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Second Order Duality in Mathematical Programming with Support Functions

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Abstract. Wolfe and Mond-Weir type second order dual programs are formulated for a non-linear programming problem in which the objective as well as each of constraint functions contain a term of a support function. Special cases are also deduced from our results.

1. **Introduction**

Many authors have studied duality for class of nonlinear programming problems in which the objective function contains a differentiable convex function along with either a positive homogenous function or the sum of positive homogenous functions, e.g., Sinha [22], Zhang and Mond [24], Mond [11, 12], Chandra and Gulati [5] and Mond and Schechter [16, 17]. These authors have introduced the square root of positive semidefinite quadratic form $(x^T Bx)^{1/2}$ or a norm term of the type $\|Px\|$ as a positive homogenous function. The popularity of this kind of problem stems from the fact that, even though the objective function and/or constraint functions are nondifferentiable, the dual problem comes out to be a differentiable problem and hence is more amenable to handle from the computational point of view. Also as demonstrated by Sinha [22], these problems have applications in the modelling of certain stochastic programming problem. While most of these studies have considered only the Wolf type of dual, Chandra *et al.* [4] studied duality for such problems in the spirit of Mond and Weir [18] in order to relax convexity conditions assumed in aforecited references.

Mangasarian [9] was the first to identify a second order dual formulation for non-linear programs under the assumptions that are complicated and somewhat difficult to verify. Mond [13] introduced the concept of second order convex functions (named as bonvex functions by Bector and Chandra [2] and studied second order duality for nonlinear programs.

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Mond and Schechter [17], studied symmetric duality for nondifferentiable problems containing support functions of certain compact convex sets instead of the usual term of the type $(x^TBx)^{1/2}$ or $\|Px\|$. Further Husain, Abha and Jabeen [7] studied the duality for nondifferentiable nonlinear programming problem in which the objective as well as the constraint functions contains a term of a support function. Subsequently, Husain and Jabeen [8] studied its fractional case.

The purpose of this paper is to formulate Wolfe and Mond-Weir type second order dual for a nonlinear programming problem in which the objective and the constraint functions contains a term of a support function and establish various duality results for each pair of dual problems. It is well known that second order dual enjoys computational advantage over a first order dual. It is pointed out that duality results obtained in [7] become special cases of our results.

2. Notations and Preliminaries

In this section, we mention some notations to be used in the analysis of our exposition and recourse some preliminaries for easy references.

Definitions. (i) *Support function*: Let C be compact convex set in R^n. The function $S(x/C)$ given by

$$S(x/C) = \text{Max}\{z^T x : z \in C\},$$

is called a support function of C.

It may be noted that the support function $S(x/C)$ is a non differentiable convex function and has sub-differential given by

$$\partial S(x/C) = \{z \in C : z^T x = S(x/C)\}.$$

(ii) *Normal cone*: For any set $x \subseteq R^n$, the normal cone to X at a point $x \in X$ is defined by

$$N_X(x) = \{y : y^T(z - x) \leq 0, \ \forall\, z \in X\}$$

It can be easily seen that for a compact convex set C, $y \in N_C(x)$ iff $S(y/C) = x^T y$, or equivalently x is subdifferential of $S(y/C)$.

(iii) *Second order convex* (*Bonvex*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order convex, if for all $x, p, u \in R^n$

$$f(x) - f(u) \geq (x - u)^T[\nabla f(u) + \nabla^2 f(u)p] - 1/2 p^T \nabla^2 f(u)p.$$

(iv) *Second order concave* (*Boncave*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order concave, if for all $x, p, u \in R^n$

$$f(x) - f(u) \leq (x - u)^T[\nabla f(u) + \nabla^2 f(u)p] - 1/2 p^T \nabla^2 f(u)p\,.$$

(v) *Second order pseudoconvex* (*Pseudobonvex*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order pseudoconvex, if for all $x, p, u \in R^n$

$$(x-u)^T[\nabla f(u)+\nabla^2 f(u)p] \geq 0 \Rightarrow f(x) \geq f(u) - 1/p^T\nabla^2 f(u)p.$$

(vi) *Second order quasiconvex* (*Quasibonvex*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order pseudoconvex, if for all $x, p, u \in R^n$

$$f(x) - f(u) + 1/2p^T\nabla^2 f(u)p \leq 0 \Rightarrow (x-u)^T[\nabla f(u)+\nabla^2 f(u)p] \leq 0.$$

(vii) *Second order quasiconcave* (*Quasiboncave*): Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be second order quasiconcave, if for all $x, p, u \in R^n$

$$f(x) - f(u) + 1/2p^T\nabla^2 f(u)p \geq 0 \Rightarrow (x-u)^T[\nabla f(u)+\nabla^2 f(u)p] \geq 0.$$

Let $f : R^n \to R$ and $g_j : R^n \to R$ $(j = 1, 2, \ldots, m)$ be subdifferentiable functions. Let C be a compact convex set in R^n. Then consider the following nonlinear programming problem:

(P) Min $f(x)$

subject to

$$g_j(x) \leq 0 \quad (j = 1, 2, \ldots, m)$$
$$x \in C$$

The following lemmas relating to (P) will be used here:

