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Orthogonal Wavelet Packets in Discrete Periodic Spaces and Applications

Research Article

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Abstract. This paper proposes the construction and application of orthogonal wavelet packets in discrete space $\ell^2(Z_N)$. First, we give the definition and construction of orthogonal wavelet packets. Moreover, the corresponding orthogonal decomposition is proved. Then, the realization of decomposition and reconstruction algorithm is studied. Finally, a numerical example for signal processing is given, which shows that signal processing based on wavelet packets in discrete spaces can gain better effect in some cases.

Keywords. Orthogonal basis; Wavelet packets; Decomposition algorithm; Reconstruction algorithm

MSC. 42C15, 42C40

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1. Introduction

Wavelet analysis has made great progress in the past twenty years, but most of them focus on the function space $L^2(R)$. In wavelet decomposition procedure, the generic step splits the approximation coefficients into two parts and obtains a vector of approximation coefficients and a vector of detail coefficients, both at a coarser scale. Then the next step consists of splitting the new approximation coefficient vector, successive details are never reanalyzed. In the corresponding wavelet packet situation ([1]), each detail coefficient vector is also decomposed into two parts using the same approach as in approximation vector splitting, this offers the richest analysis. Since the signals gathered in practice is discrete, it is more natural and more important to study the wavelet theory in discrete spaces ([2]-[8]). This paper mainly studies the wavelet packets in discrete periodic space

$$\ell^2(Z_N) = \{z = (z(0), z(1), \ldots, z(N-1)) : \ z(j) \in C, \ 0 \le j \le N-1\}.$$

In convention, $z(j+N) = z(j), \ j \in Z$. It is a Hilbert space under the inner product and norm

$$\langle z, \omega \rangle = \sum_{k=0}^{N-1} z(k)\overline{\omega(k)}, \quad \|z\| = \left(\sum_{k=0}^{N-1} |z(k)|^2\right)^{\frac{1}{2}}.$$

The discrete Fourier transform $\widehat{z} = \left(\widehat{z}(0), \widehat{z}(1), \ldots, \widehat{z}(N-1)\right) \in \ell^2(Z_N)$ is defined as

$$\widehat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-\frac{2\pi i m n}{N}}.$$

For $z \in \ell^2(Z_N)$, define the following conjugate reflection, translation operator, up-sampling and down-sampling ($N = 2M$) as

$$\widetilde{z}(n) = \overline{z(-n)} = \overline{z(N-n)}, \quad (R_k z)(n) = z(n-k).$$

$$U : \ell^2(Z_N) \to \ell^2(Z_{2N}), \quad U(z)(n) = \begin{cases} z\left(\frac{n}{2}\right), & n|2, \\ 0, & n \nmid 2; \end{cases}$$

$$D : \ell^2(Z_N) \to \ell^2(Z_M), \quad D(z)(n) = z(2n).$$

2. Construction of Wavelet Packets

In this section, we will give the specific method for constructing orthogonal wavelet packets.

Definition 2.1. Suppose N is divisible by 2^p. A p^{th}-wavelet filter sequence is a sequence of vectors $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ such that, for each $\ell = 1, 2, \ldots, p$, $u_\ell, v_\ell \in \ell^2\left(Z_{\frac{N}{2^{\ell-1}}}\right)$, and the system matrix

$$A_\ell(n) = \frac{1}{\sqrt{2}} \begin{pmatrix} \widehat{u}_\ell(n) & \widehat{v}_\ell(n) \\ \widehat{u}_\ell\left(n + \frac{N}{2^\ell}\right) & \widehat{v}_\ell\left(n + \frac{N}{2^\ell}\right) \end{pmatrix}$$

is unitary for all $n = 0, 1, \ldots, \frac{N}{2^\ell} - 1$.

As described in [2], p^{th} wavelet filter sequences can be obtained by folding method, that is, if we have $u_1, v_1 \in \ell^2(Z_N)$ such that $A_1(n)$ is unitary for $n = 0, 1, \ldots, \frac{N}{2} - 1$, then a p^{th} wavelet filter sequence is defined by

$$u_\ell(n) = \sum_{k=0}^{2^{\ell-1}-1} u_1\left(n + \frac{kN}{2^{\ell-1}}\right), \quad v_\ell(n) = \sum_{k=0}^{2^{\ell-1}-1} v_1\left(n + \frac{kN}{2^{\ell-1}}\right).$$

Definition 2.2. Suppose N is divisible by 2^p, where p is a positive integer. Let B be a set of the form

$$\left\{R_{2^p k} g_p^{(0)}\right\}_{k=0}^{\frac{N}{2^p}-1} \cup \left\{R_{2^p k} g_p^{(1)}\right\}_{k=0}^{\frac{N}{2^p}-1} \cup \ldots \cup \left\{R_{2^p k} g_p^{(2^p-1)}\right\}_{k=0}^{\frac{N}{2^p}-1}$$

for some $g_p^{(0)}, g_p^{(1)}, \ldots, g_p^{(2^p-1)} \in \ell^2(Z_N)$. If B forms an orthonormal basis for $\ell^2(Z_N)$, we call B a p^{th} stage wavelet packets for $\ell^2(Z_N)$.

