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The Sum and Product of Independence Numbers of Graphs and Their Line Graphs

Research Article

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Abstract. The bounds on the sum and product of chromatic numbers of a graph and its complement are known as Nordhaus-Gaddum inequalities. In this paper, we study the bounds on the sum and product of the independence numbers of graphs and their line graphs. We also provide a new characterization of the certain graph classes.

Keywords. Independence number; Matching number; Line graph

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1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [10]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

Many problems in extremal graph theory seek the extreme values of graph parameters on families of graphs. The classic paper of Nordhaus and Gaddum [6] study the extreme values of the sum (or product) of a parameter on a graph and its complement, following solving these problems for the chromatic number on n-vertex graphs. In this paper, we study such problems for some graphs and their associated graphs.

Definition 1.1 ([5]). A *Walk*, $W = v_0e_1v_1e_2v_2\dots v_{k-1}e_kv_k$, in a graph G is a finite sequence whose terms are alternately vertices and edges such that, for $1 \le i \le k$, the edge e_i has ends v_{i-1} and v_i.

Definition 1.2 ([5])**.** If the vertices $v_0, v_1, \dots, v_k$ of a walk W are distinct then W is called a *Path*. A path with n vertices will be denoted by P_n. P_n has length $n-1$.

Definition 1.3 ([4])**.** Two vertices that are not adjacent in a graph G are said to be *independent*. A set S of vertices is independent if any two vertices of S are independent. The *vertex independence number* or simply the *independence number*, of a graph G, denoted by $\alpha(G)$ is the maximum cardinality among the independent sets of vertices of G.

Definition 1.4 ([2])**.** A subset M of the edge set of G, is called a *matching* in G if no two of the edges in M are adjacent. In other words, if for any two edges e and f in M, both the end vertices of e are different from the end vertices of f.

Definition 1.5 ([2])**.** A *perfect matching* of a graph G is a matching of G containing $n/2$ edges, the largest possible, meaning perfect matchings are only possible on graphs with an even number of vertices. A perfect matching sometimes called a *complete matching* or 1-*factor*.

Definition 1.6 ([2])**.** The matching number of a graph G, denoted by $\nu(G)$, is the size of a maximal independent edge set. It is also known as *edge independence number*. The matching number $\nu(G)$ satisfies the inequality $\nu(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.
Equality occurs only for a perfect matching and graph G has a perfect matching if and only if $|G| = 2\,\nu(G)$, where $|G| = n$ is the vertex count of G.

Definition 1.7 ([2])**.** A *maximum independent set* in a line graph corresponds to maximum matching in the original graph.

In this paper, we discussed the sum and product of the independence numbers of certain class of graphs and their line graphs.

2. New Results

Definition 2.1 ([12])**.** The line graph $L(G)$ of a simple graph G is the graph whose vertices are in one-one correspondence with the edges of G, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent.

Theorem 2.2 ([11])**.** The independence number of the line graph of a graph G is equal to the matching number of G.

Proposition 2.3. The sum of the independence number of a complete graph and its line graph is $\left\lfloor \frac{n}{2} \right\rfloor + 1$ and their product is $\left\lfloor \frac{n}{2} \right\rfloor$.

Proof. The independence number of a complete graph K_n on n vertices is 1, since each vertex is joined with every other vertex of the graph G. By theorem 2.2, the independence number of the line graph of K_n is the matching number of $K_n = \left\lfloor \frac{n}{2} \right\rfloor$.