Lemma 2.1 ([22]). *If $\bar{x}$ is an optimal solution for* (P)*, then there exist $\lambda \in R_+$ and $\mu \in R^m_+$, such that*

$$0 \in \lambda\partial f(\bar{x}) + \sum_{j=1}^{m}\mu_j\partial g_j(\bar{x}) + N_C(\bar{x})$$
$$\lambda + \sum_{j=1}^{m}\mu_j > 0$$
$$\mu_j g_j(\bar{x}) = 0, \quad j = 1, 2, \ldots, m.$$

Lemma 2.2 ([22]). *If $\bar{x}$ is an optimal solution for* (P)*, and a suitable constraint qualification* [10] *holds for* (P)*, then there exist non negative constants μ_j* ($j = 1, 2, \ldots, m$)*, such that*

$$0 \in \partial f(x) + \sum_{j=1}^{m}\mu_j\partial g_j(\bar{x}) + N_C(\bar{x})$$
$$\mu_j g_j(\bar{x}) = 0, \quad j = 1, 2, \ldots m.$$

It is to be noted that under the conditions of convexity on the functions f and g_j $(j = 1, 2, \ldots, m)$, these necessary conditions are also sufficient for the optimality of $\bar{x}$ for (P).

3. Non Differentiable Programming Problem Containing Support Functions and Duality

Let $f : R^n \to R$ and $g_j : R^n \to R$ $(j = 1,2,\ldots,m)$ be twice differentiable functions. Let C and D_j $(j = 1,2,\ldots,)$ be compact convex sets in R^n. We consider the following nondifferentiable nonlinear programming problem:

(NP) Min $f(x) + S(x/C)$

subject to

$$g_j(x) + S(x/D_j) \leq 0, \quad (j = 1,2\ldots,m). \tag{3.1}$$

In studying duality for (NP) certain optimality conditions in the non-smooth setting will be required. These conditions which can be derived from [22] along with the application of Lemma 1 and Lemma 2 are as follow:

Theorem 3.1. *If $\bar{x}$ is an optimal solution for* (NP)*, then there exists $\bar{\alpha} \in R$, $\bar{z} \in C$, $\bar{y} \in R^m$ and $\bar{w}_j \in D_j$ $(j = 1,2,\ldots,m)$ such that*

$$\bar{\alpha}(\nabla f(\bar{x}) + \bar{z}) + \sum_{j=1}^{m} \bar{y}_j(\nabla g_j(\bar{x}) + \bar{w}_j) = 0,$$

$$\sum_{j=1}^{m} \bar{y}_j(\nabla g_j(\bar{x}) + \bar{w}_j^T(\bar{x})) = 0,$$

$$\bar{z}^T(\bar{x}) = S(\bar{x}/C) \ \ and \ \ \bar{w}_j^T(\bar{x}) = S(\bar{x}/D_j), \quad \forall\ j = 1,2,\ldots,m$$

$$(\bar{\alpha},\bar{y}) \geq 0, \ \ (\bar{\alpha},\bar{y}) \neq 0.$$

When a suitable constraint qualification holds for (NP) the above Fritz John optimality conditions reduces to the Karush-Kuhn-Tucker optimality conditions, as this asserts positiveness of the multiplier $\bar{\alpha}$ associated with the objective function.

3.1. *Wolfe type duality*

Consider the following nonlinear program, which we shall prove to be a dual program to (NP)

(WD) Max $f(u) + z^T u + \sum_{j=1}^{m} y_j(g_j(u) + w_j^T(u)) - \frac{1}{2}p^T \nabla^2(f(u) + y^T g(u))p$

subject to

$$\nabla(f(u) + z^T u) + \sum_{j=1}^{m} y_j \nabla(g_j(u) + w_j) + \nabla^2(f(u) + y^T g(u))p = 0, \tag{3.2}$$

$$y \geq 0, \tag{3.3}$$

$$z \in C,\ w_j \in D_j \ \ (j = 1,2,\ldots,m) \tag{3.4}$$

Theorem 3.2 (***Weak Duality***)**.** *Let x be feasible for* (NP) *and $(u,z,y,p,w_1,w_2,\ldots,w_m)$ be feasible for* (WD) *and let for all feasible $(x,z,y,p,w_1,w_2,\ldots,w_m)$, $f(\cdot)$ and $g_j(\cdot)$*

$(j = 1, 2, \ldots, m)$ be second order convex, then

$$f(x) + S(x/C) \geq f(u) + z^T u + \sum_{j=1}^{m} y_j(g_j(u) + w_j^T(u)) - 1/2p^T \nabla^2(f(u) + y^T g(u))p.$$

i.e.,

$$\text{infimum}\,(\text{NP}) \geq \text{supremum}\,(\text{WD})\,.$$

Proof. Let x be feasible for (NP) and $(u, z, y, p, w_1, w_2, \ldots, w_m)$ be feasible for (WD), therefore, from second order convexity of $f(\cdot)$ and $g_j(\cdot)$, $(j = 1, 2, \ldots, m)$ we have

$$\left(f(x) + y^T g(x) + \sum_{j=1}^{m} y_j w_j^T x\right) - \left(f(u) + y^T g(u) + \sum_{j=1}^{m} y_j w_j^T u\right)$$
$$\geq \sum_{j=1}^{m} y_j w_j^T (x-u) - 1/2p^T \nabla^2(f(u) + y^T g(u)p)$$
$$+ (x-u)[(\nabla f(u) + \nabla y^T g(u) + \nabla^2(f(u) + y^T g(u))p)]. \qquad (3.5)$$

