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Spectral Results for Operator Valued Functions

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Abstract. In this paper, the general spectral results of operator valued functions in particular holomorphic operator valued functions and operator polynomials are studied. Further the completeness results of quadratic operator polynomials are presented.

1. **Introduction**

In 1914 O. Faber studied the existence and asymptotic behaviour of eigenvalues and eigen functions. Later in 1955 R. J. Duffin considered over damped systems which leads to quadratic matrix eigenvalue problem of the form $A\lambda^2 + B\lambda + C$ with A, B, C are symmetric matrices, A, B positive, C nonnegative. His work was later extended to a more general matrix problem by E. H. Rogers [18]. In [23] the problem of the type $(A - B(\lambda))x = 0$ in a Hilbert space H, where A is non-negative compact operator and $B(\lambda)$ is a polynomial operator in λ having non-negative operator coefficients and satisfying $B(0) = 0$ is considered and studied the spectrum on the non-negative real axis. In [10] a completeness result is studied for $D(\lambda) = I + \lambda C_1 + \lambda^2 C_2$, where C_1 is a bounded self-adjoint operator and C_2 is a positive definite self-adjoint operator. Completeness results for the generalized eigenvectors of a polynomial operator are obtained in [9]. In [16] the completeness of root vectors of quadratic polynomial are obtained by an entirely different approach. In [14] the completeness of self-adjoint quadratic pencils are obtained by different method. In [12] the boundary value problem of a vibrating membrane is considered and associated with quadratic polynomial operator of the form $D(\lambda) = \lambda^2 I - \lambda A_1 - A_2$ with A_1, A_2 compact self-adjoint operators and obtained the completeness result for eigenvectors of $D(\lambda)$. The main aim of this paper is to consider some more boundary value problems of mathematical physics as in [12] and to study the spectral properties of them by associating with operator valued functions. Further the spectral results for operator valued functions are presented. The approach in this paper is to treat eigenvalue problems

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as eigenvalue problems for an operator polynomial $A(\lambda) = \lambda^l A_0 + \lambda^{l-1} A_1 + \cdots + A_l$, where A_i's $(i = 0, 1, \ldots, l)$ are linear operators in an appropriate function space. The coefficient operators A_i's of the operator polynomial are determined by the differential expression (operator) τ and the boundary conditions. Information about the eigenvalues and the eigen functions is obtained by analyzing the operator polynomial using the method of Functional Analysis.

2. **Preliminaries**

Let X be a complex Banach space and let $L(X)$ be the space of bounded linear operators on X. An operator polynomial is an operator valued function P defined on an open connected subset G of the complex plane $\mathbf{C}$ by $P(\lambda) = A_l \lambda^l + A_{l-1} \lambda^{l-1} + \cdots + A_1 \lambda + A_0$, where $A_0, A_1, \ldots, A_{l-1}, A_l$, are members of $L(X)$ and $\lambda \in G$. The set of all $\lambda \in G$ for which $P(\lambda)$ is one-one and surjective is called the resolvent set of P and is denoted by $\mathrm{Res}[P]$ or $\rho[P]$. The set of all $\lambda \in G$ for which $P(\lambda)$ is not one-one and surjective is called the spectrum of P and is denoted by $\mathrm{sp}[P]$ or $\sigma[P]$. The set of all λ in $\mathrm{sp}[P]$ for which $P(\lambda)$ is not one-one is called the point spectrum of P and is denoted by $\mathrm{psp}[P]$. An element in the $\mathrm{psp}[P]$ is called an eigenvalue of P.

2.1. *Fundamental Results for Holomorphic Operator Valued Functions*

Let X denote a complex Banach Space and let $L(X)$ denote the Banach space of bounded linear operators with domain and range in X. A function $U : G \subseteq \mathbf{C} \to L(X)$ is called an operator valued function, where G is connected subset of $\mathbf{C}$. An operator valued function U is said to be holomorphic if it has derivatives of all orders at every point in a neighborhood of it.

Theorem 2.1. *If $U(\lambda)$ with values in $L(X)$ is locally holomorphic operator valued function in an open connected subset of* **C**, *then* $\mathrm{Res}[U]$ *is an open connected set in* **C** *and $U^{-1}(\lambda)$ is locally holomorphic on* $\mathrm{Res}[U]$ *and*

$$\frac{d}{d\lambda} U^{-1}(\lambda) = -U^{-1}(\lambda) \left[\frac{d}{d\lambda} U(\lambda) \right] U^{-1}(\lambda) \tag{2.1.1}$$

Proof. The proof of this theorem can be found in [13]. □

The next theorem gives a sufficient condition for a pole of $U^{-1}(\lambda)$ to be an eigenvalue of U.

Theorem 2.2. *If $U^{-1}(\lambda)$ has a pole at γ and $U(\lambda)$ with values in $L(X)$ is locally holomorphic in an open neighborhood of γ then γ is an eigenvalue of U.*

Proof. Let $U_n = \frac{U^{(n)}}{n!}$ $(n = 0, 1, 2, \ldots)$ be the coefficients in the Taylor series expansion of the holomorphic operator U about γ then

$$U(\lambda) = \sum_{n=0}^{\infty} U_n (\lambda - \gamma)^n \tag{2.1.2}$$

Let γ be a pole of order m of U^{-1}. Let V_n, $(n = -m, -m+1, \dots)$, $V_{-m} \neq 0$ be the coefficients in the Laurent expansion of U^{-1} about γ. In a deleted neighborhood of γ, we have

$$(2.1.3) \qquad U^{-1}(\lambda) = \sum_{n=-m}^{\infty} (\lambda - \gamma)^n V_n$$

and

$$(2.1.4) \qquad I = U(\lambda)U^{-1}(\lambda)$$

substituting (2.1.2) and (2.1.3) in (2.1.4) we obtain

$$(2.1.5) \qquad I = \left\{ \sum_{n=0}^{\infty} (\lambda - \gamma)^n U_n \right\} \left\{ \sum_{n=-m}^{\infty} (\lambda - \gamma)^n V_n \right\}$$

$$(2.1.6) \qquad I = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} U_n V_{n-m-i} \right) (\lambda - \gamma)^n$$

from (2.1.6) we obtain, the coefficient of $(\lambda - \gamma)^n$ in the above expansion as

$$(2.1.7) \qquad \sum_{i=0}^{n} U_i V_{n-m-i} = \delta_{nm} I, \quad (n = 0, 1, \dots, m)$$

where δ_{nm} is the kronecker delta.