The following two lemmas are taken from [2]:

Lemma 2.3. *Suppose $u, v \in \ell^2(Z_N)$, the set $\left\{R_{2k} u\right\}_{k=0}^{\frac{N}{2}-1} \cup \left\{R_{2k} v\right\}_{k=0}^{\frac{N}{2}-1}$ is an orthonormal set with N elements if and only if the system matrix $A(n)$ is unitary for $n = 0, 1, \ldots, \frac{N}{2}-1$.*

Lemma 2.4. *Suppose N is divisible by 2^ℓ, $g_{\ell-1} \in \ell^2(Z_N)$, and the set $\left\{R_{2^{\ell-1}k} g_{\ell-1}\right\}_{k=0}^{\frac{N}{2^{\ell-1}}-1}$ is orthonormal with $\frac{N}{2^{\ell-1}}$ elements. Suppose $u_\ell, v_\ell \in \ell^2\left(Z_{\frac{N}{2^{\ell-1}}}\right)$ and the corresponding system matrix $A_\ell(n)$ is unitary for $n = 0, 1, \ldots, \frac{N}{2^\ell}-1$. Define*

$$f_\ell = g_{\ell-1} * U^{\ell-1}(v_\ell) \ \text{ and } \ g_\ell = g_{\ell-1} * U^{\ell-1}(u_\ell).$$

Then $\{R_{2^\ell k} f_\ell\}_{k=0}^{\frac{N}{2^\ell}-1} \cup \{R_{2^\ell k} g_\ell\}_{k=0}^{\frac{N}{2^\ell}-1}$ is an orthonormal set with $\frac{N}{2^{\ell-1}}$ elements.

Suppose $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ are p^{th} stage wavelet filter sequences, let

$$g_1^{(0)} = u_1, \quad g_1^{(1)} = v_1.$$

For $\ell \geq 2$, define

$$g_\ell^{(2n)} = g_{\ell-1}^{(n)} * U^{\ell-1}(u_\ell), \quad g_\ell^{(2n+1)} = g_{\ell-1}^{(n)} * U^{\ell-1}(v_\ell). \tag{2.1}$$

Furthermore, for $n \geq 0$, define the spaces

$$U_{-\ell}^{(n)} = \operatorname{span}\left\{R_{2^\ell k} g_\ell^{(n)} \,\middle|\, k = 0, 1, \ldots, \frac{N}{2^\ell}-1\right\}.$$

Then, we have the following orthogonal decomposition:

Theorem 2.5. *It holds that for $n = 0, 1, \ldots, 2^{\ell-1}-1$, there has*

$$U_{-\ell+1}^{(n)} = U_{-\ell}^{(2n)} \oplus U_{-\ell}^{(2n+1)}.$$

Here, $U_0^{(0)} = \ell^2(Z_N)$.

Proof. Since $A_1(n)$ is unitary for $n = 0, 1, \ldots, \frac{N}{2}-1$, by Lemma 2.3, we know

$$\left\{R_{2k} u_1\right\}_{k=0}^{\frac{N}{2}-1} \cup \left\{R_{2k} v_1\right\}_{k=0}^{\frac{N}{2}-1}$$

is orthonormal. Furthermore, $\ell^2(Z_N) = U_{-1}^{(0)} \cup U_{-1}^{(1)}$. Note that $u_2, v_2 \in \ell^2(Z_{\frac{N}{2}})$ and the system matrix $A_2(n)$ is unitary for $n = 0, 1, \ldots, \frac{N}{2^2} - 1$, moreover,

$$g_2^{(0)} = u_1 * U(u_2), \qquad g_2^{(1)} = u_1 * U(v_2);$$
$$g_2^{(2)} = v_1 * U(u_2), \qquad g_2^{(3)} = v_1 * U(v_2).$$

By Lemma 2.4, $\left\{R_{2^2k} g_2^{(0)}\right\}_{k=0}^{\frac{N}{2^2}-1} \cup \left\{R_{2^2k} g_2^{(1)}\right\}_{k=0}^{\frac{N}{2^2}-1}$ and $\left\{R_{2^2k} g_2^{(2)}\right\}_{k=0}^{\frac{N}{2^2}-1} \cup \left\{R_{2^2k} g_2^{(3)}\right\}_{k=0}^{\frac{N}{2^2}-1}$ are orthonormal sets, then $U_{-2}^{(0)} \perp U_{-2}^{(1)}$ and $U_{-2}^{(2)} \perp U_{-2}^{(3)}$. Furthermore, by induction, one has

$$U_{-\ell}^{(0)} \perp U_{-\ell}^{(1)},\ U_{-\ell}^{(2)} \perp U_{-\ell}^{(3)},\ \ldots,\ U_{-\ell}^{(2^\ell-2)} \perp U_{-\ell}^{(2^\ell-1)}.$$