Now from the dual feasibility, we have

$$(x-u)(\nabla f(u) + \nabla y^T g(u) + \nabla^2(f(u) + y^T g(u))p)$$
$$= -(x-u)^T z - \sum_{j=1}^{m} y_j w_j^T (x-u) \qquad (3.6)$$

Therefore from (3.5) and (3.6) we get

$$\left(f(x) + y^T g(x) + \sum_{j=1}^{m} y_j w_j^T x\right) - \left(f(u) + y^T g(u) + \sum_{j=1}^{m} y_j w_j^T u\right)$$
$$\geq -(x-u)^T z - 1/2p^T \nabla^2(f(u) + y^T g(u))p$$

i.e.,

$$(f(x) + z^T x) - (f(u) + z^T u + y^T g(u) + \sum_{j=1}^{m} y_j w_j^T u - 1/2p^T \nabla^2(f(u) + y^T g(u))p)$$
$$\geq \left(-y^T g(x) - \sum_{j=1}^{m} y_j w_j^T x\right)$$

but $S(x/C) \geq z^T x$, whenever $z \in C$ and $S(\bar{x}/D_j) \geq w_j^T x$, whenever $w_j \in D_j$, which implies that

$$0 \geq g_j(x) + S(\bar{x}/D_j) \geq g_j(x) + w_j^T x$$
$$0 \geq y_j(g_j) + S(\bar{x}/D_j)$$

$$0 \geq \sum y_j g_j(x) + \sum y_j w_j^T x = y^T g(x) + \sum_{j=1}^{m} y_j w_j^T x$$

$$y^T g(x) + \sum_{j=1}^{m} y_j w_j^T x \leq 0.$$

As $y \geq 0$, we get

$$\left(-y^T g(x) - \sum_{j=1}^{m} y_j w_j^T x \right) \geq 0\,.$$

Hence

$$(f(x) + z^T x) \geq \left(f(u) + z^T u + y^T g(u) + \sum_{j=1}^{m} y_j w_j^T u - 1/2 p^T \nabla^2 (f(u) + y^T g(u)) p \right)$$

infimum (NP) $\geq$ supremum (WD). □

Corollary 3.3. *Let $\bar{x}$ be feasible for* (NP) *and* $(\bar{u}, \bar{z}, \bar{y}, \bar{p}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)$ *is feasible for* (WD) *with corresponding objective functions being equal. Let the hypotheses of Theorem 2 hold. Then $\bar{x}$ is optimal for* (NP) *and* $(\bar{u}, \bar{z}, \bar{y}, \bar{p}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)$ *is optimal for* (WD).

Theorem 3.4 (***Strong Duality***)**.** *Let $\bar{x}$ be optimal for* (NP) *and the suitable constraint qualification* [10] *hold. Then there exists $\bar{z} \in C$, $\bar{y} \in R^m$, $\bar{w}_j \in D_j$, $(j = 1, 2, \ldots, m)$ such that $(\bar{x}, \bar{z}, \bar{y}, \bar{p} = 0, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)$ is feasible for* (WD) *and the objective function values of* (NP) *and* (WD) *are equal. Further if the hypotheses of Theorem* 3.2 *hold then $(\bar{x}, \bar{z}, \bar{y}, \bar{p} = 0, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)$ is an optimal solution for* (WD).

Proof. Since $\bar{x}$ be an optimal solution for (NP) and a suitable constraint qualification [10] holds for (NP), then there exists $\bar{z} \in C$, $\bar{y} \in R^m_+$, $\bar{w}_j \in D_j$, $(j = 1, 2, \ldots, m)$ such that

$$\nabla f(\bar{x}) + \bar{z} + \sum_{j=1}^{m} \bar{y}_j (\nabla g_j(\bar{x}) + \bar{w}_j) = 0,$$

$$\sum_{j=1}^{m} \bar{y}_j (g_j(\bar{x}) + \bar{w}_j^T \bar{x}) = 0,$$

$$\bar{z}^T \bar{x} = S(\bar{x}/C), \text{ and } \bar{w}_j^T \bar{x} = S(\bar{x}/D_j), \ \forall\, j = 1, 2, \ldots, m.$$

Hence $(\bar{x}, \bar{z}, \bar{y}, \bar{p} = 0, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)$ is feasible for (WD) and

$$f(\bar{x}) + \bar{z}^T \bar{x} + \sum_{j=1}^{m} \bar{y}_j (g_j(\bar{x}) + \bar{w}_j^T) = f(\bar{x}) + S(\bar{x}/C).$$

That is, the objective function values of (NP) and (WD) are equal. Remainder of the proof now immediately follows from Corollary 3.3. □

3.2. *Second order converse duality*

In this, we establish converse duality theorem which yields the solution of (P) from the solution of (WD).