Therefore

$$\alpha(K_n) + \alpha(L(K_n)) = \left\lfloor \frac{n}{2} \right\rfloor + 1$$

and

$$\alpha(K_n)\cdot\alpha(L(K_n)) = \left\lfloor \frac{n}{2} \right\rfloor. \qquad \square$$

Proposition 2.4. For a bipartite graph $B_{m,n}$,

$$\alpha(B_{m,n}) + \alpha(L(B_{m,n})) = m + n$$

and

$$\alpha(B_{m,n})\cdot\alpha(L(B_{m,n})) = mn.$$

Proof. Without the loss of generality, let $m < n$. The independence number of a bipartite graph, $\alpha(B_{m,n}) = \max(m,n) = n$ and that of its line graph, $\alpha(L(B_{m,n}))$ = matching number of $B_{m,n} = \nu(B_{m,n}) = \min(m,n) = m$.

Therefore,

$$\alpha(B_{m,n}) + \alpha(L(B_{m,n})) = m + n$$

and

$$\alpha(B_{m,n})\cdot\alpha(L(B_{m,n})) = mn. \qquad \square$$

Definition 2.5 ([10])**.** For $n \geq 3$, a *wheel graph* W_{n+1} is the graph $K_1 + C_n$. A wheel graph W_{n+1} has $n+1$ vertices and $2n$ edges.

Theorem 2.6. For a wheel graph W_{n+1}, $n \geq 3$, $\alpha(W_{n+1}) + \alpha(L(W_{n+1})) = 2\left\lfloor \frac{n}{2} \right\rfloor$ and $\alpha(W_{n+1})\cdot\alpha(L(W_{n+1})) = \left(\left\lfloor \frac{n}{2} \right\rfloor\right)^2$.

Proof. Let I be the maximal independent set of a wheel graph W_{n+1}. By definition a wheel graph is defined to be the graph $K_1 + C_n$. If $K_1 \in I$, no vertex of C_n can be in I. Hence let $K_1 \notin I$.

***Case* 1:** If n is even, C_n is an even cycle. Then $C_n = v_1, v_2, \ldots, v_n, v_1$. Without loss of generality, choose v_1 to I. Since v_2 is adjacent to v_1, $v_2 \notin I$. Now choose v_3 to I, since it is not adjacent to v_1. Now v_4 cannot be selected to I, since it is adjacent to v_3. Proceeding in this way, finite number of times, the vertices of the form v_i, $i = 1,3,5,\ldots,n-1$ belong to I. That is, $\alpha(C_n) = \frac{n}{2}$.

***Case* 2:** If n is odd, C_n is an odd cycle. Then $C_n = v_1, v_2, \ldots, v_n, v_1$. Without loss of generality, choose v_1 to I. Since v_2 is adjacent to v_1, $v_2 \notin I$. Now choose v_3 to I, since it is not adjacent to v_1. Now v_4 cannot be selected to I, since it is adjacent to v_3. Proceeding in this way, finite number of times, the vertices of the form v_i, $i = 1,3,5,\ldots,n-2$ belong to I. That is, $\alpha(C_n) = \frac{n-1}{2}$.

From the above two cases, it is clear that the independence number of a wheel graph W_{n+1} is $\left\lfloor \frac{n}{2} \right\rfloor$.

Now, consider the line graph of the wheel graph W_{n+1}. By theorem 2.2, the independence number of $L(W_{n+1})$ is equal to the matching number of W_{n+1}.

Let M be a maximal matching set of the wheel graph W_{n+1}. Then $\alpha(L(W_{n+1})) = \nu(W_{n+1})$. Let $e_1, e_2, \dots, e_n$ be the edges the outer cycle taken in the order of the wheel graph W_{n+1} and let $e'_1, e'_2, \dots, e'_n$ be the edges incident on the vertex of K_1.

***Case* 1:** If n is even, without loss of generality choose e_1 to the set M. Now, $e_2 \notin M$ as e_2 is adjacent to e_1. Now take the edge e_3 to M. Since e_4 is adjacent to e_3, $e_4 \notin M$. Proceeding in this manner, a finite number of times, an edge of the form e_i, $i = 1, 3, 5, \dots, n-1$ belong to M. In this case no edge of the form e'_j can be a member of M. That is, $|M| = \frac{n}{2}$.

