Journal of Informatics and Mathematical Sciences
Volume 5 (2013), Number 3, pp. 143–156
RGN Publications

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Parametrically Sufficient Optimality Conditions for Multiobjective Fractional Subset Programming Relating to Generalized (η,ρ,θ)-Invexity of Higher Order

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Abstract Inspired by the recent investigations, a general framework for a class of (η,ρ,θ)-invex n-set functions of higher order $r \geq 1$ is introduced, and then some optimality conditions for multiobjective fractional programming on the generalized (η,ρ,θ)-invexity are established. The obtained results are general in nature and unify various results on fractional subset programming in the literature.

1. Introduction

Based on new developments on parametric duality models and global parametric models for fractional programming to the context of the generalized invex functions, we present using the generalized (η,ρ,θ)-invexity of higher order $r \geq 1$ of differentiable functions, the following multiobjective fractional subset programming problem:

$$\text{(P)} \quad \underset{S\in X}{\text{Minimize}} \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \dots, \frac{F_p(S)}{G_p(S)}\right)$$
$$\text{subject to } H_j(S) \leq 0,\ j \in \underline{m},\ S \in X,$$

where $X = \{S \in \Lambda^n : H_j(S) \leq 0, j \in \underline{m}\}$, is the feasible set (assumed to be nonempty) of (P), Λ^n is n-fold product of σ-algebra Λ of subsets of a given set X in the measure space (X,Λ,μ), F_i, G_i, $i \in \underline{p} \equiv \{1,\dots,p\}$, $H_j(S) \leq 0$, $j \in \underline{m} \equiv \{1,\dots,m\}$, are real valued functions defined on Λ^n, and for each $G_i(S) > 0$, for each $i \in \underline{p}$, for all $S \in \Lambda^n$.

2010 *Mathematics Subject Classification*. 90C32, 49K50.
Key words and phrases. Generalized invexity of higher order; Multiobjective fractional programming; Fractional subset programming; Sufficient optimality conditions.

Next, we introduce an auxiliary problem $(\mathrm{P}\lambda)$ to (P) as follows:

$$(\mathrm{P}\lambda) \quad \underset{S\in X}{\text{Minimize}} \Big(F_1(S) - \lambda_1 G_1(S), \dots, F_p(S) - \lambda_p G_p(S) \Big),$$

where λ_i, $i \in \underline{p}$ are parameters.

Similarly, we have an auxiliary problem $(\mathrm{P}\lambda^*)$ to (P) as follows:

$$(\mathrm{P}\lambda^*) \quad \underset{S\in X}{\text{Minimize}} \Big(F_1(S) - \lambda_1^* G_1(S), \dots, F_p(S) - \lambda_p^* G_p(S) \Big),$$

where $\lambda^* = \left(\dfrac{F_1(S^*)}{G_1(S^*)}, \dots, \dfrac{F_p(S^*)}{G_p(S^*)} \right)$.

Mishra et al. [4] investigated several parametric and semi-parametric sufficient conditions for the multiobjective fractional subset programming problems based on generalized invexity assumptions. Moreover, these results are also applicable to other classes of problems with multiple, fractional, and conventional objective functions.

2. Higher Order (ρ, η, θ)-Invexity

Let (X, Λ, μ) be a finite atomless measure space with $L_1(X, \Lambda, \mu)$ separable. Let Λ^n be the n-fold product of a σ-algebra Λ for subsets in the measure space (X, Λ, μ). The function d (referred to as a pseudometric on Λ^n) on $\Lambda^n \times \Lambda^n$ is defined by

$$d(R,S) = \left\{ \sum_{i=1}^{n} [\mu^2(R_i \Delta S_i)] \right\}^{\frac{1}{2}},$$

where $R = (R_1, \dots, R_n)$, $S = (S_1, \dots, S_n) \in \Lambda^n$, and Δ denotes the symmetric difference of sets. Thus, (Λ^n, d) is a pseudometric space. Let $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ be a vector valued function.

We introduce the following definitions based on Mishra et al. [4], Lai and Huang [3] and references therein.

Definition 2.1. An n-set function $F : \Lambda^n \to \mathbb{R}$ is said to be differentiable at $S^* \in \Lambda^n$ if there exists $DF(S^*) \in L_1(X, \Lambda, \mu)$, called the derivative of F at S^* such that for each $S \in \Lambda^n$,

$$F(S) = F(S^*) + \langle DF(S^*), \eta(S, S^*) \rangle + V_F(S, S^*),$$

where $V_F(S, S^*)$ is $o(d(S, S^*))$, that is,

$$\lim_{d(S,S^*)\to 0} V_F(S, S^*)/d(S, S^*) = 0.$$

Definition 2.2. An n-set function $G : \Lambda^n \to \mathbb{R}$ is said to have a partial derivative at $S^* = (S_1^*, \dots, S_n^*) \in \Lambda^n$ with respect to its ith argument if the function

$$F(S_i) = G(S_1^*, \dots, S_{i-1}^*, S_i^*, S_{i+1}^*, \dots, S_n^*)$$

has the derivative $DF(S_i^*)$, $i \in \underline{n}$; in that case, the ith partial derivative of G at S^* is defined to be $D_iG(S^*) = DF(S_i^*) \in L_1$, $i \in \underline{n}$, which behave as continuous linear operators on L_∞.