Theorem 3.5 (***Converse Duality***). *Let* $(\bar{u},\bar{z},\bar{y},\bar{p},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_m)$ *is optimal for* (WD) *and the Hessian matrix* $\nabla^2\left(f(\bar{u})+\sum_{j=1}^{m} y_j g_j(\bar{u})\right)$ *be non-singular and* $\nabla^2\left(\nabla^2 f(\bar{u})+\nabla^2\sum_{j=1}^{m} y_j g_j(\bar{u})\right)$ *be either positive or negative definite. Then* $\sum_{j=1}^{m} y_j g_j(\bar{u})+S(\bar{u}/D)=0$*, and* $\bar{u}$ *is feasible for* (NP) *and the objective function values of* (NP) *and* (WD) *are equal. Further if the hypotheses of Theorem* 3.2 *hold then* $\bar{u}$ *is an optimal for* (NP).

Proof. First we rewrite problem (WD) in the form of (P), for this let $q=(u,z,y,p,w_1,w_2,\ldots,w_m)\in R^{(3+m)n+m}$ and

$$F(q)=(f(\bar{u})+\bar{z}^T(\bar{u}))+\sum_{j=1}^{m}\bar{y}_j(g_j(\bar{u})+\bar{w}_j^T\bar{u})-1/2\bar{p}^T\nabla^2(f(\bar{u})+y^Tg(\bar{u}))\bar{p},$$

$$G(q)=(\nabla f(\bar{u})+\bar{z})+\sum_{j=1}^{m}\bar{y}_j(\nabla g_j(\bar{u})+\bar{w}_j)+\nabla^2(f(\bar{u})+\bar{y}^Tg(\bar{u}))\bar{p},$$

$$H(q)=-y.$$

Let the set S be defined by $S=\{q:q=(u,z,y,p,w_1,w_2,\ldots,w_m),\ z\in C,\ w_j\in D_j,\ \forall\ j=1,2,\ldots m\}$, then problem (WD) may be rewritten as follows:

Max $F(q)$

subject to

$G(q)=0,$

$H(q)\leq 0,$

$q\in S.$

As $\bar{q}=(\bar{u},\bar{z},\bar{y},\bar{p},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_m)$ is optimal for (WD), from Lemma 1, there exist constants $\alpha\geq 0$, $\mu_j\geq 0$, $j=1,2,\ldots,m$ and λ_i, $i=1,2,\ldots n$, not all zero, and the normal cone to S at $\bar{q}$ as $N_S(\bar{q})$ such that

$$-\left[\alpha(\nabla f(\bar{u})+\bar{z})+\sum_{j=1}^{m}\bar{y}_j(\nabla g_j(\bar{u})+\bar{w}_j)-1/2\nabla\bar{p}^T\nabla^2(f(\bar{u})+\bar{y}^Tg(\bar{u}))\bar{p}\right]$$
$$+(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^Tg(\bar{u}))\lambda+\lambda\nabla(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^Tg(\bar{u}))\bar{p}=0 \quad (3.7)$$

$$-\alpha(-(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^Tg(\bar{u}))\bar{p})+\lambda(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^Tg(\bar{u}))=0 \quad (3.8)$$

$$-\alpha(g_j(\bar{u})+\bar{w}_j^T\bar{u}-1/2\bar{p}^T\nabla^2 g_j(\bar{u})\bar{p})+\lambda(\nabla g_j(\bar{u})+\bar{w}_j+\nabla^2 g_j(\bar{u})\bar{p})$$
$$-\mu_j=0,\ \forall\ j=1,2,\ldots m \quad (3.9)$$

$$-\alpha\bar{u}+\lambda\in N_C(\bar{z}), \tag{3.10}$$

$$-\alpha\bar{u}\bar{y}_j+\lambda\bar{y}_j\in N_{D_j}(\bar{w}_j), \tag{3.11}$$

$$\mu_j y_j=0,\ \forall\ j=1,2,\dots,m \tag{3.12}$$

From (3.8) we have,

$$(\alpha p+\lambda)(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u}))=0,$$

But from nonsingularity of the matrix $(\nabla^2 f(\bar{u})+\nabla^2 y^T g(\bar{u}))$ we have $(\alpha p+\lambda)$ $=0$. If possible, let $\alpha=0$ then $\lambda=0$. From these values, (3.9) implies $\mu_j=0$, $\forall\ j=1,2,\dots m$, which makes all the multipliers equal to zero. Since this cannot happen as it contradicts $(\alpha,\lambda,\mu)\neq 0$. So we must have $\alpha\neq 0$, so $\alpha>0$.