On using above equation (2.1.7) with $n = 0$ we have

$$(2.1.8) \qquad U_0 V_{-m} = 0 \quad \text{or} \quad U(\gamma)V_{-m} = 0.$$

This implies that the range of V_{-m} is contained in the null space of $U(\gamma)$ and V_{-m} is different from zero implies the range of V_{-m} contains non-zero elements. This implies γ is an eigenvalue of U. Hence the theorem. □

Let H will denote a complex Hilbert space with inner product $(\bullet, \bullet)$. If T is a linear operator acting in H with domain dense in H, then we denote by T^* the adjoint of T. The following results give a sufficient condition for a pole γ of U^{-1} to be a simple pole. Here U takes its values in $L(H)$ and is locally holomorphic in a neighborhood of γ. Let γ be a pole of U^{-1} of order $m > 1$. For $n = 0$, the equation (2.1.7) gives

$$(2.1.9) \qquad U(\gamma)V_{-m} = 0$$

Again on using (2.1.7) for $n = 1$, we get $\sum_{i=0}^{1} U_i V_{1-m-i} = 0$, i.e.,

$$U_0 V_{1-m} + U_1 V_{-m} = 0,$$

since $U_n = \frac{U^{(n)}}{n!}$, so for $n = 1$, $U_1 = U^{(1)}$, therefore

$$(2.1.10) \qquad U(\gamma)V_{-m+1} + U^{(1)}(\gamma)V_{-m} = 0$$

choose a non-zero vector x in the range of V_{-m} and write $x = V_{-m}w$. Then form (2.1.9), x is in the null space of $U(\gamma)$. From (2.1.10) and for the vector w,

$$U(\gamma)V_{-m+1}w + U^{(1)}(\gamma)V_{-m}w = 0.$$

Then on taking inner product with $x \neq 0$.

$$(U(\gamma)V_{-m+1}w, x) + (U^{(1)}(\gamma)V_{-m}w, x) = 0 \tag{2.1.11}$$

and hence on using the definition of adjoint operator to $U(\gamma)$ and $x = V_{-m}w$, we have

$$(V_{-m+1}w, U(\gamma)^*x) + (U^{(1)}(\gamma)x, x) = 0 \tag{2.1.12}$$

If $U(\gamma)^*x$ is orthogonal to $V_{-m+1}w$ for all w with the property that $x = V_{-m}w$, then $(U^{(1)}(\gamma)x, x) = 0$ with x different form zero. From this fact we can define the following theorem.

Theorem 2.3. *Let H be a Hilbert space. Let γ be a pole of U^{-1}, where U with values in $L(H)$ is locally holomorphic in a neighborhood of γ. If $U(\gamma)$ is a self-adjoint and $(U^{(1)}(\gamma)x, x) \neq 0$ for all non-zero x in the null space of $U(\gamma)$ then γ is a simple pole of U^{-1}.*

Proof. Suppose γ is not a simple pole of U^{-1}. Let us suppose that γ be a pole of U^{-1} of order $m > 1$. From the observation in the above discussion for a pole of order $m > 1$, we have from (2.1.11) and (2.1.12)

$$(U(\gamma)V_{-m+1}w, x) + (U^{-1}(\gamma)x, x) = 0$$

where $x \neq 0$ is in the range of V_{-m} with $x = V_{-m}w$. i.e.,

$$(V_{-m+1}w, U(\gamma)^*x) + (U^{(1)}(\gamma)x, x) = 0,$$

but $U(\gamma)^*x = 0$ for all x in the null space of $U(\gamma)$, so, the above equation becomes $(U^{(1)}(\gamma)x, x) = 0$ which is a contradiction to the given hypothesis of the theorem. So, our assumption is wrong. Thus γ is a simple pole of U^{-1}. Hence the theorem. □

2.2. *Fundamental Results for Operator Polynomial*

Let X denote a complex Banach Space and let $L(X)$ denote the space of bounded linear operators with domain and range in X.

Let H denote a complex Hilbert space with inner product $(\bullet, \bullet)$ then

(i) An operator T is said to positive (negative) if $(Tx, x) \geq 0$ $((Tx, x) \leq 0)$

(ii) An operator T is said to positive (negative) definite if for each $x \neq 0$ in the domain of T such that $(Tx, x) > 0$ $((Tx, x) < 0)$

Some fundamental results on resolvent and spectra of operator polynomial satisfying at least the following conditions are obtained.

(2.2.1) The coefficients are in the set of closed operators having domain and range in X.

(2.2.2) One of the coefficients has the domain **D** which is contained in the domains of the remaining coefficients.

(2.2.3) The values of the operator polynomial are closed operators with domain **D** and range in X.

Here we note that if the values of the operator polynomial are not in $L(X)$, then we can not use the results directly of section 2.1. However certain operator polynomials, not having values in $L(X)$ do have properties similar to those results will be presented in this section. Let $A(\lambda) = \lambda^l A_0 + \lambda^{l-1}A_1 + \cdots + A_l$ be an operator polynomial. For $(n = 0, 1, \ldots, l)$, define $\Delta^n A(\lambda)$ to be the operator polynomial as

(2.2.4) $$\Delta^n A(\lambda) = (D^n \lambda^l)A_0 + (D^n \lambda^{l-1})A_1 + \cdots + (D^n \lambda^n)A_{l-n}$$

where $D^n = \frac{d^n}{d\lambda^n}$. Let $\Delta^n A(\lambda)$ be the zero operator if $n > l$.

