When we are concerned with the convergence, it will be useful to have an inequality for Pochhammer's symbol. If n is a non-negative integer, then

$$|(a)_n| = |a(a+1)(a+2)\ldots(a+n-1)| = |a||a+1||a+2|\ldots|a+n-1|$$

$$|(a)_n| \le |a|(|a|+1)(|a|+2)\ldots(|a|+n-1) \Rightarrow |(a)_n| \le (|a|)_n. \tag{2.8}$$

If $a \in \mathbb{C}$, the Pochhammer's symbol is related to the binomial coefficients as

$$\binom{a}{0} = 1 \text{ and } \binom{a}{n} = \frac{a(a-1)(a-2)\ldots(a-n+1)}{n!} = \frac{(-1)^n(-a)_n}{n!}, \quad \text{for } n > 0 \tag{2.9}$$

and if n is negative integer, $\frac{1}{n!} = 0$ because for every $n \in \mathbb{Z}$, the relation $\frac{n}{n!} = \frac{1}{(n-1)!}$ is preserved. Therefor, the relation $(a)_n$ remains useful for negative integers n and $\binom{a}{n}$ can only be taken to vanish (not usually defined).

Now, we give the generalized version of the addition formula (2.3) and product formula (2.5).

Lemma 2.3. *Let the complex number α and the integers $m_1, m_2, \dots, m_n$ satisfy the conditions of the following relations, then generalized form of the addition formula is*

$$(\alpha)_{m_1+m_2+\dots+m_n} = (\alpha)_{m_1}(\alpha+m_1)_{m_2}\dots(\alpha+m_1+m_2+\dots+m_{n-1})_{m_n} \tag{2.10}$$

and for $r \in \mathbb{N}$, the multiplication formula is given by

$$(r\alpha)_{rn} = rrn(\alpha)_n\left(\alpha+\frac{1}{r}\right)_n\left(\alpha+\frac{2}{r}\right)_n\dots\left(\alpha+\frac{r-1}{r}\right)_n = r^{rn}\prod_{s=0}^{r-1}\left(\alpha+\frac{s}{r}\right)_n. \tag{2.11}$$

Proof. Use the definition of Pochhammer's symbol to obtain the desired proof. □

Corollary 2.4. *In terms of gamma function, the above addition formula* (2.10) *can be written as*

$$(\alpha)_{m_1+m_2+\dots+m_n} = \frac{\Gamma(\alpha+m_1+\dots+m_n)}{\Gamma(\alpha)}. \tag{2.12}$$

Proof. Applying the relation (1.2) on R.H.S of the relation (2.10), we proceed as

$$\frac{\Gamma(\alpha+m_1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+m_1+m_2)}{\Gamma(\alpha+m_1)}\dots\frac{\Gamma(\alpha+m_1+\dots+m_n)}{\Gamma(\alpha+m_1+\dots+m_{n-1})} = \frac{\Gamma(\alpha+m_1+\dots+m_n)}{\Gamma(\alpha)}.$$ □

Corollary 2.5. *If α is not multiple of any natural number, then another form of the multiplication formula* (2.11) *is given by*

$$(\alpha)_{rn} = r^{rn}\left(\frac{\alpha}{r}\right)_n\left(\frac{\alpha+1}{r}\right)_n\left(\frac{\alpha+2}{r}\right)_n\dots\left(\frac{\alpha+r-1}{r}\right)_n = r^{rn}\prod_{s=0}^{r-1}\left(\frac{\alpha+s}{r}\right)_n. \tag{2.13}$$

Proof. Apply the definition of Pochhammer's symbol and rearrange the terms to get the required result. □

Definition 2.6. If $n = -2, -1, 0, \dots$, the double and triple factorials are defined in [4] as

$$n!! = \begin{cases} n(n-2)(n-4)\dots 6.4.2 & \text{if } n \text{ is even} \\ n(n-2)(n-4)\dots 5.3.1 & \text{if } n \text{ is odd} \\ 1, & \text{if } n = 0, -1\ ;\ (-2n)!! = \infty,\ n \in \mathbb{N} \end{cases} \tag{2.14}$$

and

$$n!!! = \begin{cases} n(n-3)(n-6)\dots 9.6.3 & \text{if } n \text{ is of the form } 3n \\ n(n-3)(n-6)\dots 8.5.2 & \text{if } n \text{ is of the form } (3n-1) \\ n(n-3)(n-6)\dots 7.4.1 & \text{if } n \text{ is of the form } (3n-2) \\ 1, & \text{if } n = 0, -1, -2\ ;\ (-3n) = \infty,\ n \in \mathbb{N}. \end{cases} \tag{2.15}$$

Now, we establish a relationship between Pochhammer's symbol and multiple factorials.