Therefore, $U_{-\ell}^{(2n)} \perp U_{-\ell}^{(2n+1)}$. Moreover, for each $n = 0, 1, \ldots, 2^\ell - 1$, the set

$$\left\{R_{2^\ell k} g_\ell^{(n)} \,\middle|\, k = 0, 1, \ldots, \frac{N}{2^\ell} - 1\right\}$$

is orthonormal basis for $U_{-\ell}^{(n)}$. Next, we claim that $U_{-\ell}^{(2n)}$ and $U_{-\ell}^{(2n+1)}$ are subspaces of $U_{-\ell+1}^{(n)}$. To see this, for $k = 0, 1, \ldots, \frac{N}{2^\ell} - 1$,

$$\begin{aligned}
R_{2^\ell k} g_\ell^{(2n)}(m) &= g_\ell^{(2n)}(m - 2^\ell k) \\
&= g_{\ell-1}^{(n)} * U^{\ell-1}(u_\ell)(m - 2^\ell k) \\
&= \sum_{i=0}^{N-1} g_{\ell-1}^{(n)}(m - 2^\ell k - i) U^{\ell-1}(u_\ell)(i) \\
&= \sum_{j=0}^{\frac{N}{2^{\ell-1}}-1} g_{\ell-1}^{(n)}(m - 2^\ell k - 2^{\ell-1} j) u_\ell(j) \\
&= \sum_{j=0}^{\frac{N}{2^{\ell-1}}-1} u_\ell(j) R_{2^{\ell-1}(j+2k)} g_{\ell-1}^{(n)}(m).
\end{aligned} \tag{2.2}$$

In the same way, we obtain

$$R_{2^\ell k} g_\ell^{(2n+1)} = \sum_{j=0}^{\frac{N}{2^{\ell-1}}-1} v_\ell(j) R_{2^{\ell-1}(j+2k)} g_{\ell-1}^{(n)}. \tag{2.3}$$

Therefore, our claim is true. Furthermore, $U_{-\ell}^{(2n)}$ and $U_{-\ell}^{(2n+1)}$ each have dimension $\frac{N}{2^\ell}$, so $U_{-\ell}^{(2n)} \oplus U_{-\ell}^{(2n+1)}$ has dimension $\frac{N}{2^{\ell-1}}$, which is the dimension of $U_{-\ell+1}^{(n)}$. It follows that

$$U_{-\ell+1}^{(n)} = U_{-\ell}^{(2n)} \oplus U_{-\ell}^{(2n+1)}.$$

The proof is completed. □

From Theorem 2.5, we have the following decomposition:

$$\begin{aligned}\ell^2(Z_N) &= U_0^{(0)}\\ &= U_{-1}^{(0)} \oplus U_{-1}^{(1)}\\ &= U_{-2}^{(0)} \oplus U_{-2}^{(1)} \oplus U_{-2}^{(2)} \oplus U_{-2}^{(3)}\\ &\ \vdots\\ &= U_{-p}^{(0)} \oplus U_{-p}^{(1)} \oplus U_{-p}^{(2)} \oplus \ldots \oplus U_{-p}^{(2^p-2)} \oplus U_{-p}^{(2^p-1)}.\end{aligned} \tag{2.4}$$

Theorem 2.6. *Suppose N is divisible by 2^p, where p is a positive integer. Then the set*

$$B = \left\{R_{2^p k} g_p^{(0)}\right\}_{k=0}^{\frac{N}{2^p}-1} \cup \left\{R_{2^p k} g_p^{(1)}\right\}_{k=0}^{\frac{N}{2^p}-1} \cup \ldots \cup \left\{R_{2^p k} g_p^{(2^p-1)}\right\}_{k=0}^{\frac{N}{2^p}-1} \tag{2.5}$$

which is defined in (2.1) *is a p^{th} stage wavelet packets for $\ell^2(Z_N)$.*

In fact, it is easy to know from (2.4) the following special conclusion:

Corollary 2.7. *Suppose N is divisible by* 8, *then the set*

$$B = \left\{R_{2^2 k} g_2^{(0)}\right\}_{k=0}^{\frac{N}{2^2}-1} \cup \left\{R_{2^3 k} g_3^{(2)}\right\}_{k=0}^{\frac{N}{2^3}-1} \cup \left\{R_{2^3 k} g_3^{(3)}\right\}_{k=0}^{\frac{N}{2^3}-1} \cup \left\{R_{2^2 k} g_2^{(2)}\right\}_{k=0}^{\frac{N}{2^2}-1} \cup \left\{R_{2^2 k} g_2^{(3)}\right\}_{k=0}^{\frac{N}{2^2}-1} \tag{2.6}$$

is still an orthogonal basis for $\ell^2(Z_N)$.