Lemma 2.4. *Let X be a complex Banach space and let $A(\lambda) = \lambda^l A_0 + \lambda^{l-1}A_1 + \cdots + A_l$ be an operator polynomial with coefficients acting in X satisfying* (2.2.1)*,* (2.2.2) *and* (2.2.3) *then*

(i) *If the leading coefficient A_0 satisfies* (2.2.2) *and is one-one and surjective then* $\mathrm{sp}[A]$ *is bounded.*

(ii) *If the constant term A_l satisfies the assumption in* (i) *then* $\mathrm{Res}[A]$ *containes a neighborhood of zero.*

Theorem 2.5. *Let X be a complex Banach space and let $A(\lambda) = \lambda^l A_0 + \lambda^{l-1}A_1 + \cdots + A_l$ be an operator polynomial with coefficients acting in X satisfying* (2.2.1)*,* (2.2.2) *and* (2.2.3)*. Suppose there exists an operator E in $L(X)$ which is one-one and onto the domain of $A(\lambda)$. Let $F(\lambda) = A(\lambda)E$, then*

(i) *the values of $F(\lambda)$ are in $L(X)$ and $F(\lambda)$ is holomorphic on* ***C****.*

(ii) $\mathrm{Res}[A] = \mathrm{Res}[F]$*,* $\mathrm{sp}[A] = \mathrm{sp}[F]$ *and* $\mathrm{psp}[A] = \mathrm{psp}[F]$.

(iii) $\mathrm{Res}[A$ *is open in* ***C****, A^{-1} is locally holomorphic on* $\mathrm{Res}[A]$ *and* $\frac{dA^{-1}}{d\lambda} = -A^{-1}(\lambda)\Delta A(\lambda)A^{-1}(\lambda)$

(iv) *poles of A^{-1} are the eigenvalues of A*

(v) *γ is a pole of $A^{-1} \Leftrightarrow \gamma$ is a pole of F^{-1}.*

Theorem 2.6. *Let H be a Hilbert space and let $A(\lambda)$ be an operator polynomial with coefficients acting in H satisfying* (2.2.1)*,* (2.2.2) *and* (2.2.3)*. Suppose there exists an operator E in $L(H)$ which is one-one and onto the domain of $A(\lambda)$. Let γ be a pole of A^{-1} in H. If $A(\gamma) \le A(\gamma)^*$ and $(\Delta A(\gamma)x, x) \ne 0$ for all non-zero x in the nullspace of $A(\gamma)$ then γ is a simple pole of A^{-1}.*

Corollary 2.7. *Let H be a complex Hilbert space and let $A(\lambda) = \lambda^l A_0 + \lambda^{l-1}A_1 + \cdots + A_l$ be an operator polynomial with coefficients acting in H satisfying* (2.2.1)*,* (2.2.2) *and* (2.2.3)*. Let the domain* ***D*** *defined by* (2.2.2) *be dense in H. Assume that the coefficients of $A(\lambda)$ satisfy $A_i \le A_i^*$, $(i = 0, 1, \ldots, l)$ and let $\gamma \ne 0$ be real and a pole of A^{-1}. Suppose there exists an operator E in $L(H)$ which is one-one and onto the domain of $A(\lambda)$. Then any one of the following conditions is sufficient to assume that γ is a simple pole of A^{-1}.*

(i) $\gamma > 0$ *and* $\pm A_n$, $(n = 0, 1, \ldots, l-1)$, *positive definite with respect to the nullspace of* $A(\lambda)$.
(ii) $\gamma < 0$ *and* l *even (odd),* $\pm A_n$ *positive definite with respect to the nullspace of* $A(\lambda)$ *for* n *even (odd) and* $\mp A_n$ *positive definite with respect to the nullspace of* $A(\lambda)$ *for* n *odd (even)* $(n = 0, 1, \ldots, l-1)$.
(iii) $\gamma > 0$, $\pm A_l$ *and* $\mp A_n$, $(n = 0, 1, \ldots, l-1)$ *positive definite with respect to the nullspace of* $A(\lambda)$
(iv) $\gamma < 0$ *and* l *even (odd),* $\pm A_l$ *and* $\mp A_n$ *for even (odd) and* $\mp A_n$ *for* n *odd (even) positive definite with respect to the nullspace of* $A(\lambda)$, $(n = 0, 1, \ldots, l-2)$.

Lemma 2.8. *Let* H *be a complex Hilbert space and let* $A(\lambda) = \lambda^2 A_0 + \lambda A_1 + A_2$ *be an operator polynomial with coefficients acting in* H *satisfying* (2.2.1), (2.2.2) *and* (2.2.3). *Let the domain* $\boldsymbol{D}$ *defined by* (2.2.2) *be dense in* H. *Let* γ *be an eigenvalue of* A, *assume* $A_i \leq A_i^*$, $(i = 0, 1, 2)$, *and hence assume* $\pm A_0$ *and* $\mp A_n$ *are positive definite with respect to the nullspace of* $A(\lambda)$. *Then*

(i) γ *is real*
(ii) *If* γ *is non-zero and a pole of* A^{-1} *and there exists an operator* E *in* $L(H)$ *which is one-one and onto* $\boldsymbol{D}$, *then* γ *is a simple pole of* A^{-1}.