Definition 2.3. An n-set function $G : \Lambda^n \to \mathbb{R}$ is said to be differentiable at S^* if all the derivatives $D_iG(S^*)$, $i \in \underline{n}$ exist and

$$G(S) = G(S^*) + \sum_{i=1}^{n} \langle DG_i(S^*), \eta(S_i, S_i^*) \rangle + W_G(S, S^*),$$

where $W_G(S, S^*)$ is $o(d(S, S^*))$ for all $S \in \Lambda^n$.

Let $\theta : \Lambda^n \times \Lambda^n \to \mathbb{R}_+$ be a positive-valued function such that

$$\theta(S, S^*) = 0 \text{ only if } S = S^*, \text{ or } S \xrightarrow{w^*} S^* \in \Lambda^n \Rightarrow \theta(S, S^*) \Rightarrow 0,$$

where w^* denotes the *weak**-topology in $L_\infty^n (\approx \Lambda^n) = (L_1^n)^*$ and $S = (S_1, \ldots, S_n)$, $S^* = (S_1^*, \ldots, S_n^*) \in \Lambda^n$.

For our purpose, we define θ as a pseudometric on Λ^n in the form

$$\theta(S, S^*) = \left[\sum_{i=1}^{n} \mu^2(S_i \Delta S_i^*) \right]^{\frac{1}{2}},$$

$S = (S_1, \ldots, S_n)$, $S^* = (S_1^*, \ldots, S_n^*) \in \Lambda^n$.

Definition 2.4. A differentiable n-set function $F : \Lambda^n \to \mathbb{R}$ is said to be (ρ, η, θ)-invex at S^* if there exists a vector valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S \in \Lambda^n$, for a positive-valued function $\theta : \Lambda^n \times \Lambda^n \to \mathbb{R}_+$, for a positive integer r$\geq$ 1, and $\rho \in \mathbb{R}_+$, we have

$$F_i(S) - F_i(S^*) \geqq \langle F_i'(S^*), \eta(S, S^*) \rangle + \rho \|\theta(S, S^*)\|^r.$$

Definition 2.5. The differentiable n-set function $F : \Lambda^n \to \mathbb{R}$ is said to be (ρ, η, θ)-pseudo-invex at S^* if there exists a vector valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S \in \Lambda^n$, and for $\rho \in \mathbb{R}_+$, we have

$$\sum_{i=1}^{p} \langle F_i'(S^*), \eta(S, S^*) \rangle + \rho \|\theta(S, S^*)\|^r \geq 0 \Rightarrow \sum_{i=1}^{p} F_i(S) \geq \sum_{i=1}^{p} F_i(S^*),$$

equivalently,

$$\sum_{i=1}^{p} F_i(S) < \sum_{i=1}^{p} F_i(S^*) \Rightarrow \sum_{i=1}^{p} \langle F_i'(S^*), \eta(S, S^*) \rangle + \rho \|\theta(S, S^*)\|^r < 0.$$

Definition 2.6. The differentiable n-set function $F : \Lambda^n \to \mathbb{R}$ is said to be strictly (ρ, η, θ)-pseudo-invex at S^* if there exists a vector valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S \in \Lambda^n$, and $\rho \in \mathbb{R}_+$, we have

$$\sum_{i=1}^{p} \langle F_i'(S^*), \eta(S, S^*) \rangle + \rho \|\theta(S, S^*)\|^r \geqq 0 \Rightarrow \sum_{i=1}^{p} F_i(S) > \sum_{i=1}^{p} F_i(S^*),$$

equivalently,

$$\sum_{i=1}^{p} F_i(S) \leq \sum_{i=1}^{p} F_i(S^*) \Rightarrow \sum_{i=1}^{p} \langle F_i'(S^*), \eta(S,S^*)\rangle + \rho\|\theta(S,S^*)\|^r < 0.$$

Definition 2.7. The differentiable n-set function $F : \Lambda^n \to \mathbb{R}$ is said to be prestrictly (ρ,η,θ)-pseudo-invex at S^* if there exists a vector valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S \in \Lambda^n$, and $\rho \in \mathbb{R}_+$, we have

$$\sum_{i=1}^{p} \langle F_i'(S^*), \eta(S,S^*)\rangle + \rho\|\theta(S,S^*)\|^r > 0 \Rightarrow \sum_{i=1}^{p} F_i(S) \geqq \sum_{i=1}^{p} F_i(S^*),$$

equivalently,

$$\sum_{i=1}^{p} F_i(S) < \sum_{i=1}^{p} F_i(S^*) \Rightarrow \sum_{i=1}^{p} \langle F_i'(S^*), \eta(S,S^*)\rangle + \rho\|\theta(S,S^*)\|^r \leq 0.$$

Definition 2.8. The differentiable n-set function $F : \Lambda^n \to \mathbb{R}$ is said to be (ρ,η,θ)-quasi-invex at S^* if there exists a vector valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S \in \Lambda^n$, and $\rho \in \mathbb{R}_+$, we have

$$\sum_{i=1}^{p} F_i(S) \leqq \sum_{i=1}^{p} F_i(S^*) \Rightarrow \sum_{i=1}^{p} \langle F_i'(S^*), \eta(S,S^*)\rangle + \rho\|\theta(S,S^*)\|^r \leqq 0.$$