***Case* 2:** If n is odd, an edge of the form e_i, $i = 1, 3, 5, \dots, n-2$ belong to M. Moreover there is one edge e'_j that is incident on K_1 and is not adjacent to any of the edges in M. That is, $|M| = \frac{n-1}{2}$.

From the above two cases, we follow that $\alpha(L(W_{n+1})) = \left\lfloor \frac{n}{2} \right\rfloor$

Therefore, $\alpha(W_{n+1}) + \alpha(L(W_{n+1})) = 2\left\lfloor \frac{n}{2} \right\rfloor$ and $\alpha(W_{n+1}) \cdot \alpha(L(W_{n+1})) = \left(\left\lfloor \frac{n}{2} \right\rfloor\right)^2$. □

Definition 2.7 ([9]). *Helm graphs* are graphs obtained from a wheel by attaching one pendant edge to each vertex of the cycle.

Theorem 2.8. For a helm graph H_n, $n \geq 3$, $\alpha(H_n) + \alpha(L(H_n)) = 2n+1$ and $\alpha(H_n) \cdot \alpha(L(H_n)) = n(n+1)$.

Proof. Let I be a maximal independent set of a helm graph H_n. Then, its elements are the set of all pendent vertices together with the vertex of K_1. So I consists of $n+1$ elements. There fore, $\alpha(H_n) = n+1$.

Now consider the line graph of the helm graph H_n. By theorem 2.2, the independence number of $L(H_n)$ is equal to the matching number of H_n. Let M be a maximal matching set of the helm graph H_n. Then $\alpha(L(H_n)) = \nu(H_n)$. Let $e_1, e_2, \dots, e_n$ be the pendent edges incident with the outer cycle taken in order of the helm graph H_n. Then if we take these n edges to M, it will be a maximum matching in H_n. This means the matching number of a helm graph, $\nu(H_n) = n = \alpha(L(H_n))$.

Therefore, $\alpha(H_n) + \alpha(L(H_n)) = 2n+1$ and $\alpha(H_n) \cdot \alpha(L(H_n)) = n(n+1)$. □

Definition 2.9 ([11]). Given a vertex x and a set U of vertices, an x, U-fan is a set of paths from x to U such that any two of them share only the vertex x. A U-fan is denoted by $F_{1,n}$.

Theorem 2.10. For a fan graph $F_{1,n}$,

$$\alpha(F_{1,n}) + \alpha(L(F_{1,n})) = \begin{cases} n & ;\text{ if } n \text{ is even} \\ n+1 & ;\text{ if } n \text{ is odd,} \end{cases} \quad \text{and} \quad \alpha(F_{1,n}) \cdot \alpha(L(F_{1,n})) = \begin{cases} \frac{n^2}{4} & ;\text{ if } n \text{ is even} \\ \frac{(n+1)^2}{4} & ;\text{ if } n \text{ is odd.} \end{cases}$$

Proof. Let I be a maximal independent set of a fan graph $F_{1,n}$. By the definition a fan graph is defined to be the graph $K_1 + P_n$. If $K_1 \in I$, no vertex of P_n can be in I. Hence let $K_1 \notin I$.

***Case* 1:** If n is even, then P_n is an odd path. Then, $P_n = v_1, v_2, \dots, v_n$. Without loss of generality, choose v_1 to I. Since v_2 is adjacent to v_1, $v_2 \notin I$. Now choose v_3 to I, since it is not adjacent to v_1. Now v_4 cannot be selected to I, since it is adjacent to v_3. Proceeding in this way, finite number of times, the vertices of the form v_i, $i = 1,3,5,\dots,n-1$ belong to I. That is, $\alpha(P_n) = \frac{n}{2}$.