Proposition 2.7. *Using the above definitions, the higher order factorials can be expressed in terms of Pochhammer's symbol as*

$$(2n)!! = 2^n(1)_n\,,\quad (2n-1)!! = 2^n\left(\frac{1}{2}\right)_n\,,\quad (2n+1)!! = 2^n\left(\frac{3}{2}\right)_n \tag{2.16}$$

and

$$(3n)!! = 3^n(1)_n\,,\quad (3n-1)!! = 3^n\left(\frac{2}{3}\right)_n\,,\quad (3n-2)!! = 3^n\left(\frac{1}{3}\right)_n\,. \tag{2.17}$$

Proof. As given earlier that $n! = (1)_n$ and $\left(\frac{1}{2}\right)_n = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\dots\left(n-\frac{1}{2}\right)$. Thus, if n is even,

$$n!! = (2n)!! = 2n(2n-2)(2n-4)\dots 6.4.2 = 2^n n! = 2^n(1)_n,$$

if n is odd, i.e. of the form $2n-1$,

$$n!! = (2n-1)!! = (2n-1)(2n-3)\dots 5.3.1 = 2^n\left(n-\frac{1}{2}\right)\dots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = 2^n\left(\frac{1}{2}\right)_n,$$

if n is odd of the form $2n+1$, then

$$n!! = (2n+1)!! = (2n+1)(2n-1)\dots 5.3 = 2^n\left(n+\frac{1}{2}\right)\dots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right) = 2^n\left(\frac{3}{2}\right)_n.$$

Similarly, we have the results for triple factorials. If n is of the form $3n$,

$$n!!! = (3n)!!! = 3n(3n-3)(3n-6)\dots 9.6.3 = 3^n n! = 3^n(1)_n,$$

if n is of the form $(3n-1)$,

$$n!!! = (3n-1)!!! = (3n-1)(3n-4)\dots 8.5.2 = 3^n\left(n-\frac{1}{3}\right)\dots\left(\frac{8}{3}\right)\left(\frac{5}{3}\right)\left(\frac{2}{3}\right) = 3^n\left(\frac{2}{3}\right)_n,$$

and if n is of the form $(3n-2)$, then

$$n!!! = (3n-2)!!! = (3n-2)(3n-5)\dots 7.4.1 = 3^n\left(n-\frac{2}{3}\right)\dots\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)\left(\frac{1}{3}\right) = 3^n\left(\frac{1}{3}\right)_n. \qquad \square$$

Remarks. The above results can be generalized up to finite number of higher order factorials. If r is any natural number, the r factorials, denoted by $n!_r$, means !!!… r-times [4]. In terms of Pochhammer's symbol, it can be expressed as

$$n!_r = \begin{cases} r^n(1)_n & \text{if } n \text{ is of the form } rn \\ r^n\left(\frac{r-1}{r}\right)_n & \text{if } n \text{ is of the form } (rn-1) \\ r^n\left(\frac{r-2}{r}\right)_n & \text{if } n \text{ is of the form } (rn-2) \\ \vdots & \\ r^n\left(\frac{1}{r}\right)_n & \text{if } n \text{ is of the form } (rn-(r-1)). \end{cases} \tag{2.18}$$

Limit Formulae 2.8. The Euler gamma function can be obtained from Pochhammer's symbol by a limiting process. With the help of this symbol, we can move to the several important properties of gamma function. Here, we give some limit formulae given in [2] that will be helpful in our future work.

(i) Let $x \in \mathbb{R}$, $x > 0$ and $b + x \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, then

$$\lim_{x\to\infty} \frac{\Gamma(a+x)}{\Gamma(b+x)} x^{b-a} = 1 \tag{2.19}$$

(ii) Let $x \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ and n is a non-negative integer, then

$$\Gamma(x) = \lim_{n\to\infty} \frac{(1)_n}{(x)_n} n^{x-1} \tag{2.20}$$

(iii) From the relations (1.2) and (2.19) we can prove that

$$\lim_{n\to\infty} \frac{(a)_n}{(b)_n} n^{b-a} = \frac{\Gamma(b)}{\Gamma(a)} \tag{2.21}$$

(iv) After replacing a, b and n by $2a$, $2b$ and $2n$ respectively in the equation (2.21), we have

$$\frac{\Gamma(2b)}{\Gamma(2a)} = \frac{2^{2b}\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)}{2^{2a}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)}, \tag{2.22}$$

and setting $a = \frac{1}{2}$, $b = x$ in (2.22) implies

$$\Gamma(2x) = 2^{2x-1}(\pi)^{-1/2}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right),$$

which is the Legendre's duplication formula.