Definition 2.9. The differentiable n-set function $F : \Lambda^n \to \mathbb{R}$ is said to be strictly (ρ,η,θ)-quasi-invex at S^* if there exists a vector valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S \in \Lambda^n$, and $\rho \in \mathbb{R}_+$, we have

$$\sum_{i=1}^{p} F_i(S) \leqq \sum_{i=1}^{p} F_i(S^*) \Rightarrow \sum_{i=1}^{p} \langle F_i'(S^*), \eta(S,S^*)\rangle + \rho\|\theta(S,S^*)\|^r < 0,$$

equivalently,

$$\sum_{i=1}^{p} \langle F_i'(S^*), \eta(S,S^*)\rangle + \rho\theta(S,S^*) \geq 0 \Rightarrow \sum_{i=1}^{p} F_i(S) > \sum_{i=1}^{p} F_i(S^*).$$

Definition 2.10. The differentiable n-set function $F : \Lambda^n \to \mathbb{R}$ is said to be prestrictly (ρ,η,θ)-quasi-invex at S^* if there exists a vector valued function $\eta : \Lambda^n \times \Lambda^n \to L_\infty^n$ such that for each $S \in \Lambda^n$, and $\rho \in \mathbb{R}_+$, we have

$$\sum_{i=1}^{p} F_i(S) < \sum_{i=1}^{p} F_i(S^*) \Rightarrow \sum_{i=1}^{p} \langle F_i'(S^*), \eta(S,S^*)\rangle + \rho\|\theta(S,S^*)\|^r \leqq 0,$$

equivalently,

$$\sum_{i=1}^{p} \langle F_i'(S^*), \eta(S,S^*)\rangle + \rho\theta(S,S^*) > 0 \Rightarrow \sum_{i=1}^{p} F_i(S) \geq \sum_{i=1}^{p} F_i(S^*).$$

This subsection deals with some parametric sufficient efficiency conditions for problem (P) under the generalized frameworks for generalized invexity. First, we introduce the necessary efficiency conditions regarding the solvability for (P) and (Pλ) as follows:

Definition 2.11. An $S^* \in X$ is an efficient solution to (P) if there does not exist an $S \in X$ such that

$$\frac{F_i(S)}{G_i(S)} \leq \frac{F_i(S^*)}{G_i(S^*)}, \quad \forall\, i = 1, \ldots, p.$$

Next, we introduce the efficient solvability for $(\mathrm{P}\lambda^*)$ as follows:

Definition 2.12. An $S^* \in X$ is an efficient solution to $(\mathrm{P}\lambda^*)$ if there exist no $S \in X$ such that

$$F_i(S) - \lambda_i^* G_i(S) \leq F_i(S^*) - \lambda_i^* G_i(S^*), \quad \forall\, i = 1, \ldots, p,$$

where $\lambda^* = \left(\dfrac{F_1(S^*)}{G_1(S^*)}, \ldots, \dfrac{F_p(S^*)}{G_p(S^*)}\right)$.

3. Parametric Optimality Conditions

This section deals with some results on sufficient optimality conditions for the generalized invexity of higher order $(r \geq 1)$, where r is an integer. In the following theorem, we examine some generalized sufficiency criteria relating to (P).

Theorem 3.1. *Let $S^* \in \Lambda^n$ and let us suppose that $F_i, G_i, i \in \{1, \ldots, p\}$, and $H_j, j \in \{1, \ldots, m\}$ are differentiable at $S^* \in \Lambda^n$ and there are $u^* \in U = \left\{u \in R_+^p : \sum_{i=1}^{p} u_i = 1\right\}$ and $v^* \in R_+^m$ such that*

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)] + \sum_{j=1}^{m} v_j^* H_j'(S^*), \eta(S, S^*)\right\rangle + \rho\|\theta(S, S^*)\|^r \geq 0$$

$$\forall\, S \in X, \quad (3.1)$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \quad \text{for } i \in \{1, \ldots, p\}, \quad (3.2)$$

$$v_j^* H_j(S^*) = 0 \quad \text{for } j \in \{1, \ldots, m\}. \quad (3.3)$$

We further suppose that any one of the following sets of assumptions holds:

(a) (i) *F_i is $(\bar{\rho}, \eta, \theta)$-invex and $-G_i$ is $(\hat{\rho}, \eta, \theta)$-invex at S^* of higher order $(r \geq 1)$ $\forall\ i \in \{1, \ldots, p\}$.*

(ii) *$v_j^* H_j(S^*)$ is (ρ^*, η, θ)-quasi-invex at S^* of higher order $(r \geq 1)$ $\forall\ j \in \{1, \ldots, m\}$.*

(iii) $\sum_{i=1}^{p} u_i^*[\bar{\rho} + \lambda_i^* \hat{\rho}] + \sum_{j=1}^{m} v_j^* \rho^* \geq 0.$

(b) (i) *F_i is $(\bar{\rho}, \eta, \theta)$-invex and $-G_i$ is $(\hat{\rho}, \eta, \theta)$-invex at S^* of higher order $(r \geq 1)$ $\forall\ i \in \{1, \ldots, p\}$.*

(ii) *$\sum_{j=1}^{m} v_j^* H_j(S^*)$ is (ρ, η, θ)-quasi-invex at S^* of higher order $(r \geq 1)$ $\forall\ j \in \{1, \ldots, m\}$.*

(iii) $\sum_{i=1}^{p} u_i^*[\bar{\rho} + \lambda_i^* \hat{\rho}] + \rho^* \geq 0.$

Then S^* is an efficient solution to (P).