***Case* 2:** if n is odd, then P_n is an even path. Then, $P_n = v_1, v_2, \dots, v_n$. Without loss of generality, choose v_1 to I. Since v_2 is adjacent to v_1, $v_2 \notin I$. Now choose v_3 to I, since it is not adjacent to v_1. Now, v_4 cannot be selected to I, since it is adjacent to v_3. Proceeding in this way, finite number of times, the vertices of the form v_i, $i = 1,3,5,\dots,n$ belong to I. That is, $\alpha(P_n) = \frac{n+1}{2}$.

From the above two cases it is clear that the independence number of a fan graph $F_{1,n}$ is either $\frac{n}{2}$ or $\frac{n+1}{2}$, depending on n is even or odd.

Now consider the line graph of the fan graph $F_{1,n}$. By Theorem 2.2, the independence number of $L(F_{1,n})$ is equal to the matching number of $F_{1,n}$.

Let M be a maximal matching set of the fan graph $F_{1,n}$. Then $\alpha(L(F_{1,n})) = \nu(F_{1,n})$. Let $e_1, e_2, \dots, e_{n-1}$ be the edges the outer path taken in order of the fan graph $F_{1,n}$ and let $e'_1, e'_2, \dots, e'_n$ be the edges incident with the vertex of K_1.

***Case* 1:** Without loss of generality choose e_1 to the set M. Now, since e_2 is adjacent to e_1, $e_2 \notin M$. Now take the edge e_3 to M. Since e_4 is adjacent to e_3, $e_4 \notin M$. Proceeding in this manner, a finite number of times, an edge of the form $e_i, i = 1,3,5,\dots,n-1$ belong to M. In this case no edge of the form e'_j can be a member of M. That is, $|M| = \frac{n}{2}$.

***Case* 2:** If n is odd, an edge $e_i, i = 1,3,5,\dots,n-2$ belongs to M. Moreover there is one edge e'_j that is incident on K_1 and is not adjacent to any of the edges in M. That is, $|M| = \frac{n+1}{2}$.

From the above two cases, we follow that $\alpha(L(F_{1,n})) = \nu(F_{1,n})$ is either $\frac{n}{2}$ or $\frac{n+1}{2}$ depending on n is even or odd.

Therefore For a fan graph $F_{1,n}$,

$$\alpha(F_{1,n}) + \alpha(L(F_{1,n})) = \begin{cases} n & ; \text{if } n \text{ is even} \\ n+1 & ; \text{if } n \text{ is odd,} \end{cases} \quad \text{and} \quad \alpha(F_{1,n}) \cdot \alpha(L(F_{1,n})) = \begin{cases} \frac{n^2}{4} & ; \text{if } n \text{ is even} \\ \frac{(n+1)^2}{4} & ; \text{if } n \text{ is odd.} \end{cases}$$ □

Definition 2.11 ([1, 13])**.** An $n-$sun or a *trampoline*, denoted by S_n, is a chordal graph on $2n$ vertices, where $n \geq 3$, whose vertex set can be partitioned into two sets $U = \{u_1, u_2, u_3, \dots, u_n\}$ and $W = \{w_1, w_2, w_3, \dots, w_n\}$ such that U is an independent set of G and u_i is adjacent to w_j if and only if $j = i$ or $j = i+1$ (mod n). A *complete sun* is a sun G where the induced subgraph $\langle U \rangle$ is complete.

Theorem 2.12. For S_n, $n \geq 3$, $\alpha(S_n)+\alpha(L(S_n)) = 2n$ and $\alpha(S_n)\cdot\alpha(L(S_n)) = n^2$.