Remarks. The formula (2.20) is often attributed to Gauss, but it is only a variant of Euler's infinite product

$$\Gamma(x) = \frac{1}{x}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{x}\left(1+\frac{x}{n}\right)^{-1}, \quad x \in \mathbb{C} \setminus \{0, -1, -2, \dots\}. \tag{2.23}$$

Lemma 2.9. *If $r - 1 \in \mathbb{N}$, then we have*

$$\Gamma\left(\frac{1}{r}\right)\Gamma\left(\frac{2}{r}\right)\Gamma\left(\frac{3}{r}\right)\dots\Gamma\left(\frac{r-1}{r}\right) = r^{-\frac{1}{2}}(2\pi)^{\frac{(r-1)}{2}}. \tag{2.24}$$

Proof. In the relation (2.20), replace x by $\left(\frac{1}{r}, \frac{2}{r}, \dots, \frac{r-1}{r}, \left(\frac{r}{r} = 1\right)\right)$ and multiply all the results. Then use $(1)_n = n!$ and $(1)_{rn} = (rn)!$, in the Stiriling formula (1.6) for $n!$ and $(rn)!$ to reach the required proof. □

Lemma 2.10. *If $r - 1 \in \mathbb{N}$, and $ra, rb \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, then we have*

$$\frac{\Gamma(rb)}{\Gamma(ra)} = \frac{r^{rb}\Gamma(b)\Gamma\left(b+\frac{1}{r}\right)\Gamma\left(b+\frac{2}{r}\right)\dots\Gamma\left(b+\frac{r-1}{r}\right)}{r^{ra}\Gamma(a)\Gamma\left(a+\frac{1}{r}\right)\Gamma\left(a+\frac{2}{r}\right)\dots\Gamma\left(a+\frac{r-1}{r}\right)}. \tag{2.25}$$

Proof. Replacing a,b and n by ra,rb and rn respectively in (2.21) along with the Lemma 2.9 and relation (2.19), we have the proof. □

Corollary 2.11. *If we set $b=x$ and $a=\frac{1}{r}$ in the above lemma, we get*

$$\frac{\Gamma(rx)}{\Gamma(1)}=\frac{r^{rx}\Gamma(x)\Gamma\left(x+\frac{1}{r}\right)\Gamma\left(x+\frac{2}{r}\right)\ldots\Gamma\left(x+\frac{r-1}{r}\right)}{r\Gamma\left(\frac{1}{r}\right)\Gamma\left(\frac{2}{r}\right)\Gamma\left(\frac{3}{r}\right)\ldots\Gamma\left(\frac{r-1}{r}\right)\Gamma(1)}.$$

Using the Lemma 2.9 in the denominator, we have

$$\Gamma(rx)=r^{rx-\frac{1}{2}}(2\pi)^{\frac{(1-r)}{2}}\Gamma(x)\Gamma\left(x+\frac{1}{r}\right)\ldots\Gamma\left(x+\frac{r-1}{r}\right)$$

which is the Gauss multiplication theorem valid for $rx\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$ and if $r=2$, it will be the Legendre's duplication formula.

3. k-Pochhammer's Symbol and k-Gamma Function

Recently, Diaz and Pariguan [5] introduced the generalized k-gamma function as

$$\Gamma_k(x)=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\quad k>0,\ x\in\mathbb{C}\setminus kZ^- \tag{3.1}$$

and also gave the properties of said function. The Γ_k is one parameter deformation of the classical gamma function such that $\Gamma_k\to\Gamma$ as $k\to1$. The Γ_k is based on the repeated appearance of the expression of the following form

$$\alpha(\alpha+k)(\alpha+2k)(\alpha+3k)\ldots(\alpha+(n-1)k). \tag{3.2}$$

The function of the variable α given by the statement (3.2), denoted by $(\alpha)_{n,k}$, is called the Pochhammer k-symbol. We obtain the usual Pochhammer symbol $(\alpha)_n$ by taking $k=1$. This product of n factors, beginning with any complex number α and increasing each step by k, as a special case $(\alpha)_{0,k}=1$, and $(\alpha)_{n,k}=(\alpha+nk-k)(\alpha)_{n-1,k}$. Also, the above definition becomes $(\alpha)_{-n,k}=\frac{1}{(\alpha-k)(\alpha-2k)\ldots(\alpha-nk)}$ for $\alpha\neq k,2k,\ldots,nk$ and a link between k-gamma function and k-pochhammer's symbol is given by

$$(x)_{n,k}=\frac{\Gamma_k(x+nk)}{\Gamma_k(x)}. \tag{3.3}$$

The definition given in relation (3.1), is the generalization of $\Gamma(x)$ and the integral form of Γ_k is given by

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt,\quad \mathrm{Re}(x)>0. \tag{3.4}$$

From relation (3.4), we can easily show that

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right). \tag{3.5}$$

Also, the researchers [6–11] have worked on the generalized k-gamma function and discussed the following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x) \tag{3.6}$$

$$\Gamma_k(k) = 1, \qquad k > 0 \tag{3.7}$$

$$\Gamma_k(x) = a^{\frac{x}{k}} \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}a} dt, \qquad a \in \mathbb{R} \tag{3.8}$$

$$\Gamma_k(\alpha k) = k^{\alpha-1}\Gamma(\alpha), \qquad k > 0,\ \alpha \in \mathbb{R} \tag{3.9}$$

$$\Gamma_k(nk) = k^{n-1}(n-1)!, \qquad k > 0,\ n \in \mathbb{N} \tag{3.10}$$

$$\Gamma_k\left((2n+1)\frac{k}{2}\right) = k^{\frac{2n-1}{2}} \frac{(2n)!\sqrt{\pi}}{2^n n!}, \qquad k > 0,\ n \in \mathbb{N}. \tag{3.11}$$