3. Main Results

3.1. *General Spectral Results*

Let X be a complex Banach space and let X' denote the conjugate space of X, let $L(X)$ be the space of bounded linear operators on X. For an operator $T \in L(X)$ define the conjugate of T, which we denote by T', as the operator with domain and range in X^* with the property that $T'f' = f'T$, for $T \in X'$. We will use the notions of Fredholm operator. If T is a linear operator acting in X, let $n(T)$ denote the dimension of the null space of T and let $d(T)$ denote the dimension of the quotient space of X modulo range of T. If $n(T)$ and $d(T)$ are not both infinite then we say that T has index. The index of T which we denote by $\operatorname{ind}(T)$, is defined by $\operatorname{ind}(T) = n(T) - d(T)$. An operator T in $L(X)$ which has closed range and which has finite index is called Fredholm operator.

In this section, we examine operator valued functions $W(\lambda)$ with the property that the spectrum consists of isolated eigenvalues which are poles of the resolvent operator W^{-1}. Let $H(\lambda)$ and $J(\lambda)$ be operator valued functions defined on an open set G of the complex plane $\mathbf{C}$. Let $H(\lambda)$ take its values in $L(X)$ and $J(\lambda)$ take its values in $K(X)$ ($K(X)$ is the set of compact operators with domain X and range in X). Let $H(\lambda)$ and $J(\lambda)$ be locally holomorphic on G and we assume that $\operatorname{Res}[H] = G$. Let

$$(3.1.1) \qquad W(\lambda) = H(\lambda) + J(\lambda)$$

$W(\lambda)$ defined on G, takes its values in $L(X)$ and is locally holomorphic on G. We have the following theorem which describes about the spectrum of operator W.

Theorem 3.1. *Let $W(\lambda) = H(\lambda) + J(\lambda)$ be defined as above. The spectrum of W consists of eigenvalues, and either all λ in G are eigenvalues or there are only a finite number of eigenvalues in each compact subset of G.*

Proof. Express $W(\lambda)$ as

$$W(\lambda) = H(\lambda)(I + H^{-1}(\lambda)J(\lambda)) \tag{3.1.2}$$

H^{-1} with values in $L(X)$ is locally holomorphic on G by (2.2.1), and hence $H^{-1}(\lambda)J(\lambda)$ with values in $K(X)$ is locally holomorphic on G. Let γ be in $\mathrm{sp}[W]$. Suppose 1 is not an eigenvalue of $\lambda I + H^{-1}(\gamma)J(\gamma)$, where γ is fixed and λ is the parameter. By the classical spectral theory for compact operators it follows that 1 is in the resolvent set of $\lambda I + H^{-1}(\gamma)J(\gamma)$. Then since

$$W^{-1}(\gamma) = (I + H^{-1}(\gamma)J(\gamma))^{-1}H^{-1}(\gamma) \tag{3.1.3}$$

γ is in the $\mathrm{Res}[W]$, which is a contradiction. Thus $\gamma \in \mathrm{sp}[W] \Rightarrow 1$ is an eigenvalue of $\lambda I + H^{-1}(\gamma)J(\gamma)$ and on using (3.1.2) γ is an eigenvalue of operator W. Thus $\mathrm{sp}[W]$ consists entirely of eigenvalues of W. Further more, the set of eigenvalues of W is equal to the set of γ with the property that 1 is an eigenvalue of $\lambda I + H^{-1}(\gamma)J(\gamma)$. By Theorem 1.9 of Chapter 7 of [11] either 1 is an eigenvalue of $\lambda I + H^{-1}(\gamma)J(\gamma)$ for all γ in G or 1 is an eigenvalue of $\lambda I + H^{-1}(\gamma)J(\gamma)$ for only a finite number of γ in each compact subset of G. Hence the theorem. □

Lemma 3.2. *Let $W(\lambda) = H(\lambda) + J(\lambda)$ be defined as above. Let γ be an eigenvalue of W then*

(i) *the null spaces of $W(\gamma)$ and $I + H^{-1}(\gamma)J(\gamma)$ coincide.*
(ii) *the null spaces of $W(\gamma)'$ and $(I + H^{-1}(\gamma)J(\gamma))'$ have the same dimension.*
(iii) *$W(\gamma)$ has closed range if and only if $I + H^{-1}(\gamma)J(\gamma)$ has closed range.*

Proof. (i) From (3.1.2), we have $W(\gamma) = H(\gamma)(I + H^{-1}(\gamma)J(\gamma))$,
Since H^{-1} with values in $L(X)$ is locally holomorphic on G, so $H(\gamma)$ is one-one. Hence the result.
(ii) $W(\gamma) = H(\gamma)(I+H^{-1}(\gamma)J(\gamma))$ on using the properties of the conjugate operator we get

$$W(\gamma)' = (I + H^{-1}(\gamma)J(\gamma))'H(\gamma)' \tag{3.1.4}$$

since $H(\gamma)$ is onto, so $H(\gamma)'$ is one-one, hence the result follows (ii) from (3.1.4) as in result (i).
(iii) since $H(\gamma)$ and $H^{-1}(\gamma)$ are in $L(X)$, so result (iii) follows by using (3.1.2). □

Theorem 3.3. *Let $W(\lambda) = H(\lambda) + J(\lambda)$ be defined as above. Let γ be an eigenvalue of W then*

(i) *the null spaces of $W(\gamma)$ and $W(\gamma)'$ are finite dimensional and have the same dimension.*
(ii) *the range of $W(\gamma)$ is closed.*
(iii) *$W(\gamma)$ is a Fredholm operator with index zero.*

Proof. Assume $\lambda = 1$ is an eigenvalue. On applying the Riesz-Schauder theory to an operator $\lambda I + H^{-1}(\gamma)J(\gamma)$ with $\lambda = 1$, the null space of $I + H^{-1}(\gamma)J(\gamma)$ is finite dimensional [21, 3]. By (i) of Lemma 3.2, the nullspaces of $W(\gamma)$ and $I + H^{-1}(\gamma)J(\gamma)$ coincide. Similarly the nullspace of $(I + H^{-1}(\gamma)J(\gamma))'$ is finite dimensional and (ii) of Lemma 3.2 implies

$$n(W(\gamma)') = n((I + H^{-1}(\gamma)J(\gamma))' \tag{3.1.5}$$

The nullspace of $I + H^{-1}(\gamma)J(\gamma)$ and $(I + H^{-1}(\gamma)J(\gamma))'$ have the same dimension. Hence the result (i) and an operator $I + H^{-1}(\gamma)J(\gamma)$ maps bounded closed sets onto closed sets. Hence $I + H^{-1}(\gamma)J(\gamma)$ has closed range from Theorem 1.10 of Chapter IV of [11]. Hence the result (ii) follows from (iii) of Lemma 3.2.