Proof. Let S_n be a sun graph on $2n$ vertices. Let $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertex set of K_n and $U = \{u_1, u_2, u_3, \dots, u_n\}$ be the set of vertices attached to the edges of the outer ring of K_n. Clearly, all the vertices in U are independent and U itself is the maximum independent set in S_n. Therefore the independence number of S_n, $\alpha(S_n) = n$. Let $E_1 = \{e_1, e_2, e_3, \dots, e_n\}$ be the edge set of the outer rings of K_n. Now, corresponding to each edge of the outer ring of K_n, there exist two edges connecting its end vertices in S_n. Therefore, let $E_2 = \{e'_1, e'_2, e'_3, \dots, e'_n\}$ be the edge set in S_n such that the pair (e'_j, e'_k) corresponds to the edge e_i of the outer rings of K_n. Clearly, one edge among each pair (e'_j, e'_k) contributes to a maximal matching of S_n. That is, $\nu(S_n) = n$. Therefore, for a sun graph S_n, $n \geq 3$, $\alpha(S_n)+\alpha(L(S_n)) = 2n$ and $\alpha(S_n)\cdot\alpha(L(S_n)) = n^2$. □

Definition 2.13 ([13])**.** The n-sunlet graph is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n and is denoted by L_n.

Theorem 2.14. For a sunlet graph L_n on $2n$ vertices, $n \geq 3$, $\alpha(L_n)+\alpha(L(L_n)) = 2n$ and $\alpha(L_n)\cdot\alpha(L(L_n)) = n^2$.

Proof. Let L_n be a sunlet graph on $2n$ vertices. Let $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertex set of the cycle C_n and $U = \{u_1, u_2, u_3, \dots, u_n\}$ be the set of pendent vertices attached to the vertices of the cycle C_n. Clearly, all the vertices in U are independent and U itself is the maximum independent set in L_n. Therefore, $\alpha(L_n) = n$. Let $E_1 = \{e_1, e_2, e_3, \dots, e_n\}$ be the edge set of C_n. Now, corresponding to each edge of C_n, there exist two edges connecting its end vertices in L_n. Therefore, let $E_2 = \{e'_1, e'_2, e'_3, \dots, e'_n\}$ be the edge set in L_n such that the pair (e'_j, e'_k) corresponds to the edge e_i of the cycle C_n. Clearly, the set $E_2 = \{e'_1, e'_2, e'_3, \dots, e'_n\}$ contributes to a maximal matching of L_n. That is, $\alpha(L(L_n)) = \nu(L_n) = n$. Therefore, for a sunlet graph L_n on $2n$ vertices, $n \geq 3$, $\alpha(L_n)+\alpha(L(L_n)) = 2n$ and $\alpha(L_n)\cdot\alpha(L(L_n)) = n^2$. □

Definition 2.15 ([10])**.** The *armed crown* is a graph G obtained by adjoining a path P_m to every vertex of a cycle C_n.

Theorem 2.16. For an armed crown graph G with a path P_m and a cycle C_n,

$$\alpha(G)+\alpha(L(G)) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor \left[\frac{m+1}{2}+1\right] + \left[\frac{m-1}{2}\right]\left[n+\left\lceil \frac{n}{2}\right\rceil\right] ; & \text{if } m, n \text{ are odd} \\ mn ; & \text{otherwise,} \end{cases}$$

and

$$\alpha(G)\cdot\alpha(L(G)) = \begin{cases} \left[\left\lfloor \frac{n}{2} \right\rfloor \left[\frac{m+1}{2}\right] + \left\lceil \frac{n}{2}\right\rceil \left[\frac{m-1}{2}\right]\right] \cdot \left[\left\lfloor \frac{n}{2} \right\rfloor + n\left[\frac{m-1}{2}\right]\right] ; & \text{if } m, n \text{ are odd} \\ \frac{n^2m^2}{4} ; & \text{otherwise.} \end{cases}$$

Proof. Note that the number of vertices of P_m is m. Let $u_1, u_2, \dots, u_n$ be the vertices of the cycle C_n. Let $u_i^1, u_i^2, \dots, u_i^m$ be the vertices of the paths of length m attached with u_i, $1 \leq i \leq n$ with identification of u_i and u_i^m.