In [12], it is proved that gamma function $\Gamma(z)$ is analytic on $\mathbb{C}$ except the poles at $z = 0, -1, -2, \dots$ and the residue at $z = -n$ is equal to $\frac{(-1)^n}{n!}$, $n = 0, 1, 2, \dots$. Recently, Mubeen et al. [13] proved that for $k > 0$, the function $\Gamma_k(x)$ is analytic on $\mathbb{C}$, except the single poles at $x = 0, -k, -2k, \dots$ and the residue at $x = -nk$ is $\frac{1}{(-1)^n k^n n!}$.

Proposition 3.1. *If $(\alpha)_n$ and $(\alpha)_{n,k}$ shows the Pochhammer's symbol and k-Pochhammer's symbol respectively, then we have*

$$(\alpha)_{n,k} = k^n \left(\frac{\alpha}{k}\right)_n. \tag{3.12}$$

Proof. From the relation (3.2), we have

$$\begin{aligned} (\alpha)_{n,k} &= \alpha(\alpha+k)(\alpha+2k)(\alpha+3k)\dots(\alpha+(n-1)k) \\ &= k^n \left(\frac{\alpha}{k}\right)\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)\dots\left(\frac{\alpha}{k}+n-1\right) \\ \Rightarrow \qquad (\alpha)_{n,k} &= k^n \left(\frac{\alpha}{k}\right)_n. \end{aligned}$$

□

Remarks. From the above conclusion, we see that $(k)_{n,k} = k^n(1)_n = k^n n!$ and $(1)_{n,k} = k^n \left(\frac{1}{k}\right)_n$.

Theorem 3.2. *Let the complex number α and the integers m and n be such that both sides of the following equations are satisfied, then for $k > 0$, we have*

$$(\alpha)_{m+n,k} = (\alpha)_{m,k}(\alpha+mk)_{n,k} \qquad (\textit{addition formula}) \tag{3.13}$$

$$(\alpha)_{-n,k} = \frac{(-1)^{-nk}}{(k-\alpha)_{n,k}} \tag{3.14}$$

$$(2\alpha)_{2n,k} = 2^{2n}(\alpha)_{n,k}\left(\alpha+\frac{k}{2}\right)_{n,k} \qquad (\textit{multiplication formula}). \tag{3.15}$$

Proof. To prove the addition formula, we use the definition of k-Pochhammer symbol on R.H.S of the equation (3.13) and obtain

$$(\alpha)_{m,k}(\alpha+mk)_{n,k} = \alpha(\alpha+k)\dots(\alpha+(m-1)k)(\alpha+mk)\dots(\alpha+mk+(n-1)k) = (\alpha)_{m+n,k}.$$

Now, we prove the relation (3.14). For $n = 0$, the case is trivial and if n is a positive integer, by definition of k-Pochhammer symbol, we have

$$(\alpha)_{-n,k} = \frac{1}{(\alpha-k)(\alpha-2k)\dots(\alpha-nk)} = \frac{(-1)^{-nk}}{(k-\alpha)_{n,k}},$$

and if n is negative integer i.e. $n = -N$. We apply the preceding result on R.H.S of the relation (3.14) as

$$\frac{(-1)^{-nk}}{(k-\alpha)_{n,k}} = \frac{(-1)^{Nk}}{(k-\alpha)_{-N,k}} = (k-k+\alpha)_{N,k} = (\alpha)_{-n,k},$$

which completes the proof. To prove the multiplication formula (3.15), we proceed as

$$(2\alpha)_{2n,k} = (2\alpha)(2\alpha+k)\dots(2\alpha+(n-1)k)(2\alpha+nk)\dots(2\alpha+(2n-2)k)(2\alpha+(2n-1)k).$$

Separating the even and odd terms and taking common 2^n from each group, we get

$$(2\alpha)_{2n,k} = 2^n(\alpha)(\alpha+k)\dots(\alpha+(n-1)k)\cdot 2^n\left(\alpha+\frac{k}{2}\right)\left(\alpha+\frac{k}{2}+k\right)\dots\left(\alpha+\left(n-\frac{1}{2}\right)k\right)$$

which implies that

$$(2\alpha)_{2n,k} = 2^{2n}(\alpha)_{n,k}\left(\alpha+\frac{k}{2}\right)_{n,k}.$$ □

Theorem 3.3. *For $k > 0$, let the complex number α and the integers $m_1, m_2, \dots, m_n$ satisfy the conditions of the following relations, then generalized form of the addition formula is given by*

$$(\alpha)_{m_1+\ldots+m_n,k} = (\alpha)_{m_1,k}(\alpha+m_1k)_{m_2,k}\dots(\alpha+k(m_1+\ldots+m_{n-1}))_{m_n,k} \tag{3.16}$$

and for $r \in \mathbb{N}$, the multiplication formula in generalized form is given by

$$(r\alpha)_{rn,k} = r^{rn}\prod_{s=0}^{r-1}\left(\alpha+\frac{sk}{r}\right)_{n,k}. \tag{3.17}$$