Using the equality constraint of the dual problem in equation (3.7) we have,

$$\alpha[(\nabla^2(f(\bar{u})+\bar{y}^T g(\bar{u}))p)-1/2\bar{p}^T\nabla^2(f(\bar{u})+\bar{y}^T g(\bar{u}))\bar{p}]$$
$$+(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u}))\lambda+\lambda\nabla(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u}))\bar{p}=0\,.$$

This can be written as

$$(\alpha p+\lambda)(\nabla^2(f(\bar{u})+y^T g(\bar{u}))p)+\left(\lambda-\frac{\alpha p}{2}\right)\nabla(\nabla^2(f(\bar{u})+y^T g(\bar{u}))\bar{p}))=0.$$

This along with $\alpha p+\lambda=0$ yields,

$$\frac{\alpha p}{2}\nabla(\nabla^2(f(\bar{u})+\bar{y}^T g(\bar{u}))\bar{p})=0.$$

Because of positiveness of α. This equation is simplified as

$$p^T\nabla(\nabla^2(f(\bar{u})+\bar{y}^T g(\bar{u}))\bar{p})=0$$

which by the condition of ∇ $(\nabla^2(f\bar{u}+y^T g(\bar{u})))$ to be either positive or negative definite implies $p=0$. Now $(\alpha p+\lambda)=0$, hence $\lambda=0$. Then equation (3.9) implies that

$$(-\alpha+\lambda)\{g_j(\bar{u})+w_j\}+\left(-\frac{\alpha p}{2}+\lambda\right)\nabla^2 g_j(\bar{u})p-\mu_j$$
$$-\alpha(\nabla g_j(\bar{u})+w_j)+0=\mu_j$$
$$\nabla g_j(\bar{u})+w_j=-\frac{\mu_j}{\alpha}\leq 0,$$
$$g_j(\bar{u})+\bar{w}_j^T\bar{u}\leq 0,\ \forall\ j=1,2,\dots m.$$

Now from (3.10) and (3.11) we have $\bar{u}\in N_C(\bar{z})$ and $\bar{u}\in N_{D_j}(\bar{w}_j)$ so that $\bar{z}^T\bar{u}=S(\bar{u}/C)$, $\bar{w}_j^T\bar{u}=S(\bar{u}/D_j)$, $\forall\ j=1,2,\dots,m$. Hence

$$g_j(\bar{u})+\bar{w}_j^T\bar{u}=g_j(\bar{u})+S(\bar{u}/D_j)\leq 0,\ \forall\ j=1,2,\dots,m,$$

which implies that $\bar{u}$ is feasible for problem (NP). Also from (3.9) and (3.12) we get

$$\bar{y}_j(g_j(\bar{u})+\bar{w}_j^T\bar{u})=0,\quad j=1,2,\dots m.$$

Therefore

$$(f(\bar{u})+\bar{z}^T\bar{u})+\sum_{j=1}^{m}\bar{y}_j(g_j(\bar{u})+\bar{w}_j^T\bar{u})-1/2\bar{p}^T\nabla^2(f(\bar{u})+\bar{y}^Tg(\bar{u}))\bar{p}$$
$$=f(\bar{u})+S(\bar{u}/C).$$

This by Corollary 1 implies that $\bar{u}$ is optimal for (NP). □

3.3. *Mond and Weir type duality*

We state the following problem as a Mond-Weir type second order dual for the problem (NP).

(SMWD) Max $f(u)+z^Tu-1/2p^T\nabla^2(f(u))p$

subject to

$$\nabla f(u)+z+\sum_{j=1}^{m}y_j(\nabla g_j(u)+w_j)+\nabla^2(f(u)+y^Tg(u))p=0, \quad (3.13)$$

$$\sum_{j=1}^{m}y_j(g_j(u)+w_j^Tu)-1/2p^T\nabla^2(y^Tg(u))p\geq 0, \quad (3.14)$$

$$y\geq 0, \quad (3.15)$$

$$z\in C, w_j\in D, \ \forall\ j=1,2,\dots m \quad (3.16)$$

Theorem 3.6 (***Weak Duality***)**.** *Let x be feasible for* (NP) *and* $(u,z,y,p,w_1,w_2,\dots,w_m)$ *be feasible for* (SMWD) *and let for all feasible* $(x,u,z,y,p,w_1,w_2,\dots,w_m)$ *to* (NP) *and* (SMWD)*, $f(\cdot)+(\cdot)^Tz$ is second order pseudoconvex and $\sum_{j=1}^{m}y_j(g_j(\cdot)+(\cdot)^Tw_j)$ is second order quasiconvex, then*

$$f(x)+S(x/D_j)\geq f(u)+z^Tu-\frac{1}{2}p^T\nabla^2 f(u)p\,.$$

Proof. By the primal feasibility of x and dual feasibility of $(u,z,y,p,w_1,w_2,\dots,w_m)$, we have

$$\sum_{j=1}^{m}y_j(g_j(x)+S(x/D_j))\leq\sum_{j=1}^{m}y_j(g_j(x)+w_j^Tu)-\frac{1}{2}p^T\nabla^2(y^Tg(u))p\,.$$

This in view of $w_j^Tx\leq S(x/D_j)$, $\forall\ j=1,2,\dots,m$, gives

$$\sum_{j=1}^{m}y_j(g_j(x)+w_j^Tx)\leq\sum_{j=1}^{m}y_j(g_j(x)+w_j^Tu)-\frac{1}{2}p^T\nabla^2(y^Tg(u))p\,. \quad (3.17)$$

Because of second order quasiconvexity of $\sum_{j=1}^{m}y_j(g_j(\cdot)+(\cdot)^Tw_j)$, (3.17) yields,

$$(x-u)^T\left(\sum_{j=1}^{m}y_j(\nabla g_j(u)+w_j)+\nabla^2(y^Tg(u))p\right)\leq 0\,.$$