Remark 2.8. We call the basis in the form of (2.6) as 3-stage incomplete wavelet packets, the corresponding basis in (2.5) is called complete wavelet packets.

3. Decomposition and Reconstruction Algorithms

This section will give the realization of decomposition and reconstruction algorithm.

$$U_{-\ell+1}^{(n)} = U_{-\ell}^{(2n)} \oplus U_{-\ell}^{(2n+1)} \Longleftrightarrow f_{-\ell+1}^{(n)} = f_{-\ell}^{(2n)} + f_{-\ell}^{(2n+1)}$$

$$\sum_{k=0}^{\frac{N}{2^{\ell-1}}-1} c_{-\ell+1}^{(n)} R_{2^{\ell-1}k} g_{\ell-1}^{(n)} = \sum_{k=0}^{\frac{N}{2^\ell}-1} c_{-\ell}^{(2n)} R_{2^\ell k} g_\ell^{(2n)} + \sum_{k=0}^{\frac{N}{2^\ell}-1} c_{-\ell}^{(2n+1)} R_{2^\ell k} g_\ell^{(2n+1)}. \tag{3.1}$$

In the both sides of (3.1), do inner product with $R_{2^\ell m} g_\ell^{(2n)}$:

$$c_{-\ell}^{(2n)}(m) = \sum_{k=0}^{\frac{N}{2^{\ell-1}}-1} c_{-\ell+1}^{(n)}(k) \langle R_{2^{\ell-1}k} g_{\ell-1}^{(n)},\ R_{2^\ell m} g_\ell^{(2n)} \rangle.$$

By (2.2), $R_{2^\ell m} g_\ell^{(2n)} = \sum_{j=0}^{\frac{N}{2^{\ell-1}}-1} u_\ell(j) R_{2^{\ell-1}(j+2m)} g_{\ell-1}^{(n)}$, then we have

$$c_{-\ell}^{(2n)}(m) = \sum_{k=0}^{\frac{N}{2^{\ell-1}}-1} c_{-\ell+1}^{(n)}(k) u_\ell(k-2m) = D(c_{-\ell+1}^{(n)} * \widetilde{u}_\ell)(m). \tag{3.2}$$

In the same way, the both sides do inner product with $R_{2^\ell m} g_\ell^{(2n+1)}$:

$$c_{-\ell}^{(2n+1)}(m) = \sum_{k=0}^{\frac{N}{2^{\ell-1}}-1} c_{-\ell+1}^{(n)}(k)\langle R_{2^{\ell-1}k} g_{\ell-1}^{(n)},\ R_{2^\ell m} g_\ell^{(2n+1)}\rangle.$$

By (2.3), $R_{2^\ell m} g_\ell^{(2n+1)} = \sum_{j=0}^{\frac{N}{2^{\ell-1}}-1} v_\ell(j) R_{2^{\ell-1}(j+2m)} g_{\ell-1}^{(n)}$, then we have

$$c_{-\ell}^{(2n+1)}(m) = \sum_{k=0}^{\frac{N}{2^{\ell-1}}-1} c_{-\ell+1}^{(n)}(k) v_\ell(k-2m) = D(c_{-\ell+1}^{(n)} * \tilde{v}_\ell)(m). \tag{3.3}$$

Decomposition Algorithms. since $\left\{R_{2k} u_1\right\}_{k=0}^{\frac{N}{2}-1} U \left\{R_{2k} v_1\right\}_{k=0}^{\frac{N}{2}-1}$ is the orthonormal basis for $\ell^2(Z_N)$, then for any $z \in \ell^2(Z_N)$, one has

$$z = \sum_{k=0}^{\frac{N}{2}-1} \langle z, R_{2k} u_1\rangle R_{2k} u_1 + \sum_{k=0}^{\frac{N}{2}-1} \langle z, R_{2k} v_1\rangle R_{2k} v_1 = \sum_{k=0}^{\frac{N}{2}-1} c_{-1}^{(0)} R_{2k} g_1^{(0)} + \sum_{k=0}^{\frac{N}{2}-1} c_{-1}^{(1)} R_{2k} g_1^{(1)}.$$

$$c_{-1}^{(0)}(k) = \langle z, R_{2k} u_1\rangle = z * \tilde{u}_1(2k) = D(z * \tilde{u}_1)(k);$$

$$c_{-1}^{(1)}(k) = \langle z, R_{2k} v_1\rangle = z * \tilde{v}_1(2k) = D(z * \tilde{v}_1)(k).$$

Furthermore, c_{-2}^0, c_{-2}^1, c_{-2}^2 and c_{-2}^3 are obtained from (3.2) and (3.3) with fast algorithm.