Proof. The procedure adopted in the proof of the Theorem 3.2 is applicable here for n terms to get the generalized result. □

To prove the multiplication formula (3.17), just use the definition of Pochhammer k-symbol as $(r\alpha)_{rn,k} = (r\alpha)(r\alpha+k)(r\alpha+2k)\dots(r\alpha+(n-1)k)(r\alpha+nk)(r\alpha+nk+k)\dots(r\alpha+(2n-1)k)(r\alpha+2n)(r\alpha+2nk+k)\dots(r\alpha+(rn-1)k)$ and rearrange the terms to get the required result.

Corollary 3.4. *The above addition formula* (3.16)*, can be written in terms of k-gamma function as*

$$(\alpha)_{m_1+m_2+\ldots+m_n,k} = \frac{\Gamma_k(\alpha+(m_1+\ldots+m_n)k)}{\Gamma_k(\alpha)}. \tag{3.18}$$

Corollary 3.5. *If α is not multiple of any natural number, then another form of the multiplication formula* (3.17) *is given by*

$$(\alpha)_{rn,k} = r^{rn}\left(\frac{\alpha}{r}\right)_{n,k}\left(\frac{\alpha+k}{r}\right)_{n,k}\cdots\left(\frac{\alpha+(r-1)k}{r}\right)_{n,k} = r^{rn}\prod_{s=0}^{r-1}\left(\frac{\alpha+sk}{r}\right)_{n,k}. \tag{3.19}$$

Theorem 3.6. *Let $n\in\mathbb{N}$ and $a,b\in\mathbb{C}$. Then for $k>0$, we have*

$$\sum_{m=0}^{n}\frac{(a)_{m,k}(b)_{n-m,k}}{m!(n-m)!} = \frac{(a+b)_{n,k}}{n!}. \tag{3.20}$$

Proof. Consider the binomial series

$$\begin{aligned}(1-x)^{-\frac{a}{k}} &= 1+\left(\frac{a}{k}\right)x+\frac{\left(-\frac{a}{k}\right)\left(-\frac{a}{k}-1\right)}{2!}(-x)^2+\frac{\left(-\frac{a}{k}\right)\left(-\frac{a}{k}-1\right)\left(-\frac{a}{k}-2\right)}{3!}(-x)^3+\dots\\ &= 1+\left(\frac{a}{k}\right)x+\frac{\left(\frac{a}{k}\right)\left(\frac{a+k}{k}\right)}{2!}x^2+\frac{\left(\frac{a}{k}\right)\left(\frac{a+k}{k}\right)\left(\frac{a+2k}{k}\right)}{3!}x^3+\dots\\ &= \sum_{m=0}^{\infty}\frac{1}{k^m}(a)_{m,k}\frac{x^m}{m!}.\end{aligned}$$

Similarly, we have

$$(1-x)^{-\frac{b}{k}} = 1+\left(\frac{b}{k}\right)x+\frac{\left(\frac{b}{k}\right)\left(\frac{b+k}{k}\right)}{2!}x^2+\dots = \sum_{m'=0}^{\infty}\frac{1}{k^{m'}}(b)_{m',k}\frac{x^{m'}}{m'!}$$

and

$$(1-x)^{-\frac{a+b}{k}} = \sum_{n=0}^{\infty}\frac{1}{k^n}(a+b)_{n,k}\frac{x^n}{n!}.$$

By substituting these values in $(1-x)^{-\frac{a}{k}}(1-x)^{-\frac{b}{k}} = (1-x)^{-\frac{(a+b)}{k}}$, we get

$$\sum_{m=0}^{\infty}\frac{1}{k^m}(a)_{m,k}\frac{x^m}{m!}\sum_{m'=0}^{\infty}\frac{1}{k^{m'}}(b)_{m',k}\frac{x^{m'}}{m'!} = \sum_{n=0}^{\infty}\frac{1}{k^n}(a+b)_{n,k}\frac{x^n}{n!},$$

where the summation extends over all nonnegative integers m and m' whose sum is n and $m'=n-m$. Thus, we have

$$\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\frac{(a)_{m,k}(b)_{m',k}}{m!m'!}x^n = \sum_{n=0}^{\infty}(a+b)_{n,k}\frac{x^n}{n!},$$

and equating the coefficients of x^n, we get

$$\sum\frac{(a)_{m,k}(b)_{n-m,k}}{m!(n-m)!} = \frac{(a+b)_{n,k}}{n!}.$$ □

Corollary 3.7. *Theorem* 3.6 *contains the binomial theorem as a limiting case.*

Proof. If we set $a = cx$, then definition of Pochhammer k-symbol implies

$$(cx)_{m,k} = cx(cx+k)(cx+2k)\dots(cx+(m-1)k).$$

Dividing both sides by c^m and taking limit $c \to \infty$, we get

$$\frac{(cx)_{m,k}}{c^m} = \frac{cx}{c}\frac{(cx+k)}{c}\frac{(cx+2k)}{c}\dots\frac{(cx+(m-1)k)}{c} \to x^m.$$