To prove (iii), use results (i) and (ii) i.e.,

$$d(W(\gamma)) = n(W(\gamma)') = n(W(\gamma)) \tag{3.1.6}$$

on using Theorem 2.3 of Chapter IV of [11] and (3.1.6), we have

$$\operatorname{ind}(W(\gamma)) = n(W(\gamma)) - d(W(\gamma)) = 0 \tag{3.1.7}$$

Hence the result (iii). □

The notions of generalized ascent and descent have been formulated in Chapter III of [2] in such a way as to be useful in characterizing poles of locally holomorphic operator valued functions. The definitions of ascent and descent for locally holomorphic operator valued functions are given as follows.

Let X be a complex Banach space and let $U(\lambda)$ defined on as open neighborhood of γ with values in $L(X)$ be locally holomorphic. Let U_n be the nth coefficient of Taylor's expansion of U at γ. i.e.,

$$U_n = \frac{U^{(n)}}{n!} \quad (n = 0, 1, 2, \dots) \tag{3.1.8}$$

For $(m = 0, 1, 2, \dots)$, let H_m be the set of all x in X with the property that there exist $x_0, x_1, \dots, x_m$ in X such that $x_0 = x$ and

$$\sum_{i=0}^{n} U_i x_{n-i} = 0, \quad (n = 0, 1, 2, \dots, m) \tag{3.1.9}$$

Further, let H'_m be the set of all y in X with the property that there exists $x_0, x_1, \dots, x_m$ in X such that

$$\sum_{i=0}^{n} U_i x_{n-i} = \delta_{nm} y, \quad (n = 0, 1, 2 \dots, m) \tag{3.1.10}$$

The extended integer $\alpha(U)$, defined by

$$\alpha(U) = \min\{m : H_m = \{0\}\} \tag{3.1.11}$$

will be called the ascent of U at γ. The extended integer $\delta(U)$, defined by

(3.1.12) $\quad \delta(U) = \min\{m : H'_m = X\}$

will be called the descent of U at γ.

For the locally holomorphic operator valued function $W(\lambda) = H(\lambda) + J(\lambda)$ defined as above, we show that the eigenvalues of W are poles of W^{-1}. We know that the poles of W^{-1} are eigenvalues of W in Theorem 2.2.

Theorem 3.4. *Let $W(\lambda) = H(\lambda) + J(\lambda)$ be defined as above. Assume* Res$[W]$ *is not empty. Then γ is an eigenvalue of W if and only if γ is a pole of W^{-1}.*

Proof. Let γ be an eigenvalue of W. By Theorem 3.3, $W(\gamma)$ is a Fredholm operator with index zero. Hence by Theorem 1.3 of Chapter III of [2]. We have $\alpha(W) = \delta(W)$. If $\alpha(W) = +\infty$ then γ is an interior point of sp$[W]$ by proposition 5.5 of Chapter III of [2]. This is impossible by Theorem 3.1 Since we are assuming Res$[W]$ is not empty. Thus $\alpha(W) < +\infty$ and $\alpha(W) = \delta(W) = m$ for some positive integer m. It follows from Theorem 5.2 of Chapter III of [2] that γ is a pole of W^{-1} of order m. □

3.2. *Spectral Results for Operator Polynomial*

Let X be a complex Banach space and let $K(X)$ denote the closed subspace of $L(X)$ consisting of compact operators. In this section, we restrict the discussion to operator polynomial which satisfy the following conditions

(3.2.1) One of the coefficients is a one-one closed operator with domain $\mathbf{D}$ in X and range X.

(3.2.2) The coefficients with the exception of the one described in (3.2.1) are in $L(X)$

Note that these conditions imply that the values of operator polynomial are in the set of closed operators having domain $\mathbf{D}$ and range in X. Hence any operator polynomial satisfying (3.2.1) and (3.2.2) also satisfies (2.2.1), (2.2.2) and (2.2.3).

Now we state some spectral results for operator polynomials satisfying the above conditions (3.2.1) and (3.2.2).

Theorem 3.5. *Let X be complex Banach space and let $A(\lambda)$ be an operator polynomial on X satisfying* (3.2.1) *and* (3.2.2)*. Assume* Res$[A]$ *is not empty and assume there exists an operator E in $K(X)$ which is one-one and onto the domain of $A(\lambda)$, then*

(i) *the non-zero elements of* sp$[A]$ *are eigenvalues of A and the eigenvectors only accumulate at zero or at ∞.*

(ii) *the non-zero eigenvalues of A are poles of A^{-1} and poles of A^{-1} are eigenvalues of A.*

Proof. Let $A(\lambda) = \lambda^l A_0 + \lambda^{l-1} A_1 + \cdots + A_l$ and let A_i be the coefficients of operator polynomial $A(\lambda)$ satisfying (3.2.1) and (3.2.2). Let $A(\lambda)E = H(\lambda) + J(\lambda)$, Where $H(\lambda) = \lambda^i A_i E$ and where $J(\lambda) = \lambda^l A_0 E + \cdots + \lambda^{l-i+1} A_{i+1} E + \cdots + A_l E$. Let us assume that the operators H and J are defined on $\mathbf{C} - \{0\}$. The assumptions on A_i

and E imply A_iE and $(A_iE)^{-1} = E^{-1}A_i^{-1}$ are both in $L(X)$. Thus the values of H are in $L(X)$ and $\mathrm{Res}[H] = \mathbf{C} - \{0\}$. The coefficients of J are in $K(X)$ as the product of a bounded operator and a compact operator is compact. Hence the operators H and J are holomorphic on $\mathbf{C} - \{0\}$. Hence we can apply Theorem 3.1 to $A(\lambda)E$ and then on using Theorem 2.5, we obtain the result (i). The proof of (ii) is obtained by applying the Lemma 3.2 to $A(\lambda)E$ and then on using Theorem 2.5. Hence the required result. □

Remark 3.6. Let $A(\lambda)$ be an operator polynomial with coefficients acting in X satisfying (3.2.1) and (3.2.2). If $\mathrm{Res}[A]$ is not empty and the coefficients A_i satisfying (3.2.1) have compact inverse, then the assumptions of Theorem 3.5 are satisfied with $E = A_i^{-1}$.