Proof. (a) If (ii) holds, and if $S \in X$ is an arbitrary point, then it follows (using the $(\bar{\rho}, \eta, \theta)$-invexity of F_i and $(\hat{\rho}, \eta, \theta)$-invexity of $-G_i$ at S^*) that

$$\begin{aligned}
&\sum_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \\
&= \sum_{i=1}^{p} u_i^*\{[F_i(S) - F_i(S^*)] - \lambda_i^*[G_i(S) - G_i(S^*)]\} \\
&\geq \sum_{i=1}^{p} u_i^*\{\langle [F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S, S^*)\rangle + [\bar{\rho} + \lambda_i^*\hat{\rho}]\|\theta(S, S^*)\|^r\} \qquad (3.4)
\end{aligned}$$

Since $v^* \geq 0$, $S \in X$ and (3.3) holds, we have

$$\sum_{j=1}^{m} v_t^* H_t(S) \leq \sum_{j=1}^{m} v_j^* H_j(S^*),$$

so in light of the (ρ, η, θ)-quasi-invexity, we arrive at

$$\left\langle \sum_{j=1}^{m} v_j^* H_j'(S^*), \eta(S, S^*) \right\rangle \leq -\rho^* \|\theta(S, S^*)\|^r. \tag{3.5}$$

It follows from (3.5) that

$$\begin{aligned}
&\sum_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \\
&\quad = \sum_{i=1}^{p} u_i^*\{[F_i(S) - F_i(S^*)] - \lambda_i^*[G_i(S) - G_i(S^*)]\} \\
&\quad \geq \sum_{i=1}^{p} u_i^*\{\langle [F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S, S^*)\rangle + [\bar{\rho} + \lambda_i^*\hat{\rho}]\|\theta(S, S^*)\|^r\} \\
&\quad = -\left\langle \sum_{j=1}^{m} v_j^* H_j'(S^*), \eta(S, S^*) \right\rangle + \sum_{i=1}^{p} u_i^*[\bar{\rho} + \lambda_i^*\hat{\rho}]\|\theta(S, S^*)\|^r \\
&\quad \geq \left\{ \sum_{j=1}^{m} v_j^* \rho^* + \sum_{i=1}^{p} u_i^*[\bar{\rho} + \lambda_i^*\hat{\rho}] \right\} \|\theta(S, S^*)\|^r \\
&\quad \geq 0 \qquad \text{(by (iii))}.
\end{aligned}$$

It follows that

$$(F_1(S) - \lambda_1^* G_1(S), \ldots, F_p(S) - \lambda_1^* G_p(S)) \nleq (0, \ldots, 0).$$

Thus, we conclude that

$$\phi(S) = \left(\frac{F_1(S)}{G_1(S)}, \ldots, \frac{F_p(S)}{G_p(S)} \right) \nleq \lambda^*$$

At this stage, as we observe that $\lambda^* = \phi(S^*)$ and $S \in X$ is arbitrary, it implies that S^* is an efficient solution to (P). Similar proof holds for (b). □

At this stage, we introduce the following notations for the next theorem as follows:

$$A_i(\cdot\,;\lambda,u)=u_i[F_i(S)-\lambda G_i(S)],\quad i\in \underline{p},$$
$$B_j(\cdot\,,v)=v_jH_j(S),\quad j\in \underline{m}.$$

Theorem 3.2. *Let $S^*\in\Lambda^n$ and let us suppose that F_i, G_i, $i\in\{1,\dots,p\}$, and H_j, $j\in\{1,\dots,m\}$ are differentiable at $S^*\in\Lambda^n$ and there are $u^*\in U=\left\{u\in R^p_+:\sum_{i=1}^{p}u_i=1\right\}$ and $v^*\in R^m_+$ such that*

$$\left\langle \sum_{i=1}^{p}u_i^*[F_i'(S^*)-\lambda_i^*G_i'(S^*)]+\sum_{j=1}^{m}v_j^*H_j'(S^*),\eta(S,S^*)\right\rangle+\rho\|\theta(S,S^*)\|^r\ge 0 \quad \forall\ S\in X, \tag{3.6}$$

$$F_i(S^*)-\lambda_i^*G_i(S^*)=0\quad \text{for } i\in\{1,\dots,p\}, \tag{3.7}$$

$$v_j^*H_j(S^*)=0\quad \text{for } j\in\{1,\dots,m\}. \tag{3.8}$$

We further suppose that any one of the following sets of assumptions holds:

(a) (i) $A_i(\cdot\,;\lambda^*,u^*)$ $(\forall\ i=1,\dots,p)$ *are (ρ,η,θ)-pseudo-invex of higher order at S^*;*

(ii) $B_j(\cdot\,;v^*)$ $(\forall\ j\in\{1,\dots,m\}$ *are (ρ,η,θ)-quasi-invex of higher order at S^*.*

(b) (i) $A_i(\cdot\,;\lambda^*,u^*)$ $(\forall\ i\in\{1,\dots,p\}$ *are (ρ,η,θ)-prestrictly-pseudo-invex of higher order at S^**

(ii) $B_j(\cdot\,;v^*)$ $(\forall\ j\in\{1,\dots,m\}$ *are (ρ,η,θ)-strictly-quasi-invex of higher order at S^*.*

(c) (i) $A_i(\cdot\,;\lambda^*,u^*,v^*)$ $(\forall\ i\in\{1,\dots,p\}$ *are (ρ,η,θ)-prestrictly-quasi-invex of higher order at S^**

(ii) $B_j(\cdot\,;v^*)$ $(\forall\ j\in\{1,\dots,m\}$ *are (ρ,η,θ)-strictly-pseudo-invex of higher order at S^*.*

Then S^* is an efficient solution to (P).