Similarly, by setting $b = cy$, we have

$$\frac{(cy)_{n-m,k}}{c^{n-m}} \to y^{n-m} \quad \text{and} \quad \frac{(cx+cy)_{n,k}}{c^n} \to (x+y)^n.$$

Now, Theorem 3.6 becomes

$$\sum_{m=0}^{n} \frac{x^m y^{n-m}}{m!(n-m)!} = \frac{(x+y)^n}{n!} \quad \Rightarrow \quad (x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m},$$

which is the usual form of the binomial theorem. □

Remarks. An important summation formula was proved by Vandermonde (1772 [14]). The Chines mathematician Chu published a less general form of the theorem in 1303 [15], we will call it Vandermonde's theorem and for convenience will use the same name to designate an extension to multiple sums. The above Theorem 3.6 is the k-analogue of Vandermonde's theorem and it contains the binomial theorem as a limiting case.

Theorem 3.8. *If $a_1, a_2, \dots, a_n$ are complex numbers and $k > 0$, then we have the generalized form of multiplication theorem as*

$$\sum \frac{(a_1)_{m_1,k}(a_2)_{m_2,k}\dots(a_r)_{m_r,k}}{m_1!m_2!\dots m_r!} = \frac{(a_1+a_2+\dots+a_r)_{n,k}}{n!} \tag{3.21}$$

where the summation extends over all non negative integers $m_1, m_2, \dots, m_r$ whose sum is n.

Proof. Consider the binomial series as in previous theorem, so using the results in $(1-x)^{-\frac{a_1}{k}}(1-x)^{-\frac{a_2}{k}}\dots(1-x)^{-\frac{a_r}{k}} = (1-x)^{-\frac{(a_1+a_2+\dots+a_r)}{k}}$, we get

$$\frac{1}{k^{m_1+m_2+\dots+m_r}} \sum_{m_1=0}^{\infty}(a_1)_{m_1,k}\frac{x^{m_1}}{m_1!}\dots\sum_{m_r=0}^{\infty}(a_r)_{m_r,k}\frac{x^{m_r}}{m_r!} = \frac{1}{k^n}\sum_{n=0}^{\infty}(a_1+\dots+a_r)_{n,k}\frac{x^n}{n!}.$$

By comparing the coefficients of x^n, we get the desired result. □

Remarks. The above theorem is the k-analogue of Vandermondes's theorem with multiple sums. If $k = 1$, we have the classical Vandermonde's theorem with multiple sums. Also, the Vandermonde's theorem contains the multi nomial theorem as a limiting case which can be expressed in the form of the relation (3.22) (by setting $a_i = cx_i$, $i = 1, 2, \dots, r$ in the Theorem 3.8)

$$(x_1+x_2+\dots+x_r)^n = \sum \frac{n!}{m_1!m_2!\dots m_r!} x_1^{m_1} x_2^{m_2}\dots x_r^{m_r}, \tag{3.22}$$

where the summation extends over all nonnegative integers $m_1, m_2, \dots, m_r$ whose sum is n.

Lemma 3.9. *Let I be an interval in $\mathbb{R}$ and let the functions $f : I \to \mathbb{R}^+$ and $g : I \to \mathbb{C}$ satisfy the following conditions*

(i) *f attains its maximum at a point y in the interior of I, the supremum of f in any closed interval not containing y is strictly less than $f(y)$ and there is a neighborhood of y in which f'' exist, continuous and strictly negative,*

(ii) *g is continuous at y and $g(y) \neq 0$,*

(iii) *f and g are Lebesgue measurable and there exist $r \in \mathbb{R}$ such that $|g|f'$ is integrable on I, then for $x \in \mathbb{R}$ and $x \to \infty$ (see [16–18])*

$$\int_I g(t)[f(t)]^x dt \sim g(y)[f(y)]^{x+\frac{1}{2}} \left[\frac{-2\pi}{xf''(y)}\right]^{\frac{1}{2}}. \tag{3.23}$$

The k-gamma function can be obtained from the Pochhammer k-symbol by a limiting process. With the help of these limit formulas, we can prove several important properties of k-gamma function. Here we introduce some k-analogue limit formulae that will be helpful in proving our coming results.