This is conjunction with (3.13), we get

$$(x-u)^T(\nabla f(u)+z+\nabla^2(f(u))p)\geq 0,$$

which by second order pseudoconvexity of $f(\cdot)+(\cdot)^T z$ gives

$$f(x)+z^T x\geq f(u)+z^T u-\frac{1}{2}p^T\nabla^2 f(u)p.$$

Since $z^T x\leq S(x/C)$, as earlier, we have

$$f(x)+S(x/C)\geq f(u)+z^T u-\frac{1}{2}p^T\nabla^2 f(u)p. \qquad \square$$

Corollary 3.7. *Let $\bar{x}$ be feasible for* (NP) *and $(\bar{u},\bar{z},\bar{y},\bar{p},\bar{w}_1,\bar{w}_2,\dots,\bar{w}_m)$ is feasible for* (SMWD) *with corresponding objective function being equal. Let the hypotheses of Theorem* 3.6 *hold. Then $\bar{x}$ is optimal for* (NP) *and $(\bar{u},\bar{z},\bar{y},\bar{p},\bar{w}_1,\bar{w}_2,\dots,\bar{w}_m)$ is optimal for* (SMWD).

Theorem 3.8 (***Strong Duality***)**.** *Let $\bar{x}$ be optimal for* (NP) *and the suitable constraint qualification holds for* (NP). *Then there exists $\bar{z}\in C$, $\bar{y}\in R^m$, $\bar{w}_j\in D_j$ $(j=1,2,\dots,m)$ such that $(\bar{x},\bar{z},\bar{y},\bar{p}=0,\bar{w}_1,\bar{w}_2,\dots,\bar{w}_m)$ is feasible for* (SMWD) *and the objective function values of* (NP) *and* (MWD) *are equal. Further if the hypotheses of Theorem* 3.6 *hold then $(\bar{x},\bar{z},\bar{y},\bar{p}=0,\bar{w}_1,\bar{w}_2,\dots,\bar{w}_m)$ is optimal for* (SMWD).

Proof. Since $\bar{x}$ be optimal for (NP) and the suitable constraint qualification holds for (NP), then there exists $\bar{z}\in C$, $\bar{y}\in R^m_+$, $\bar{w}_j\in D_j$ $(j=1,2,\dots,m)$ such that

$$\nabla f(\bar{x})+\bar{z}+\sum_{j=1}^{m}\bar{y}_j(\nabla g_j(\bar{x})+\bar{w}_j)=0,$$

$$\sum_{j=1}^{m}\bar{y}_j(g_j(\bar{x})+\bar{w}_j^T\bar{x})=0,$$

$$\bar{z}^T\bar{x}=S(\bar{x}/C),\ \text{and}\ \ \bar{w}_j^T\bar{x}=S(\bar{x}/D_j),\ \forall\ j=1,2\dots,m$$

Hence $(\bar{x},\bar{z},\bar{y},\bar{p}=0,\bar{w}_1,\bar{w}_2,\dots,\bar{w}_m)$ is feasible for (MWD) and

$$f(\bar{x})+\bar{z}^T\bar{x}-\frac{1}{2}\bar{p}^T\nabla^2 f(\bar{x})\bar{p}=f(\bar{x})+S(x/C).$$

Therefore the objective function values of (NP) and (SMWD) are equal. Rest of the proof now follows from Corollary 3.7. $\square$

3.4. *Second order converse duality*

In this section, we shall validate a second order converse duality theorem.

Theorem 3.9 (***Converse Duality***)**.** *Let $(\bar{x},\bar{z},\bar{y},\bar{w},\bar{p})$ be an optimal solution to* (SMWD) *at which*

(H_1): (a) *the* $n \times n$ *Hessian matrix* $\nabla^2\left(\sum_{j=1}^{m} \bar{y}_j g_j(\bar{x})\right)$ *is positive definite and* $\bar{p}^T \sum_{j=1}^{m} \bar{y}_j (g_j(\bar{x}) + \bar{w}_j) \geq 0$, *or*

(b) *the Hessian matrix* $\nabla^2(\bar{y}_j^T g_j(\bar{x}))$ *is negative definite and* $\bar{p}^T \nabla \sum_{j=1}^{m} \bar{y}_j (\nabla g_j(\bar{x}) + \bar{w}_j) \leq 0$,

(H_2): *the set* $\{[\nabla^2 f(\bar{x})]_i, [\nabla^2(\bar{y} g(\bar{x}))]_i \mid i = 1, 2, \dots, n\}$, *of vectors is linearly independent, where* $[\nabla^2 f(\bar{x})]_i$ *is the ith row of* $[\nabla^2 f(\bar{x})]$ *and* $[\nabla^2(\bar{y}^T g(\bar{x}))]_i$ *is ith row of the matrix* $[\nabla^2(\bar{y}^T g(\bar{x}))]$.