***Case* 1: (when m is even and n is even)** Since $u_1^1, u_1^2, \dots u_1^m$ be the vertices of the first path attached to the first vertex u_1 of the cycle C_n, the maximal independent set consists of exactly $\frac{m}{2}$ elements. Since there are n number of paths attached to every vertex u_i, $1 \le i \le n$ of the cycle C_n, which contributes $n\frac{m}{2}$ number of elements to the maximal independent set I. Therefore, $\alpha(G) = \frac{nm}{2}$. Let $E_1 = \{e_1, e_2, \dots, e_n\}$ be the edge set of the cycle C_n and let $E_2 = \{e_i^1, e_i^2, \dots, e_i^m\}$ be the edge set of the path P_m. Now for every u_i in C_n, there exists a path with m number of edges. Clearly, $\frac{m}{2}$ edges contributes to a maximal matching of P_m. For each n vertices u_i, $(1 \le i \le n)$ of C_n, there is a path P_m adjoined to it, so that the maximal matching in G, $\nu(G) = \frac{nm}{2}$.

Therefore, $\alpha(G) + \alpha(L(G)) = \frac{nm}{2} + \frac{nm}{2} = nm$ and $\alpha(G) \cdot \alpha(L(G)) = \frac{nm}{2} \cdot \frac{nm}{2} = \frac{n^2m^2}{4}$.

***Case* 2: (m is odd and n is even)** Since m is odd, the maximal independent set of P_m adjoined with the vertex u_1 of C_n is $\frac{(m+1)}{2}$. But the vertex u_1 is an element of the maximal independent set, u_2, the vertex adjacent to u_1 of C_n cannot be in I. So the maximal independent set from the path P_m adjoined with the vertex u_2 consists of $\frac{(m-1)}{2}$ elements. Proceeding like this, in all the paths of G, the maximal independent set corresponding to u_i, $1 \le i \le n$ of the cycle C_n, is alternately $\frac{m+1}{2}$ and $\frac{m-1}{2}$. Since there are n vertices in C_n, the independence number of G, $\alpha(G) = \frac{n}{2}\left[\frac{(m-1)}{2} + \frac{(m+1)}{2}\right] = \frac{nm}{2}$. Let $E_1 = \{e_1, e_2, \dots, e_n\}$ be the edge set of the cycle C_n and let $E_2 = \{e_i^1, e_i^2, \dots, e_i^m\}$ be the edge set of the path P_m. Now, for every u_i in C_n, there exists a path with m number of edges. Clearly, $\frac{m}{2}$ edges contributes to a maximal matching of P_m. For each n vertices u_i, $1 \le i \le n$ of C_n, there is a path P_m adjoined to it, so that the maximal matching in G, $\nu(G) = \frac{n}{2} + n[\frac{m-1}{2}] = \frac{nm}{2}$.

Therefore, $\alpha(G) + \alpha(L(G)) = \frac{nm}{2} + \frac{nm}{2} = nm$ and $\alpha(G) \cdot \alpha(L(G)) = \frac{nm}{2} \cdot \frac{nm}{2} = \frac{n^2m^2}{4}$.

***Case* 3: (m is even and n is odd)** Since m is even, and since $u_1^1, u_1^2, \dots, u_1^m$ be the vertices of the first path attached to the first vertex u_1 of the cycle C_n, the maximal independent set consists of exactly $\frac{m}{2}$ elements. Since there are n number of paths attached to every vertex u_i, $i \le 1 \le n$ of the cycle C_n, which contributes $n\frac{m}{2}$ number of elements to the maximal independent set I. Therefore $\alpha(G) = \frac{nm}{2}$. Let $E_1 = \{e_1, e_2, \dots, e_n\}$ be the edge set of the cycle C_n and let $E_2 = \{e_i^1, e_i^2, \dots, e_i^m\}$ be the edge set of the path P_m. Now, for every u_i in C_n, there exists a path with m number of edges. Clearly, $\frac{m}{2}$ edges contributes to a maximal matching of P_m. For all n vertices of the cycle C_n, there adjoined paths P_m, so that the maximal matching in G is $\nu(G) = \frac{nm}{2}$.