Theorem 3.10. *If $n \in \mathbb{N}$, $k > 0$ and $a + nk, b + nk \in \mathbb{C} \setminus \{0, -k, -2k, \ldots\}$, then*

$$\lim_{n\to\infty} \frac{\Gamma_k(a+nk)}{\Gamma_k(b+nk)}(nk)^{\frac{b}{k}-\frac{a}{k}} = 1. \tag{3.24}$$

Proof. Using the integral form of k-gamma function (3.4), we have

$$\Gamma_k(a+nk) = \int_0^\infty \tau^{a+nk-1} e^{-\frac{\tau^k}{k}} d\tau.$$

Setting $\frac{\tau^k}{k} = nt \Rightarrow \tau = (nkt)^{1/k}$ and above equation becomes

$$\Gamma_k(a+nk)(nk)^{\frac{-a}{k}-n} = \frac{1}{k}\int_0^\infty t^{\frac{a}{k}-1}(te^{-t})^n dt.$$

As te^{-t} has a single maximum at $t = 1$ and $t^{\frac{a}{k}-1}$ is continuous at that point, so for large n, the value of the integral can be estimated by Lemma 3.9. Thus, we have

$$\Gamma_k(a+nk)(nk)^{\frac{-a}{k}-n} \sim \frac{1}{k}(1)^{\frac{a}{k}-1}h(n) = h(n), \quad n \to \infty \tag{3.25}$$

and

$$\Gamma_k(b+nk)(nk)^{\frac{-b}{k}-n} \sim \frac{1}{k}(1)^{\frac{b}{k}-1}h(n) = h(n), \quad n \to \infty. \tag{3.26}$$

Dividing the equation (3.25) by (3.26), we have the required proof. □

Corollary 3.11. *If $n - 1 \in \mathbb{N}, k > 0$ and $x \in \mathbb{C} \setminus \{0, -k, -2k, \ldots\}$, then*

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{(k)_{n,k}}{(x)_{n,k}}(nk)^{\frac{x}{k}-1}. \tag{3.27}$$

Proof. Using the relation (3.3), we have

$$\frac{(a)_{n,k}}{(b)_{n,k}} = \frac{\Gamma_k(a+nk)}{\Gamma_k(a)} \frac{\Gamma_k(b)}{\Gamma_k(b+nk)}.$$

Multiplying both sides by $n^{\frac{b}{k}-\frac{a}{k}}$, taking limit $n \to \infty$ and using the Theorem 3.10, we get

$$\lim_{n\to\infty} \frac{(a)_{n,k}}{(b)_{n,k}} (nk)^{\frac{b}{k}-\frac{a}{k}} = \lim_{n\to\infty} \frac{\Gamma_k(a+nk)}{\Gamma_k(b+nk)} (nk)^{\frac{b}{k}-\frac{a}{k}} \frac{\Gamma_k(b)}{\Gamma_k(a)} = \frac{\Gamma_k(b)}{\Gamma_k(a)}. \tag{3.28}$$

Setting $a = k$ and $b = x$ with $\Gamma_k(k) = 1$, we approaches our result. □

Remarks. If we use $x = \frac{k}{2}$, we find the important result $\Gamma_k\left(\frac{k}{2}\right) = \sqrt{\frac{\pi}{k}}$ which is a conclusion of the relation (3.11). Also, if $k = 1$, we have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ proved [12].

Theorem 3.12. *If $r - 2 \in \mathbb{N}$, $k > 0$, then we have*

$$\Gamma_k\left(\frac{k}{r}\right)\Gamma_k\left(\frac{2k}{r}\right)\Gamma_k\left(\frac{3k}{r}\right)\ldots\Gamma_k\left(\frac{(r-1)k}{r}\right) = k^{\frac{(1-r)}{2}} r^{-\frac{1}{2}} (2\pi)^{\frac{(r-1)}{2}}. \tag{3.29}$$

Proof. Replacing x by $\frac{k}{r}, \frac{2k}{r}, \ldots, \frac{(r-1)k}{r}$ and $\frac{rk}{r}$ in the relation (3.5) respectively, we have

$$\Gamma_k\left(\frac{k}{r}\right) = k^{\frac{1}{r}-1}\Gamma\left(\frac{1}{r}\right), \quad \Gamma_k\left(\frac{2k}{r}\right) = k^{\frac{2}{r}-1}\Gamma\left(\frac{2}{r}\right)$$

$$\Gamma_k\left(\frac{(r-1)k}{r}\right) = k^{\frac{r-1}{r}-1}\Gamma\left(\frac{r-1}{r}\right) \quad \ldots \quad \Gamma_k\left(\frac{rk}{r}\right) = k^{1-1}\Gamma(1) = 1.$$

Multiplying all above equations and applying the Lemma 2.9, we get

$$\begin{aligned}
\Gamma_k\left(\frac{k}{r}\right)\Gamma_k\left(\frac{2k}{r}\right)\ldots\Gamma_k\left(\frac{(r-1)k}{r}\right) &= k^{\frac{1}{r}+\frac{2}{r}+\ldots+\frac{r-1}{r}-(r-1)}\,\Gamma\left(\frac{1}{r}\right)\Gamma\left(\frac{2}{r}\right)\ldots\Gamma\left(\frac{r-1}{r}\right) \\
&= k^{\frac{1+2+3+\ldots+(r-1)}{r}-(r-1)} r^{-\frac{1}{2}} (2\pi)^{\frac{(r-1)}{2}} \\
&= k^{\frac{(1-r)}{2}} r^{-\frac{1}{2}} (2\pi)^{\frac{(r-1)}{2}}.
\end{aligned}$$