Remark 3.7. Let $A(\lambda)$ be an operator polynomial with coefficients acting in X satisfying (3.2.1) and (3.2.2). If the leading coefficient or the constant coefficient of polynomial operator A satisfies (3.2.1), then $\mathrm{Res}[A]$ is not empty by Lemma 2.4.

3.3. *A Completeness Results for a Quadratic Operator Polynomial*

In this section, let H be a complex Hilbert space with inner product $(\bullet, \bullet)$. Let K_1 and K_2 be compact self-adjoint operators in H and assume K_2 positive definite. Let $C(\lambda)$ be the operator polynomial defined by

$$C(\lambda) = \lambda^2 I - \lambda K_1 - K_2 \tag{3.3.1}$$

then $C(\lambda)$ satisfies the conditions (2.2.1), (2.2.2) and (2.2.3). Since the leading coefficient of $C(\lambda)$ is bijective, so the spectrum of operator polynomial C is bounded by Lemma 2.4. Since K_2 is assumed to be positive definite by Lemma 2.8, so the eigenvalues of C are real and zero is not an eigenvalue of C. We know from Theorem 3.5 that the spectrum of C consists of atmost a countable set of eigenvalues with zero as the only possible limit point.

In this section, we obtain a completeness result for $C(\lambda)$. To do this we linearize the equation

$$C(\lambda)x = \lambda^2 x - \lambda K_1 x - K_2 x = 0, \quad \text{where } x \in H \tag{3.3.2}$$

as follows. Equation (3.3.2) for λ non-zero is equivalent to the pair of equations

$$\lambda x - K_1 x - K_2^{\frac{1}{2}} y = 0, \quad \text{where } \ x, y \in H \tag{3.3.3}$$

$$y = \frac{1}{\lambda} K_2^{\frac{1}{2}} x \tag{3.3.4}$$

where $K_2^{\frac{1}{2}}$ denotes the positive square root of K_2.

The pair of equations (3.3.3) and (3.3.4) can be treated as a single equation in the product space $H \times H$, i.e.,

$$LY = \lambda Y, \tag{3.3.5}$$

where

$$(3.3.6) \qquad L = \begin{pmatrix} K_1 & K_2^{\frac{1}{2}} \\ K_2^{\frac{1}{2}} & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} x \\ y \end{pmatrix} \in H \times H$$

The operator $K_2^{\frac{1}{2}}$ is positive and self-adjoint. Since $K_2^{\frac{1}{2}}$ is self-adjoint and $(K_2^{\frac{1}{2}})^2 = K_2$, which is compact. So, $K_2^{\frac{1}{2}}$ is compact. Thus the operator L defined by (3.3.6) is a compact self-adjoint operator in $H \times H$.

The eigenvalue problems (3.3.2) and (3.3.5) are equivalent in the following sense. If non-zero λ is an eigenvalue of (3.3.2) with corresponding eigenvector x in H, then λ is an eigenvalue of (3.3.5) with corresponding eigenvector $\begin{pmatrix} x \\ \frac{1}{\lambda} K_2^{\frac{1}{2}} x \end{pmatrix}$ in $H \times H$. Also, if $\lambda \neq 0$ is an eigenvalue of (3.3.5) with corresponding eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $H \times H$, then λ is an eigenvalue of (3.3.2) with corresponding eigenvector x in H.

The following results are obtained by applying the classical theory for compact symmetric operators in a Hilbert space to the newly defined operator L, for the classical theorey of Chapter IV of [17].

Let I denote the identity operator in $H \times H$. The operator $L - \lambda I$ defined by (3.3.5) has both positive and negative eigenvalues as

$$(3.3.7) \qquad \langle LY, Y \rangle = (K_1 x, x) + 2Re(K_2^{\frac{1}{2}} x, y)$$

assume both positive and negative values. Here $\langle \bullet, \bullet \rangle$ denote the inner product in a $H \times H$ and $(\bullet, \bullet)$ denotes the inner product in H and we define

$$(3.3.8) \qquad \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = (x_1, y_1) + (x_2, y_2)$$

$Re(\bullet, \bullet)$ denotes the real part of $(\bullet, \bullet)$.

If λ is an eigenvalue of $L - \lambda I$ then the nullspace of $L - \lambda I$ is finite dimensional and closed, multiplicity of λ is the dimension of the nullspace of $L - \lambda I$. Let $\{\lambda_i\}$ be the sequence of eigenvalues of $L - \lambda I$, where the sequence includes eigenvalues repeated according to their multiplicity. There exists a corresponding set of orthonormal eigenvectors $\left\{ \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right\}$ with the property that for any $Y = \begin{pmatrix} x \\ y \end{pmatrix}$ in $H \times H$.

$$(3.3.9) \qquad \langle LY, Y \rangle = \sum_i \lambda_i |(x, x_i) + (y, y_i)|^2$$

where the sum is taken over all the eigenvalues, (multiplicities included). The set $\left\{ \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right\}$ is complete in $H \times H$, since zero is not eigenvalue of $L - \lambda I$.