Therefore $\alpha(G) + \alpha(L(G)) = \frac{nm}{2} + \frac{nm}{2} = nm$ and $\alpha(G) \cdot \alpha(L(G)) = \frac{nm}{2} \cdot \frac{nm}{2} = \frac{n^2m^2}{4}$.

***Case* 4: (m is odd and n is odd)** Since m is odd, the maximal independent set I of P_m adjoined with the vertex u_1 of C_n is $\frac{(m+1)}{2}$. But the vertex u_1 is an element of the maximal independent set of P_m, u_2, the vertex adjacent to u_1 of C_n cannot be in I. So the maximal independent set from the path P_m adjoined with the vertex u_2 consists of $\frac{(m-1)}{2}$ elements. Proceeding like this, in all the paths of G, the maximal independent set corresponding to u_i, $1 \le i \le n$ of the cycle C_n, is alternately $\frac{m+1}{2}$ and $\frac{m-1}{2}$. Obviously, there are $\left\lfloor \frac{n}{2} \right\rfloor$ number of paths which contributes $\frac{m+1}{2}$ and $\left\lceil \frac{n}{2} \right\rceil$ number of paths which contributes $\frac{m-1}{2}$ number of vertices to the maximal independent set, the independence number of G, $\alpha(G) = \left\lfloor \frac{n}{2} \right\rfloor \left[\frac{m+1}{2} \right] + \left\lceil \frac{n}{2} \right\rceil \left[\frac{m-1}{2} \right]$. Let $E_1 = \{e_1, e_2, \dots, e_n\}$ be the edge set of the cycle C_n and let $E_2 = \{e_i^1, e_i^2, \dots, e_i^m\}$ be the edge set of the path P_m. Now, for every u_i in C_n, there exists a path with m number of edges. Clearly, since m is odd, $n \left[\frac{m-1}{2} \right]$ edges contributes to a maximal matching of P_m for all paths of G. Also, since n is also odd, which contributes $\left\lfloor \frac{n}{2} \right\rfloor$ number of edges to the maximal matching of G, so that the independence number of $L(G)$ is equal to the matching number of G is $\nu(G) = n \left[\frac{m-1}{2} \right] + \left\lfloor \frac{n}{2} \right\rfloor$.

Therefore,

$$\begin{aligned} \alpha(G) + \alpha(L(G)) &= \left\lfloor \frac{n}{2} \right\rfloor \left[\frac{m+1}{2} \right] + \left\lceil \frac{n}{2} \right\rceil \left[\frac{m-1}{2} \right] + \left\lfloor \frac{n}{2} \right\rfloor + n \left[\frac{m-1}{2} \right] \\ &= \left\lfloor \frac{n}{2} \right\rfloor \left[\frac{m+1}{2} + 1 \right] + \left[\frac{m-1}{2} \right] \left[n + \left\lceil \frac{n}{2} \right\rceil \right] \end{aligned}$$

and

$$\alpha(G) \cdot \alpha(L(G)) = \left[\left\lfloor \frac{n}{2} \right\rfloor \left[\frac{m+1}{2} \right] + \left\lceil \frac{n}{2} \right\rceil \left[\frac{m-1}{2} \right] \right] \cdot \left[\left\lfloor \frac{n}{2} \right\rfloor + n \left[\frac{m-1}{2} \right] \right]. \qquad \square$$

3. Conclusion

The theoretical results obtained in this research may provide a better insight into the problems involving matching number and independence number by improving the known lower and upper bounds on sums and products of independence numbers of a graph G and an associated graph of G. More properties and characteristics of operations on independence number and also other graph parameters are yet to be investigated. The problems of establishing the inequalities on sums and products of independence numbers for various graphs and graph classes still remain unsettled. All these facts highlight a wide scope for further studies in this area.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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