(H_3): *the vectors* $\sum_{j=1}^{m} \bar{y}_j (g_j(\bar{x}) + \bar{w}_j) \neq 0$

If, for all feasible $(x, z, y, u, w_1, w_2, \dots, w_m, p)$, $f(\cdot) + (\cdot)^T$ *is second order pseudoconvex and* $\sum_{j=1}^{m} \bar{y}_j (g_j(\cdot) + (\cdot)^T w_j)$ *is second order quasiconvex, then* $\bar{x}$ *is an optimal solution of the problem* (NP).

Proof. Since $(\bar{x}, \bar{z}, \bar{y}, \bar{w})$, where $\bar{w} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_m)$ is an optimal solution of (SM-WD), by generalized Fritz John necessary optimality conditions [10], there exists, $\alpha \in R$, $\beta \in R^n$, $\theta \in R$, and $\mu \in R^m$, such that

$$\begin{aligned}
&\alpha\left\{-(f(\bar{x}) + \bar{z}) + \frac{1}{2}\bar{p}^T \nabla\left[\nabla^2(f(\bar{x}))\bar{p}\right]\right\} \\
&\quad + \beta^T\{\nabla^2(f(\bar{x}) + \bar{y}^T g(\bar{x})) + \nabla(\nabla^2(f(\bar{x}) + \bar{y}^T g(\bar{x}))\bar{p})\} \\
&\quad - \theta\left\{\sum_{j=1}^{m} \bar{y}_j(\nabla g_j(\bar{x}) + \bar{w}_j) - \frac{1}{2}\bar{p}^T \nabla[(\nabla^2(\bar{y}^T g(\bar{x})))\bar{p}]\right\} = 0,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\beta\{\nabla(g_j(\bar{x}) + \bar{w}_j) + \nabla^2 g_j(\bar{x})\bar{p}\} \\
&\quad - \theta^T\left\{g_j(\bar{x}) + \bar{x}_j^T \bar{w}_j - \frac{1}{2}\bar{p}^T \nabla^2 g_j(\bar{x})\bar{p}\right\} - \mu_j = 0, \ j = 1(1)m,
\end{aligned} \tag{3.19}$$

$$(\alpha\bar{p} + \beta)^T \nabla f(\bar{x}) + (\theta\bar{p} + \beta)^T \nabla^2(\bar{y} g(\bar{x})) = 0, \tag{3.20}$$

$$\theta\left\{\sum_{j=1}^{m} \bar{y}_j(g_j(\bar{x}) + \bar{x}_j^T \bar{w}_j) - \frac{1}{2}\bar{p}^T \nabla^2 \bar{y}_j (g_j(\bar{x}))\bar{p}\right\} = 0, \tag{3.21}$$

$$\mu^T \bar{y} = 0, \tag{3.22}$$

$$-\alpha\bar{x} + \beta \in N_c(\bar{z}), \tag{3.23}$$

$$(\beta - \theta)\bar{y}_j, \ \bar{x} \in N_{D_j}(\bar{w}_j), \quad j = 1, (1)m, \tag{3.24}$$

$$(\alpha, \theta, \mu) \geq 0, \tag{3.25}$$

$$(\alpha, \beta, \theta, \mu) \neq 0. \tag{3.26}$$

The relation (3.20), in view of assumption (A_2) yields,

$$\alpha\bar{p} + \beta = 0 \quad \text{and} \quad \theta\bar{p} + \beta = 0. \tag{3.27}$$

Multiplying (3.19) by $\bar{y}_j$, and summing over j, we get

$$\beta^T\left\{\sum_{j=1}^{m}\bar{y}_j(\nabla(g_j(\bar{x})+\bar{w}_j)+\nabla^2(\bar{y}^Tg(\bar{x}))\bar{p})\right\}$$
$$-\theta\left\{\sum_{i=1}^{m}\bar{y}_j(g_j(\bar{x})+\bar{x}_j^T\bar{w}_j)-\frac{1}{2}\bar{p}\nabla^2(\bar{y}^Tg(\bar{x}))\bar{p}\right\}=0,\quad j=1(1)m. \tag{3.28}$$

Using (3.21) in the above relation, we get,

$$\beta\left\{\sum_{j=1}^{m}\bar{y}_j(\nabla(g_j(\bar{x})+\bar{w}_j)+\nabla^2\bar{y}^Tg(\bar{x})\bar{p})\right\}=0. \tag{3.29}$$

The relation (3.18) together with the equality constraint of the dual, yields

$$(\alpha-\theta)\left\{\sum_{j=1}^{m}\bar{y}_j(\nabla(g_j(\bar{x})+\bar{w}_j))\right\}+(\alpha\bar{p}+\beta)^T[\nabla^2f(\bar{x})+\nabla(\nabla^2f(\bar{x})\bar{p})]$$
$$+(\beta+\alpha\bar{p})^T[\nabla^2(\bar{y}g(\bar{x}))+\nabla(\nabla^2(\bar{y}g(\bar{x}))\bar{p})]$$
$$+\frac{1}{2}(\alpha\bar{p})^T\nabla(\nabla^2f(\bar{x})\bar{p})-(\alpha\bar{p})^T\nabla(\nabla^2f(\bar{x})\bar{p})$$
$$+\left(\frac{\theta\bar{p}}{2}\right)^T\nabla(\nabla^2(\bar{y}g(\bar{x}))\bar{p})-\alpha p\nabla(\nabla^2(\bar{y}g(\bar{x}))\bar{p})=0\,.$$