Write $\{\lambda_i\} = \{\lambda_i^+\} + \{\lambda_i^-\}$, where $\{\lambda_i^+\}$ is the set of positive eigenvalues (repeated according to multiplicity) enumerated in non-increasing order and $\{\lambda_i^-\}$

is the set of negative eigenvalues (repeated according to multiplicity) enumerated in non-decreasing order.

Let the subset of $\left\{\begin{pmatrix} x_i \\ y_i \end{pmatrix}\right\}$ corresponding to eigenvalues $\{\lambda_i^+\}$ be denoted by $\left\{\begin{pmatrix} x_i^+ \\ y_i^+ \end{pmatrix}\right\}$ and let the subset $\left\{\begin{pmatrix} x_i \\ y_i \end{pmatrix}\right\}$ of corresponding to eigenvalues $\{\lambda_i^-\}$ be denoted by $\left\{\begin{pmatrix} x_i^- \\ y_i^- \end{pmatrix}\right\}$. The equation (3.3.9) can be expressed as

$$\langle LY, Y\rangle = \sum_i \lambda_i^+ |(x, x_i^+) + (y, y_i^+)|^2 + \sum_i \lambda_i^- |(x, x_i^-) + (y, y_i^-)|^2 \tag{3.3.10}$$

Lemma 3.8. *Let K_2, $\{x_n^+\}$ and $\{x_n^-\}$ be described as above, then $\{K_2^{\frac{1}{2}} x_n^+\}$ is complete in Hilbert space H in the sense that the span of $\{K_2^{\frac{1}{2}} x_n^+\}$ is dense in H. The similar conclusion holds for $\{K_2^{\frac{1}{2}} x_n^-\}$ as well.*

Proof. We show that

$$(y, K_2^{\frac{1}{2}} x_n^+) = 0 \tag{3.3.11}$$

for all n implies $y = 0$. To prove this let y_n^+ be described as above, i.e., $y_n^+ = \frac{1}{\lambda_n^+} K_2^{\frac{1}{2}} x_n^+$.

By (3.3.11), we have

$$(y, y_n^+) = 0 \quad \text{for all } \; n \tag{3.3.12}$$

(i) If $(y, y_n^-) = 0$ for all n, then the vector $\begin{pmatrix} 0 \\ y \end{pmatrix}$ is orthogonal to the complete set $\left\{\begin{pmatrix} x_i^+ \\ y_i^+ \end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix} x_i^- \\ y_i^- \end{pmatrix}\right\}$ and hence $y = 0$.

(ii) $(y, y_n^-) \neq 0$ for some n, then from the equations (3.3.10) and (3.3.12), we get

$$\left\langle L\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right\rangle = \sum_i \lambda_i^- |(y, y_i^-)|^2 < 0. \tag{3.3.13}$$

But from (3.3.7), we have

$$\left\langle L\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right\rangle = 0 \tag{3.3.14}$$

contradicting the relation (3.3.13). Thus result (i) holds. Hence the relation (3.3.11) implies $y = 0$.

Let M be the closed linear span of $\{K_2^{\frac{1}{2}} x_n^+\}$ and we have in the Hilbert space $H = M \bigoplus M^\perp$, where $M^\perp$ denotes orthogonal complement of M. Since $M^\perp = \{0\}$, so the span of $\{K_2^{\frac{1}{2}} x_n^+\}$ is dense in Hilbert space H. Similarly we can show that the span of $\{K_2^{\frac{1}{2}} x_n^-\}$ is dense in H. □

This lemma shows that both the sequences $\{\lambda_i^+\}$ and $\{\lambda_i^-\}$ are infinite if H is infinite dimensional space.

Lemma 3.9. *Let K_1, K_2, $\{x_n^+\}$ and $\{x_n^-\}$ be described as above. If in addition K_1 is positive (negative), then $\{x_n^+\}$ ($\{x_n^-\}$) span a dense subset of H.*

Proof. Suppose K_1 is positive. We show $(x, x_n^+) = 0$ for all n implies $x = 0$. Assume

$$(x, x_n^+) = 0 \quad \text{for all } n \tag{3.3.15}$$

(i) If $(x, x_n^-) = 0$ for all n then the vector $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is orthogonal to the complete set $\left\{\begin{pmatrix} x_i^+ \\ y_i^+ \end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix} x_i^- \\ y_i^- \end{pmatrix}\right\}$ and hence $x = 0$

(ii) $(x, x_n^-) \neq 0$ for some n, then form (3.3.10), we have

$$\begin{aligned} \left\langle L\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right\rangle &= \sum_i \lambda_i^+ |(x, x_i^+) + (0, y_i^+)|^2 + \sum_i \lambda_i^- |(x, x_i^-) + (0, y_i^-)|^2 \\ &= \sum_i \lambda_i^+ |(x, x_i^+)|^2 + \sum_i \lambda_i^- |(x, x_i^-)|^2. \end{aligned} \tag{3.3.16}$$

Since $(x, x_n^+) = 0$ for all n, so, we have

$$\left\langle L\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right\rangle = \sum_i \lambda_i^- |(x, x_i^-)|^2 < 0. \tag{3.3.17}$$

From (3.3.7) and the assumption that K_1 is positive, we have

$$\left\langle L\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right\rangle \geq 0 \tag{3.3.18}$$

which is a contradiction to (3.3.17). Thus $x = 0$. Hence $\{x_n^+\}$ span a dense subset of H. Similarly if K_1 is negative, we can show that $\{x_n^-\}$ span a dense subset of H. Hence the theorem. □

Remark 3.10. The boundary value problems that we are going to consider subsequently give rise to an operator polynomial of the form $A(\lambda) = L - \lambda M_1 - \lambda^2 M_2$ with coefficients acting in an infinite dimensional Hilbert space H satisfying the following conditions.