Reconstruction Algorithms. the both sides of (3.1) do inner product with $R_{2^{\ell-1}m} g_{\ell-1}^{(n)}$:

$$\begin{aligned}
c_{-\ell+1}^{(n)}(m) &= \sum_{k=0}^{\frac{N}{2^\ell}-1} c_{-\ell}^{(2n)}(k)\langle R_{2^\ell k} g_\ell^{(2n)},\ R_{2^{\ell-1}m} g_{\ell-1}^{(n)}\rangle \\
&\quad + \sum_{k=0}^{\frac{N}{2^\ell}-1} c_{-\ell}^{(2n+1)}(k)\langle R_{2^\ell k} g_\ell^{(2n+1)},\ R_{2^{\ell-1}m} g_{\ell-1}^{(n)}\rangle \\
&= \sum_{k=0}^{\frac{N}{2^\ell}-1} c_{-\ell}^{(2n)}(k)\left\langle \sum_{j=0}^{\frac{N}{2^{\ell-1}}-1} u_\ell(j) R_{2^{\ell-1}(j+2k)} g_{\ell-1}^{(n)},\ R_{2^{\ell-1}m} g_{\ell-1}^{(n)}\right\rangle \\
&\quad + \sum_{k=0}^{\frac{N}{2^\ell}-1} c_{-\ell}^{(2n+1)}(k)\left\langle \sum_{j=0}^{\frac{N}{2^{\ell-1}}-1} v_\ell(j) R_{2^{\ell-1}(j+2k)} g_{\ell-1}^{(n)},\ R_{2^{\ell-1}m} g_{\ell-1}^{(n)}\right\rangle \\
&= \sum_{k=0}^{\frac{N}{2^\ell}-1} c_{-\ell}^{(2n)}(k) u_\ell(m-2k) + \sum_{k=0}^{\frac{N}{2^\ell}-1} c_{-\ell}^{(2n+1)}(k) v_\ell(m-2k) \\
&= U(c_{-\ell}^{(2n)}) * u_\ell(m) + U(c_{-\ell}^{(2n+1)}) * v_\ell(m).
\end{aligned}$$

Example 3.1 (Haar wavelet packets).

$$g_1^{(0)} =: u_1 = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0,\dots,0\right), \qquad g_1^{(1)} = v_1 = \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0,\dots,0\right).$$

For $2 \le \ell \le p$, by folding method,

$$u_\ell(0) = \frac{1}{\sqrt{2}},\ u_\ell(1) = \frac{1}{\sqrt{2}},\ u_\ell(n) = 0, \quad 2 \le n \le \frac{N}{2^{\ell-1}} - 1;$$
$$v_\ell(0) = \frac{1}{\sqrt{2}},\ v_\ell(1) = -\frac{1}{\sqrt{2}},\ v_\ell(n) = 0, \quad 2 \le n \le \frac{N}{2^{\ell-1}} - 1.$$

The generators of 2^{th} stage wavelet packets are

$$g_2^{(0)} = g_1^{(0)} * U(u_2) =: \begin{cases} \frac{1}{2}, & n = 0,1,2,3; \\ 0, & n = 4,\dots,N-1. \end{cases}$$

$$g_2^{(1)} = g_1^{(0)} * U(v_2) =: \begin{cases} \frac{1}{2}, & n = 0,1; \\ -\frac{1}{2}, & n = 2,3; \\ 0, & n = 4,\dots,N-1. \end{cases}$$

$$g_2^{(2)} = g_1^{(1)} * U(u_2) =: \begin{cases} \frac{1}{2}, & n = 0,2; \\ -\frac{1}{2}, & n = 1,3; \\ 0, & n = 4,\dots,N-1. \end{cases}$$

$$g_2^{(3)} = g_1^{(1)} * U(v_2) =: \begin{cases} \frac{1}{2}, & n = 0,3; \\ -\frac{1}{2}, & n = 1,2; \\ 0, & n = 4,\dots,N-1. \end{cases}$$

Then the 2^{th} stage wavelet packets is

$$\left\{R_{2^2k}g_2^{(0)}\right\}_{k=0}^{\frac{N}{2^2}-1} U \left\{R_{2^2k}g_2^{(1)}\right\}_{k=0}^{\frac{N}{2^2}-1} U \left\{R_{2^2k}g_2^{(2)}\right\}_{k=0}^{\frac{N}{2^2}-1} U \left\{R_{2^2k}g_2^{(3)}\right\}_{k=0}^{\frac{N}{2^2}-1}.$$