Using (3.27) in this equation, we have,

$$(\alpha-\theta)\left\{\sum_{j=1}^{m}\bar{y}_j(\nabla(g_j(\bar{x})+\bar{w}_j))\right\}$$
$$-\left(\frac{\beta}{2}\right)^T(\nabla(\nabla^2(\bar{y}^Tf(\bar{x})p))+\nabla(\nabla^2(\bar{y}^Tg(\bar{x}))\bar{p}))=0\,. \tag{3.30}$$

If $(\alpha,\theta)=0$, then (3.27) implies $\beta=0$ and $\mu=0$ from (3.19) consequently we get $(\alpha,\beta,\theta,\mu)=0$ contradicting (3.26). Thus, $(\alpha,\theta)\neq 0$, this implies that at least one of these multipliers α and θ must be positive. We claim $\bar{p}=0$. Suppose that $\bar{p}\neq 0$, then (3.27) yields,

$$(\alpha-\theta)\bar{p}=0\,.$$

This implies $\alpha=\theta>0$. So from (3.29) along with (3.27), we have

$$\bar{p}^T\left\{\sum_{j=1}^{m}\bar{y}_j(\nabla(g_j(\bar{x})+\bar{w}_j)+\nabla^2(\bar{y}^Tg(\bar{x}))\bar{p})\right\}=0\,. \tag{3.31}$$

Since $\nabla^2\left(\sum_{j=1}^{m}\bar{y}_j g_j(\bar{x})\right)$ is positive definite, i.e., $\bar{p}^T\nabla^2\left(\sum_{j=1}^{m}\bar{y}_j g_j(\bar{x})\right)\bar{p} > 0$ and $\bar{p}^T\sum_{j=1}^{m}\bar{y}_j(g_j(\bar{x})+\bar{w}_j) \geq 0$, we have

$$\bar{p}^T\left\{\sum_{j=1}^{m}\bar{y}_j(\nabla(g_j(\bar{x})+\bar{w}_j)+\nabla^2(\bar{y}^T g(\bar{x}))\bar{p})\right\} > 0.$$

This is contradicted by (3.31). Hence $\bar{p}=0$. By this, (3.27) implies $\beta=0$.
From (3.19), we have

$$\Rightarrow \qquad g_j(\bar{x})+\bar{w}_j^T\bar{x} = -\frac{\mu_j}{\theta} \leq 0, \quad j=1,2,\ldots,m. \tag{3.32}$$

From (3.24), we have

$$\bar{x}^T\bar{w}_j = S(\bar{x}\,|\,D_j), \quad j=1,2,\ldots,m.$$

Using this in (3.32), we obtain

$$\Rightarrow \qquad g_j(\bar{x})+S(\bar{x}\,|\,D_j) \leq 0, \quad j=1,2,\ldots,m.$$

This implies is feasible for (NP).
Multiplying (3.32) by $\bar{y}_i$ and adding over i we have

$$\sum_{j=1}^{m}\bar{y}_j(g(\bar{x})+\bar{w}_j\bar{x}) = 0. \tag{3.33}$$

Now consider

$$(f(\bar{x})+\bar{x}^T\bar{z}) - \frac{1}{2}\bar{p}^T[\nabla^2(f(\bar{x}))\bar{p}] = f(\bar{x})+\bar{x}^T\bar{z} \text{ using } p=0.$$

From (3.23), we have

$$\bar{x}^T\bar{z} = S(\bar{x}|C).$$

Thus

$$(f(\bar{x})+\bar{x}\bar{z}) - \frac{1}{2}\bar{p}^T[\nabla^2(f(\bar{x}))\bar{p}] = f(\bar{x})+s(\bar{x}|C).$$

If for all feasible $(x,z,y,u,w_1,w_2,\ldots w_m,p)$, $f(\cdot)+(\cdot)^T z$ is second order pseudoconvex and $\sum_{j=1}^{m}\bar{y}_j(g_j(\cdot)+(\cdot)^T w_j)$, is second order quasiconvex, by Theorem 3.6, then $\bar{x}$ is an optimal solution of the problem (NP). □

4. Special cases

Now for $p = 0$, the dual program (WD) and (MD), becomes the Wolfe and Mond Weir type programs for (NP) studied by Husain *et al.* [7]

(WD) $$\text{Max } (f(u) + z^T u) + \sum_{j=1}^{m} y_j (g_j(u) + w_j^T u)$$

subject to

$$(\nabla f(u) + z) + \sum_{j=1}^{m} y_j (\nabla g_j(u) + w_j) = 0$$

$$y \geq 0,$$

$$z \in C, w_j \in D_j, \quad j = 1, 2, \ldots, m.$$

(MD) $$\text{Max } (f(u) + z^T u)$$

subject to

$$(\nabla f(u) + z) + \sum_{j=1}^{m} y_j (\nabla g_j(u) + w_j) = 0$$

$$y \geq 0,$$

$$z \in C, w_j \in D_j, \quad j = 1, 2, \ldots, m.$$

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