Proof. Let $S\in X$. Then it follows from (3.6) that it follows that

$$\left\langle \sum_{i=1}^{p}u_i^*[F_i'(S^*)-\lambda_i^*G_i'(S^*)],\eta(S,S^*)\right\rangle+\left\langle \sum_{j=1}^{m}v_j^*H_j'(S^*),\eta(S,S^*)\right\rangle +\rho\|\theta(S,S^*)\|^r\ge 0\quad \forall\ S\in Q. \tag{3.9}$$

Since $v^*\geq 0$, $S\in X$ and (3.7) holds, we have

$$\sum_{j=1}^{m}v_j^*H_j'(S)\le\sum_{j=1}^{m}v_j^*H_j'(S^*),$$

and this implies applying (ii) that

$$\left\langle \sum_{j=1}^{m}v_j^*H_j'(S^*),\eta(S,S^*)\right\rangle+\rho\|\theta(S,S^*)\|^r\le 0. \tag{3.10}$$

Next, applying (3.9) and (3.10), we find

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S,S^*) \right\rangle + \rho\|\theta(S,S^*)\|^r \geq 0.$$

Now, by using (i), we have

$$\sum_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geq \sum_{i=1}^{p} u_i^*[F_i(S^*) - \lambda_i^* G_i(S^*)].$$

On applying (3.7), this results in

$$\sum_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geq 0.$$

Since $u_i^* > 0$ for each $i \in \underline{p}$, the above inequality implies

$$\Big(F_1(S) - \lambda_1^* G_1(S), \ldots, F_p - \lambda_p^* G_p(S)\Big) \nleq (0, \ldots, 0),$$

which in turn, implies

$$\phi(S) = \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \lambda^*.$$

Since $\lambda^* = \phi(S^*)$ and $S \in X$ is arbitrary, we conclude that S^* is an efficient solution to (P).

To prove (b), we have

$$\sum_{j=1}^{m} v_j^* H_j'(S) \leq \sum_{j=1}^{m} v_j^* H_j'(S^*).$$

Now applying (ii), we have

$$\left\langle \sum_{j=1}^{m} v_j^* H_j'(S^*), \eta(S,S^*) \right\rangle + \rho\|\theta(S,S^*)\|^r < 0. \tag{3.11}$$

Next, applying (3.9), we find

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S,S^*) \right\rangle + \rho\|\theta(S,S^*)\|^r > 0.$$

Now, by using (i), we have

$$\sum_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geq \sum_{i=1}^{p} u_i^*[F_i(S^*) - \lambda_i^* G_i(S^*)].$$

On applying (3.7), this results in

$$\sum_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geq 0.$$

This implies in light of (a) that S^* is an efficient solution to (P).

Similar proof holds for (c). □

We need to introduce the following notations as we proceed to establishing results on the generalized pseudo-invexity. Let $\{J_0, J_1, \dots, J_q\}$ be a partition of the partition of the index set $\{1,\dots,m\}$. Then $J_r \subset \{1,\dots,m\}$ for $r \in \{0,1,\dots,q\}$, $J_r \cap J_s = \emptyset$ for each $r,s \in \{0,1,\dots,q\}$ with $r \neq s$ and $\cup_{r=0}^{q} J_r = \{1,\dots,m\}$. Next, we define the following real-valued functions (for fixed λ, u, v on X as follows:

$$\Gamma_i(.;\lambda,u,v) = u_i[F_i(S) - \lambda_i G_i(S) + \sum_{j\in J_0} v_j H_j(S)] \quad \text{for } i \in \{1,\dots,p\},$$

$$\Delta_t(S,v) = \sum_{j\in J_t} v_j H_j(S) \quad \text{for } t \in \{1,\dots,q\}.$$

Theorem 3.3. *Suppose that $S \in X$ and that F_i, G_i, $i \in \{1,\dots,p\}$, H_j, $j \in \{1,\dots,m\}$ are differentiable at $S^* \in X$ and there are $u^* \in U$ and $v^* \in R_+^m$ such that*

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)] + \sum_{j=1}^{m} v_j^* H_j'(S^*), \eta(S,S^*) \right\rangle + \rho\|\theta(S,S^*)\|^r \geq 0 \quad \forall\ S \in X, \tag{3.12}$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \quad \textit{for } i \in \{1,\dots,p\}, \tag{3.13}$$

$$v_j^* H_j(S^*) = 0 \quad \textit{for } j \in \{1,\dots,m\}. \tag{3.14}$$

We further suppose that any one of the following sets of assumptions holds:

(a) (i) $\Gamma_i(\cdot;\lambda^*,u^*,v^*)$ $(\forall\ i = 1,\dots,p)$ *are (ρ,η,θ)-pseudo-invex of higher order at S^*;*

(ii) $\Delta_j(\cdot;v^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are (ρ,η,θ)-quasi-invex of higher order at S^*.*

(b) (i) $\Gamma_i(\cdot;\lambda^*,u^*,v^*)$ $(\forall\ i \in \{1,\dots,p\}$ *are (ρ,η,θ)-prestrictly-pseudo-invex of higher order at S^**

(ii) $\Delta_j(\cdot;v^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are (ρ,η,θ)-strictly-quasi-invex of higher order at S^*.*

(i) $\Gamma_i(\cdot;\lambda^*,u^*,v^*)$ $(\forall\ i \in \{1,\dots,p\}$ *are (ρ,η,θ)-prestrictly-quasi-invex of higher order at S^**

(ii) $\Delta_j(\cdot;v^*)$ $(\forall\ j \in \{1,\dots,m\}$ *are (ρ,η,θ)-strictly-pseudo-invex of higher order at S^*.*

Then S^* is an efficient solution to (P).

Proof. (a) If (i) holds, and if $S \in X$, then it follows from (3.9) that

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S,S^* \right\rangle + \rho\|\eta(S,S^*)\|^r + \left\langle \sum_{j\in J_0} v_j^* H_j'(S^*), \eta(S,S^*) \right\rangle + \left\langle \sum_{t=1}^{q}\sum_{j\in J_t} v_j^* H_j'(S^*), \eta(S,S^*) \right\rangle \geq 0 \quad \forall\ S \in \Lambda^n. \tag{3.15}$$

Since $v^* \geq 0$, $S \in X$ and (3.11) holds, we have

$$\sum_{t \in J_t} v_t^* H_t(S) \leq \sum_{t \in J_t} v_t^* H_t(S^*),$$

so we have

$$\sum_{t=1}^{q} \Delta_t(S, v^*) \leq \sum_{t=1}^{q} \Delta_t(S^*, v^*).$$

Then in light of the (ρ, η, θ)-quasi-invexity, we arrive at

$$\left\langle \sum_{t=1}^{q} \sum_{j \in J_t} v_j^* H_j'(S^*), \eta(S, S^*) \right\rangle \leq -\rho \|\theta(S, S^*)\|^r. \tag{3.16}$$

It follows from (3.12) and (3.13) that

$$\left\langle \sum_{i=1}^{p} u_i^* [F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S, S^*) \right\rangle + \rho \|\theta(S, S^*)\|^r$$
$$+ \left\langle \sum_{j \in J_0} v_j^* H_j'(S^*), \eta(S, S^*) \right\rangle \geq \rho \|\theta(S, S^*)\|^r.$$

Since $\sum_{i=1}^{p} u_i = 1$, it follows that

$$\left\langle \sum_{i=1}^{p} u_i^* [F_i'(S^*) - \lambda_i^* G_i'(S^*)], \eta(S, S^*) \right\rangle + \left\langle u_i^* \left[\sum_{j \in J_0} v_j^* H_j'(S^*) \right], \eta(S, S^*) \right\rangle$$
$$\geq 0$$
$$\geq -\rho \|\theta(S, S^*)\|^r.$$

This, in turn, implies

$$\sum_{i=1}^{p} \Gamma_i(S, \lambda^*, u^*, v^*) \geq \sum_{i=1}^{p} \Gamma_i(S^*, \lambda^*, u^*, v^*) = 0. \tag{3.17}$$

It follows from (3.17) that

$$\sum_{i=1}^{p} u_i^* [F_i(S) - \lambda_i^* G_i(S)] \geq 0. \tag{3.18}$$

Since $u_i^* > 0$ for each $i \in \{1, \ldots, p\}$, we have from (3.7) that

$$(F_1(S) - \lambda_1^* G_1(S), \ldots, F_p(S) - \lambda_1^* G_p(S)) \nleq (0, \ldots, 0).$$

Thus, we conclude that

$$\phi(S) = \left(\frac{F_1(S)}{G_1(S)}, \ldots, \frac{F_p(S)}{G_p(S)} \right) \nleq \lambda^*$$

At this stage, as we observe that $\lambda^* = \phi(S^*)$ and $S \in \Xi$ is arbitrary, it implies that S^* is an efficient solution to (P). Similar proofs hold for (b) and (c). □

Before we examine some more sufficient optimality conditions using a variant form of the partition scheme applied for Theorem 3.3, we need introduce some additional notations. Suppose that $\{I_0, I_1, \ldots, I_q\}$ be a partition of the index set $\{1, \ldots, p\}$ such that $K = \{0, 1, \ldots, k\} \subset Q = \{0, 1, \ldots, q\}$ for $k < q$, and the function $\Theta_t = (., \lambda^*, u^*, v^*) : \Lambda^n \to R$ is defined by $J_r = \{1, \ldots, m\}$. The real-valued functions are defined as follows:

$$\Theta_t(\cdot\,; \lambda^*, u^*, v^*) = \sum_{i \in I_t} u_i^* [F_i(S) - \lambda_i G_i(S)] + \sum_j \in J_t v_j^* H_j(S)],\ t \in K.$$