The generators of 3^{rd} stage wavelet packets are

$$g_3^{(0)} = g_2^{(0)} * U^2(u_3) =: \begin{cases} 2^{-\frac{3}{2}}, & n = 0,1,2,3,4,5,6,7; \\ 0, & n = 8,\dots,N-1. \end{cases}$$

$$g_3^{(1)} = g_2^{(0)} * U^2(v_3) =: \begin{cases} 2^{-\frac{3}{2}}, & n = 0,1,2,3; \\ -2^{-\frac{3}{2}}, & n = 4,5,6,7; \\ 0, & n = 8,\dots,N-1. \end{cases}$$

$$g_3^{(2)} = g_2^{(1)} * U^2(u_3) =: \begin{cases} 2^{-\frac{3}{2}}, & n = 0,1,4,5; \\ -2^{-\frac{3}{2}}, & n = 2,3,6,7; \\ 0, & n = 8,\dots,N-1. \end{cases}$$

$$g_3^{(3)} = g_2^{(1)} * U^2(v_3) =: \begin{cases} 2^{-\frac{3}{2}}, & n = 0,1,6,7; \\ -2^{-\frac{3}{2}}, & n = 2,3,4,5; \\ 0, & n = 8,\dots,N-1. \end{cases}$$

$$g_3^{(4)} = g_2^{(2)} * U^2(u_3) =: \begin{cases} 2^{-\frac{3}{2}}, & n = 0,2,4,6; \\ -2^{-\frac{3}{2}}, & n = 1,3,5,7; \\ 0, & n = 8,\dots,N-1. \end{cases}$$

$$g_3^{(5)} = g_2^{(2)} * U^2(v_3) =: \begin{cases} 2^{-\frac{3}{2}}, & n = 0,2,5,7; \\ -2^{-\frac{3}{2}}, & n = 1,3,4,6; \\ 0, & n = 8,\dots,N-1. \end{cases}$$

$$g_3^{(6)} = g_2^{(3)} * U^2(u_3) =: \begin{cases} 2^{-\frac{3}{2}}, & n = 0,3,4,7; \\ -2^{-\frac{3}{2}}, & n = 1,2,5,6; \\ 0, & n = 8,\dots,N-1. \end{cases}$$

$$g_3^{(7)} = g_2^{(3)} * U^2(v_3) =: \begin{cases} 2^{-\frac{3}{2}}, & n = 0,3,5,6; \\ -2^{-\frac{3}{2}}, & n = 1,2,4,7; \\ 0, & n = 8,\dots,N-1. \end{cases}$$

Then the 3^{rd} stage wavelet packets is

$$\left\{R_{2^3k} g_3^{(0)}\right\}_{k=0}^{\frac{N}{2^3}-1} U \left\{R_{2^3k} g_3^{(1)}\right\}_{k=0}^{\frac{N}{2^3}-1} U \left\{R_{2^3k} g_3^{(2)}\right\}_{k=0}^{\frac{N}{2^3}-1} U \dots U \left\{R_{2^3k} g_3^{(6)}\right\}_{k=0}^{\frac{N}{2^3}-1} U \left\{R_{2^3k} g_3^{(7)}\right\}_{k=0}^{\frac{N}{2^3}-1}.$$

Example 3.2 (Shannon wavelet packets)**.**

$$\widehat{g}_1^{(0)} = \widehat{u}_1 =: \begin{cases} \sqrt{2}, & n = 0,1,\dots,\frac{N}{4}-1 \text{ or } \frac{3N}{4},\dots,N-1; \\ 0, & n = \frac{N}{4},\dots,\frac{3N}{4}-1. \end{cases}$$

$$\widehat{g}_1^{(1)} = \widehat{v}_1 =: \begin{cases} 0, & n = 0,1,\dots,\frac{N}{4}-1 \text{ or } \frac{3N}{4},\dots,N-1; \\ \sqrt{2}, & n = \frac{N}{4},\dots,\frac{3N}{4}-1. \end{cases}$$

By folding method, we have

$$\widehat{u}_\ell(n) =: \begin{cases} \sqrt{2}, & 0 \le n \le \frac{N}{2^{\ell+1}}-1;\ \frac{3N}{2^{\ell+1}} \le n \le \frac{N}{2^{\ell-1}}-1; \\ 0, & \frac{N}{2^{\ell+1}} \le n \le \frac{3N}{2^{\ell+1}}-1. \end{cases}$$

$$\widehat{v}_\ell(n) =: \begin{cases} 0, & 0 \le n \le \frac{N}{2^{\ell+1}}-1;\ \frac{3N}{2^{\ell+1}} \le n \le \frac{N}{2^{\ell-1}}-1; \\ \sqrt{2}, & \frac{N}{2^{\ell+1}} \le n \le \frac{3N}{2^{\ell+1}}-1. \end{cases}$$