□

Theorem 3.13. *If $r - 1 \in \mathbb{N}$, $k > 0$, and $ra, rb \in \mathbb{C} \setminus \{0, -k, -2k, \ldots\}$, then we have*

$$\frac{\Gamma_k(rb)}{\Gamma_k(ra)} = \frac{r^{\frac{rb}{k}}\Gamma_k(b)\Gamma_k\left(b+\frac{k}{r}\right)\Gamma_k\left(b+\frac{2k}{r}\right)\ldots\Gamma_k\left(b+\frac{(r-1)k}{r}\right)}{r^{\frac{ra}{k}}\Gamma_k(a)\Gamma_k\left(a+\frac{k}{r}\right)\Gamma_k\left(a+\frac{2k}{r}\right)\ldots\Gamma_k\left(a+\frac{(r-1)k}{r}\right)}. \tag{3.30}$$

Proof. Replacing a, b and n by ra, rb and rn respectively in the relation (3.28) along with the relation (3.17), we obtain

$$\begin{aligned}
\frac{\Gamma_k(rb)}{\Gamma_k(ra)} &= \lim_{n\to\infty} \frac{(ra)_{rn,k}}{(rb)_{rn,k}} (rnk)^{\frac{rb}{k}-\frac{ra}{k}} \\
&= \lim_{n\to\infty} \frac{(a)_{n,k}\left(a+\frac{k}{r}\right)_{n,k}\left(a+\frac{2k}{r}\right)_{n,k}\cdots\left(a+\frac{(r-1)k}{r}\right)_{n,k}}{(b)_{n,k}\left(b+\frac{k}{r}\right)_{n,k}\left(b+\frac{2k}{r}\right)_{n,k}\cdots\left(b+\frac{(r-1)k}{r}\right)_{n,k}} (rnk)^{\frac{rb}{k}-\frac{ra}{k}},
\end{aligned}$$

which is equivalent to

$$= \frac{r^{\frac{rb}{k}}}{r^{\frac{ra}{k}}} \lim_{n\to\infty} \frac{(a)_{n,k}}{(b)_{n,k}} (nk)^{\frac{b}{k}-\frac{a}{k}} \lim_{n\to\infty} \frac{\left(a+\frac{k}{r}\right)_{n,k}}{\left(b+\frac{k}{r}\right)_{n,k}} (nk)^{\frac{b}{k}-\frac{a}{k}} \dots \lim_{n\to\infty} \frac{\left(a+\frac{(r-1)k}{r}\right)_{n,k}}{\left(b+\frac{(r-1)k}{r}\right)_{n,k}} (nk)^{\frac{b}{k}-\frac{a}{k}}.$$

By using (3.28), the proof will be completed. □

Corollary 3.14. *Setting $b = x$ and $a = \frac{k}{r}$ in the above theorem, we have*

$$\frac{\Gamma_k(rx)}{\Gamma_k(k)} = \frac{r^{\frac{rx}{k}} \Gamma_k(x)\Gamma_k\left(x+\frac{k}{r}\right)\Gamma_k\left(x+\frac{2k}{r}\right)\dots\Gamma_k\left(x+\frac{(r-1)k}{r}\right)}{r\Gamma_k\left(\frac{k}{r}\right)\Gamma_k\left(\frac{2k}{r}\right)\Gamma_k\left(\frac{3k}{r}\right)\dots\Gamma_k\left(\frac{(r-1)k}{r}\right)\Gamma_k(k)},$$

and use of the Theorem 3.12 along with $\Gamma_k(k) = 1$ implies

$$\Gamma_k(rx) = \frac{r^{\frac{rx}{k}-1} \Gamma_k(x)\Gamma_k\left(x+\frac{k}{r}\right)\Gamma_k\left(x+\frac{2k}{r}\right)\dots\Gamma_k\left(x+\frac{(r-1)k}{r}\right)}{k^{\frac{(1-r)}{2}} r^{-\frac{1}{2}} (2\pi)^{\frac{(r-1)}{2}}}$$

which is equivalent to

$$\Gamma_k(rx) = r^{\frac{rx}{k}-\frac{1}{2}} k^{\frac{(r-1)}{2}} (2\pi)^{\frac{(1-r)}{2}}\ \Gamma_k(x)\ \Gamma_k\left(x+\frac{k}{r}\right)\ \Gamma_k\left(x+\frac{2k}{r}\right)\ \dots\ \Gamma_k\left(x+\frac{(r-1)k}{r}\right).$$

Remarks. The above Corollary is the k-analogue of Gauss multiplication theorem. If we use $r = 2$, we have k-analogue of Legendre duplication formula proved in [6]. Also, if $k = 1$, we have the classical Gauss multiplication and Legendre duplication Theorems [2].

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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