Theorem 3.4. *Suppose that $S \in \Lambda^n$ and that F_i, G_i, $i \in \{1, \ldots, p\}$, H_j, $j \in \{1, \ldots, m\}$ are differentiable at $S^* \in \Lambda^n$ and there are $u^* \in U$ and $v^* \in R_+^m$ such that*

$$\left\langle \sum_{i=1}^p u_i^* [F_i'(S^*) - \lambda_i^* G_i'(S^*)] + \sum_{j=1}^m v_j^* H_j'(S^*), \eta(S, S^*) \right\rangle + \rho \|\theta(S, S^*)\|^2 \geq 0 \quad \forall\, S \in \Lambda^n, \tag{3.19}$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \quad \textit{for } i \in \{1, \ldots, p\}, \tag{3.20}$$

$$v_j^* H_j(S^*) = 0 \quad \textit{for } j \in \{1, \ldots, m\}. \tag{3.21}$$

We further suppose that any one of the following sets of assumptions holds:

(a) (i) $\Theta_t(\cdot\,; \lambda^*, u^*, v^*)$ ($\forall\ t \in \{1, \ldots, p\}$ *are* (ρ, η, θ)*-pseudo-invex at* S^*,
   (ii) $\Delta_j(\cdot\,; v^*)$ ($\forall\ j \in \{1, \ldots, m\}$ *are* (ρ, η, θ)*-quasi-invex at* S^*.
(b) (i) $\Theta_t(\cdot\,; \lambda^*, u^*, v^*)$ ($\forall\ t \in \{1, \ldots, k\}$ *are* (ρ, η, θ)*-prestrictly-pseudo-invex at* S^*
   (ii) $\Delta_j(\cdot\,; v^*)$ ($\forall\ j \in \{1, \ldots, m\}$ *are* (ρ, η, θ)*-strictly-quasi-invex at* S^*.
(c) (i) $\Theta_i(\cdot\,; \lambda^*, u^*, v^*)$ ($\forall\ t \in \{1, \ldots, k\}$ *are* (ρ, η, θ)*-prestrictly-quasi-invex at* S^*
   (ii) $\Delta_j(\cdot\,; v^*)$ ($\forall\ j \in \{1, \ldots, m\}$ *are* (ρ, η, θ)*-strictly-pseudo-invex at* S^*.

Then S^* is an efficient solution to (P).

Proof. (a) We apply the method of contradiction. Suppose that S^* is not an efficient solution to (P). Then there exists an $S^\dagger \in \Lambda^n$ such that

$$\left(\frac{F_1(S^\dagger)}{G_1(S^\dagger)}, \frac{F_2(S^\dagger)}{G_2(S^\dagger)}, \ldots, \frac{F_p(S^\dagger)}{G_p(S\dagger)} \right) \leq \left(\frac{F_1({}^*S)}{G_1(S^*)}, \frac{F_2(S^*)}{G_2(S^*)}, \ldots, \frac{F_p(S^*)}{G_p(S^*)} \right).$$

This, in turn, using (3.17) implies $F_i(S^\dagger) - \lambda_i^* G_i(S^\dagger) \leq 0\ \ \forall\, i \in \{0, 1, \ldots, p\}$, while strict inequality holds for at least one index $l \in \{0, 1, \ldots, p\}$. Since $u^* > 0$, it further implies

$$\sum_{i \in I_t} u_i^* [F_i(S^\dagger) - \lambda_i^* G_i(S^\dagger]) \leq 0 \quad \text{for } t \in K. \tag{3.22}$$

Since $v^* \geq 0$, $S, S^* \in \Lambda^n$ and, (3.16),(3.17) and (3.19) hold, we have

$$\begin{aligned}\Theta_t(S^\dagger,\lambda^*,u^*,v^*) &= \sum_{i\in I_t} u_i^*[F_i(S^\dagger)-\lambda_i^* G_i(S^\dagger)] + \sum_{j\in J_0} v_j^* H_j(S^\dagger)\\ &\leq \sum_{i\in I_t} u_i^*[F_i(S^\dagger)-\lambda_i^* G_i(S^\dagger)] \leq 0\\ &= \sum_{i\in I_t} u_i^*[F_i(S^*)-\lambda_i^* G_i(S^*)] + \sum_{j\in J_0} v_j^* H_j(S^*)\\ &= \Theta_t(S^*,\lambda^*,u^*,v^*)\end{aligned}$$

and

$$\Theta_t(S^\dagger,\lambda^*,u^*,v^*) < \Theta_t(S^*,\lambda^*,u^*,v^*).$$

Now we have using (i) that

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*)-\lambda_i^* G_i'(S^*)] + \sum_{t\in K}\sum_{j\in J_t} v_j^* H_j'(S^*), \eta(S^\dagger,S^*)\right\rangle + \rho\|\theta(S^\dagger,S^*)\|^r < 0. \tag{3.23}$$

We observe, for each $t \in M \setminus K$ and $S \in \Lambda^n$, we have

$$\sum_{t\in M\setminus K} \Delta_t(S,v^*) \leq 0 = \sum_{t\in M\setminus K} \Delta_t(S^*,v^*),$$

and hence, we find

$$\left\langle \sum_{t\in M\setminus K}\sum_{j\in J_t} v_j^* H_j'(S^*), \eta(S^\dagger,S^*)\right\rangle + \rho\|\theta(S^\dagger,S^*)\|^r \leq 0. \tag{3.24}$$

Next, it follows from (3.19), (3.20) and (3.21) that

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*)-\lambda_i^* G_i'(S^*)] + \sum_{j=1}^{m} v_j^* H_j'(S^*), \eta(S^\dagger,S^*)\right\rangle + \rho\|\theta(S^\dagger,S^*)\|^r < 0, \tag{3.25}$$

that is a contradiction to (3.19). Hence, S^* is an efficient solution to (P). Similar proofs hold for (b) and (c). □

Theorems 3.3 and 3.2 reduce to the case of the results on the generalized (η,ρ,θ)-invexity, that is, $r = 2$.