The generators of 2^{th} stage wavelet packets are

$$\widehat{g}_2^{(0)}(n) = \widehat{u}_1(n)\widehat{u}_2(n) =: \begin{cases} 2, & 0 \le n \le \frac{N}{8}-1,\ \frac{7N}{8} \le n \le N-1; \\ 0, & \frac{N}{8} \le n \le \frac{7N}{8}-1. \end{cases}$$

$$\widehat{g}_2^{(1)}(n)=\widehat{u}_1(n)\widehat{v}_2(n)=:\begin{cases}2, & \frac{N}{8}\le n\le\frac{N}{4}-1,\ \frac{3N}{4}\le n\le\frac{7N}{8}-1;\\ 0, & 0\le n\le\frac{N}{8}-1,\ \frac{N}{4}\le n\le\frac{3N}{4}-1,\ \frac{7N}{8}\le n\le N-1.\end{cases}$$

$$\widehat{g}_2^{(2)}(n)=\widehat{v}_1(n)\widehat{u}_2(n)=:\begin{cases}2, & \frac{3N}{8}\le n\le\frac{5N}{8}-1;\\ 0, & 0\le n\le\frac{3N}{8}-1,\ \frac{5N}{8}\le n\le N-1.\end{cases}$$

$$\widehat{g}_2^{(3)}(n)=\widehat{v}_1(n)\widehat{v}_2(n)=:\begin{cases}2, & \frac{N}{4}\le n\le\frac{3N}{8}-1,\ \frac{5N}{8}\le n\le\frac{7N}{8}-1;\\ 0, & 0\le n\le\frac{N}{4}-1,\ \frac{3N}{8}\le n\le\frac{5N}{8}-1,\ \frac{7N}{8}\le n\le N-1.\end{cases}$$

The generators of 3rd stage wavelet packets are

$$\widehat{g}_3^{(0)}(n)=\widehat{g}_2^{(0)}(n)\widehat{u}_3(n)=:\begin{cases}2^{\frac{3}{2}}, & 0\le n\le\frac{N}{16}-1,\ \frac{15N}{16}\le n\le N-1;\\ 0, & \frac{N}{16}\le n\le\frac{15N}{16}-1.\end{cases}$$

$$\widehat{g}_3^{(1)}(n)=\widehat{g}_2^{(0)}(n)\widehat{v}_3(n)=:\begin{cases}2^{\frac{3}{2}}, & \frac{N}{16}\le n\le\frac{N}{8}-1,\ \frac{7N}{8}\le n\le\frac{15N}{16}-1;\\ 0, & 0\le n\le\frac{N}{16}-1,\ \frac{N}{8}\le n\le\frac{7N}{8}-1,\ \frac{15N}{16}\le n\le N-1.\end{cases}$$

$$\widehat{g}_3^{(2)}(n)=\widehat{g}_2^{(1)}(n)\widehat{u}_3(n)=:\begin{cases}2^{\frac{3}{2}}, & \frac{3N}{16}\le n\le\frac{N}{4}-1,\ \frac{3N}{4}\le n\le\frac{13N}{16}-1;\\ 0, & 0\le n\le\frac{3N}{16}-1,\ \frac{N}{4}\le n\le\frac{3N}{4}-1,\ \frac{13N}{16}\le n\le N-1.\end{cases}$$

$$\widehat{g}_3^{(3)}(n)=\widehat{g}_2^{(1)}(n)\widehat{v}_3(n)=:\begin{cases}2^{\frac{3}{2}}, & \frac{N}{8}\le n\le\frac{3N}{16}-1,\ \frac{13N}{16}\le n\le\frac{7N}{8}-1;\\ 0, & 0\le n\le\frac{N}{8}-1,\ \frac{3N}{16}\le n\le\frac{13N}{16}-1,\ \frac{7N}{8}\le n\le N-1.\end{cases}$$

$$\widehat{g}_3^{(4)}(n)=\widehat{g}_2^{(2)}(n)\widehat{u}_3(n)=:\begin{cases}2^{\frac{3}{2}}, & \frac{7N}{16}\le n\le\frac{9N}{16}-1;\\ 0, & 0\le n\le\frac{7N}{16}-1,\ \frac{9N}{16}\le n\le N-1.\end{cases}$$

$$\widehat{g}_3^{(5)}(n)=\widehat{g}_2^{(2)}(n)\widehat{v}_3(n)=:\begin{cases}2^{\frac{3}{2}}, & \frac{3N}{8}\le n\le\frac{7N}{16}-1,\ \frac{9N}{16}\le n\le\frac{5N}{8}-1;\\ 0, & 0\le n\le\frac{3N}{8}-1,\ \frac{7N}{16}\le n\le\frac{9N}{16}-1,\ \frac{5N}{8}\le n\le N-1.\end{cases}$$