(3.3.19) L has dense domain

(3.3.20) L is one-one, self-adjoint, surjective operator having compact inverse.

(3.3.21) M_1 and M_2 are bounded self-adjoint operators defined on all of H.

In some cases we also have

(3.3.22) M_1 is positive or negative and L, M_2 are positive definite.

In such cases we have the following theorem

Theorem 3.11. *Let $A(\lambda) = L - \lambda M_1 - \lambda^2 M_2$ be an operator polynomial with coefficients acting in an infinite dimensional Hilbert space H, satisfying conditions* (3.3.19) *to* (3.3.22) *then*

(i) *the eigenvalues of A are real*

(ii) *the positive (negative) eigenvalues enumerated in non-decreasing (non-increasing) order form an infinite sequence converging to $+\infty$ $(-\infty)$.*

Proof. The result (i) is true obviously by Lemma 2.8. To prove the result (ii) we require the following discussion

An operator L is positive, self-adjoint, so, L^{-1} is positive self-adjoint. Let $L^{\frac{1}{2}}$ and $L^{\frac{-1}{2}}$ denote the positive self-adjoint square roots of L and L^{-1} respectively. For information about square roots see Lemma 7.3 of Chapter XII of [8]. The domain of $L^{\frac{1}{2}}$ contains the domain of L and $L^{\frac{-1}{2}}$ is defined on all of H, since L is surjective, so, $L^{\frac{-1}{2}}$ is compact. Since $L^{\frac{-1}{2}}$ is self-adjoint, so $(L^{\frac{-1}{2}})^2 = L^{-1}$, is compact. If M_i $(i = 1, 2)$ is a positive self-adjoint multiplicative operator, then

$$L^{\frac{-1}{2}} M_i L^{\frac{-1}{2}} = (L^{\frac{-1}{2}} M_i^{\frac{1}{2}})(M_i^{\frac{1}{2}} L^{\frac{-1}{2}}) = (M_i^{\frac{1}{2}} L^{\frac{-1}{2}})^*(M_i^{\frac{1}{2}} L^{\frac{-1}{2}})$$

is positive self-adjoint. Since $M_i L^{\frac{-1}{2}}$ is bounded and $L^{\frac{-1}{2}}$ is compact, it follows that $L^{\frac{-1}{2}} M_i$ is compact.

With the help of above defined operator, consider the operator polynomial

$$K(\mu) = L^{\frac{-1}{2}} M_2 L^{\frac{-1}{2}} - \mu L^{\frac{-1}{2}} M_1 L^{\frac{-1}{2}} - \mu^2 I \tag{3.3.23}$$

The operator polynomial A and newly defined operator polynomial K are equivalent in the following sense.

If μ is an eigenvalue of K with corresponding eigenvector y, then $\frac{1}{\mu}$ is an eigenvalue of A and $L^{\frac{-1}{2}} y$ is a corresponding eigenvector. On the other hand if λ is an eigenvalue of A with corresponding eigenvector x, then $\frac{1}{\lambda}$ is an eigenvalue of K and $L^{\frac{1}{2}} x$ is a corresponding eigenvector. Now on using result (i) of Theorem 3.5 and the preceding discussion, the result (ii) is obtained. Hence the theorem. □

Theorem 3.12. *$A(\lambda) = L - \lambda M_1 - \lambda^2 M_2$ be an operator polynomial with coefficients acting in an infinite dimensional Hilbert space H satisfying* (3.3.19) *to* (3.3.22) *then*

(i) *if M_1 is positive (negative), the set of eigenvectors corresponding to the positive (negative), eigenvalues span a dense subset of H.*

(ii) *if M_2 is multiplication by a positive constant then each of the sets of eigenvectors corresponding to the positive and negative eigenvalues span a dense subset of H.*

Proof. Let $\{x_n^+\}$ $(\{x_n^-\})$ denote the subset of eigenvectors corresponding to the positive (negative) eigenvalues of A.

Suppose M_1 is positive. From Lemma 3.9 and the equivalence relation between operator polynomials A and K described in above theorem, we have

$$(x, L^{\frac{1}{2}} x_n^+) = 0 \quad \text{for all } \ n \text{ implies } x = 0. \tag{3.3.24}$$

We show that $(z, x_n^+) = 0$ for all n implies $z = 0$. Since $L^{\frac{1}{2}}$ is surjective, let $z = L^{\frac{1}{2}} x$ then $(z, x_n^+) = 0$ for all n implies $(L^{\frac{1}{2}} x, x_n^+) = 0$ for all n. Since $L^{\frac{1}{2}}$ is self-adjoint, so we have $(x, L^{\frac{1}{2}} x_n^+) = 0$. Hence by (3.3.24) we have $x = 0$ and hence $z = L^{\frac{1}{2}} x = 0$. Similar result holds for M_1 negative. Thus part (i) is proved.

From Lemma 3.8 and the equivalence relation between operator polynomials A and K described in above it follows that each of the sets $\{(L^{\frac{-1}{2}} M_2 L^{\frac{-1}{2}})^{\frac{1}{2}} L^{\frac{1}{2}} x_n^+\}$ and $\{(L^{\frac{-1}{2}} M_2 L^{\frac{-1}{2}})^{\frac{1}{2}} L^{\frac{1}{2}} x_n^-\}$ are complete in H. If M_2 is multiplication by a positive constant then it follows immediately that $\{x_n^+\}$ and $\{x_n^-\}$ are complete in H. Hence they span a dense subset of H. Thus result (ii) is obtained. Hence the theorem. □

Completeness results for a quadratic operator polynomial of the form $D(\lambda) = I + \lambda B + \lambda^2 C$, where C is a positive definite self-adjoint operator and B is a bounded self-adjoint operator are given in [10]. In [10], assuming a strong damping condition, completeness for eigenvectors, associated with eigenvalues of D are given. Completeness results of different nature for the generalized eigenvectors of an operator polynomial are obtained in [9].

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