Theorem 3.5. *Suppose that $S \in \Lambda^n$ and that F_i, G_i, $i \in \{1,\dots,p\}$, H_j, $j \in \{1,\dots,m\}$ are differentiable at $S^* \in X$ and there are $u^* \in U$ and $v^* \in R_+^m$ such that*

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*)-\lambda_i^* G_i'(S^*)] + \sum_{j=1}^{m} v_j^* H_j'(S^*), \eta(S,S^*)\right\rangle + \rho\|\theta(S,S^*)\|^2 \geq 0 \quad \forall\, S \in \Lambda^n, \tag{3.26}$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \quad \text{for } i \in \{1,\dots,p\}, \tag{3.27}$$

$$v_j^* H_j(S^*) = 0 \quad \text{for } j \in \{1,\dots,m\}. \tag{3.28}$$

We further suppose that any one of the following sets of assumptions holds:

(a) (i) $\Gamma_i(\cdot;\lambda^*,u^*,v^*)$ ($\forall\ i=1,\dots,p)$ *are* (ρ,η,θ)*-pseudo-invex at* S^**;*

(ii) $\Delta_j(\cdot;v^*)$ ($\forall\ j\in\{1,\dots,m\}$ *are* (ρ,η,θ)*-quasi-invex at* S^**.*

(b) (i) $\Gamma_i(\cdot;\lambda^*,u^*,v^*)$ ($\forall\ i\in\{1,\dots,p\}$ *are* (ρ,η,θ)*-prestrictly-pseudo-invex at* S^*

(ii) $\Delta_j(\cdot;v^*)$ ($\forall\ j\in\{1,\dots,m\}$ *are* (ρ,η,θ)*-strict-quasi-invex at* S^**.*

(c) (i) $\Gamma_i(\cdot;\lambda^*,u^*,v^*)$ ($\forall\ i\in\{1,\dots,p\}$ *are* (ρ,η,θ)*-prestrictly-quasi-invex at* S^*

(ii) $\Delta_j(\cdot;v^*)$ ($\forall\ j\in\{1,\dots,m\}$ *are* (ρ,η,θ)*-strictly-pseudo-invex at* S^**.*

Then S^* *is an efficient solution to* (P).

Theorem 3.6. *Suppose that* $S\in\Lambda^n$ *and that* $F_i, G_i, i\in\{1,\dots,p\}$, $H_j, j\in\{1,\dots,m\}$ *are differentiable at* $S^*\in\Lambda^n$ *and there are* $u^*\in U$ *and* $v^*\in R^m_+$ *such that*

$$\left\langle \sum_{i=1}^{p} u_i^*[F_i'(S^*)-\lambda_i^* G_i'(S^*)]+\sum_{j=1}^{m} v_j^* H_j'(S^*),\eta(S,S^*)\right\rangle+\rho\|\theta(S,S^*)\|^2\geq 0 \quad \forall\ S\in\Lambda^n, \tag{3.29}$$

$$F_i(S^*)-\lambda_i^* G_i(S^*)=0 \quad \text{for } i\in\{1,\dots,p\}, \tag{3.30}$$

$$v_j^* H_j(S^*)=0 \quad \text{for } j\in\{1,\dots,m\}. \tag{3.31}$$

We further suppose that any one of the following sets of assumptions holds:

(a) (i) $\Theta_t(\cdot;\lambda^*,u^*,v^*)$ ($\forall\ t\in\{1,\dots,p\}$ *are* (ρ,η,θ)*-pseudo-invex at* S^**,*

(ii) $\Delta_j(\cdot;v^*)$ ($\forall\ j\in\{1,\dots,m\}$ *are* (ρ,η,θ)*-quasi-invex at* S^**.*

(b) (i) $\Theta_t(\cdot;\lambda^*,u^*,v^*)$ ($\forall\ t\in\{1,\dots,k\}$ *are* (ρ,η,θ)*-prestrictly-pseudo-invex at* S^*

(ii) $\Delta_j(\cdot;v^*)$ ($\forall\ j\in\{1,\dots,m\}$ *are* (ρ,η,θ)*-strictly-quasi-invex at* S^**.*

(c) (i) $\Theta_i(\cdot;\lambda^*,u^*,v^*)$ ($\forall\ t\in\{1,\dots,k\}$ *are* (ρ,η,θ)*-prestrictly-quasi-invex at* S^*

(ii) $\Delta_j(\cdot;v^*)$ ($\forall\ j\in\{1,\dots,m\}$ *are* (ρ,η,θ)*-strictly-pseudo-invex at* S^**.*

Then S^* *is an efficient solution to* (P).

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Received September 2, 2011
Accepted October 19, 2013