$$\widehat{g}_3^{(6)}(n)=\widehat{g}_2^{(3)}(n)\widehat{u}_3(n)=:\begin{cases}2^{\frac{3}{2}}, & \frac{N}{4}\le n\le\frac{5N}{16}-1,\ \frac{11N}{16}\le n\le\frac{13N}{16}-1;\\ 0, & 0\le n\le\frac{N}{4}-1,\ \frac{5N}{16}\le n\le\frac{11N}{16}-1,\ \frac{13N}{16}\le n\le N-1.\end{cases}$$

$$\widehat{g}_3^{(7)}(n)=\widehat{g}_2^{(3)}(n)\widehat{v}_3(n)=:\begin{cases}2^{\frac{3}{2}}, & \frac{5N}{16}\le n\le\frac{3N}{8}-1, \frac{5N}{8}\le n\le\frac{11N}{16}-1, \frac{13N}{16}\le n\le\frac{7N}{8}-1;\\ 0, & 0\le n\le\frac{5N}{16}-1,\ \frac{3N}{8}\le n\le\frac{5N}{8}-1,\\ & \frac{11N}{16}\le n\le\frac{13N}{16}-1,\ \frac{7N}{8}\le n\le N-1.\end{cases}$$

4. Numerical Examples

The purpose of decomposition is to project signals onto the wavelet packets and obtain a series of coefficients, then we will use these coefficients to characterize the feature of signals. If the difference between these coefficients is small, then it is difficult to find the feature and the corresponding decomposition is not the best one. Therefore, the best decomposition of wavelet packets is the decomposition that the difference for the final coefficients is the largest.

The difference for the coefficients can be evaluated by Shannon entropy as

$$S=\sum_k S_k=-\sum_k\sum_i V_i^{(k)}\log V_i^{(k)}.$$

Here, $V_i^{(k)} = \dfrac{|u_i^{(k)}|^2}{\sum_k \sum_i |u_i^{(k)}|^2}$, $u_i^{(k)}$ denotes the value of ith coefficient for the kth node. If S is the smallest, then the corresponding decomposition is the best one.

Let $N = 512$, take a signal in $\ell^2(Z_N)$ as in Figure 1 and add Gauss white noise as in Figure 2. Now, we will use Haar wavelet packets in Example 3.1 to de-noise for this signal.

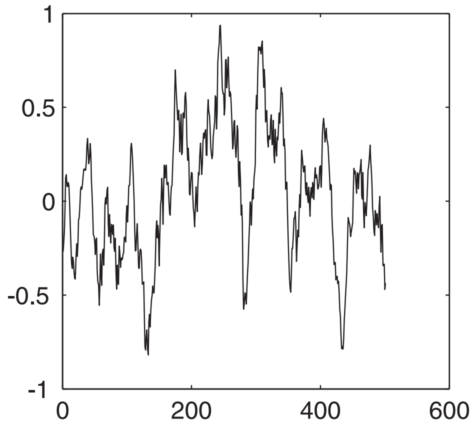

Figure 1. Original signal

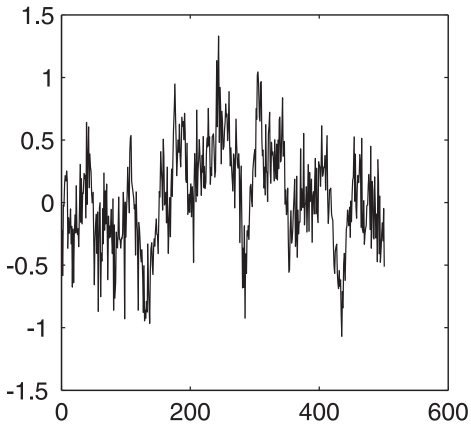

Figure 2. Noise signal

To evaluate the denoising effect, we mainly consider the following three constantly used index:

$$\text{(Signal Noise Ratio)}\ \ \mathrm{SNR} = 10\log\frac{\sum\limits_k |z(k)|^2}{\sum\limits_k |z(k)-\tilde{z}(k)|^2}\ ,$$

$$\text{(Rest Mean Square Error)}\ \ \mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_k |z(k)-\tilde{z}(k)|^2}\ ,$$

$$\text{(Rest Noise Standard Deviation)}\ \ \mathrm{RNSD} = \sqrt{\frac{1}{N-1}\sum_k |\tilde{z}_n(k)-\overline{z}_n|^2}\ ,$$

here, $\widetilde{z}_n(k) = \widetilde{z}(k) - z(k)$ and $\overline{z}_n = \frac{1}{N}\sum_k \widetilde{z}_n(k)$.

Table 1. Denoising based on Haar

Index	Wavelet decomposition	Complete wavelet packets	Best wavelet packets
SNR	18.7524	19.0821	19.7206
RMSE	0.3463	0.3334	0.3097
RNSD	0.3460	0.3331	0.3094

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Competing Interests

I declare that I have no significant competing financial, professional or personal interests that might have influenced the performance or presentation of the work described in this manuscript.

Author's Contributions

The author has wrote this article. The author has read and approved the final manuscript.

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