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Stochastic Derivation of An Integral Equation for Characteristic Functions

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Abstract. Functional, integral and differential equations of transformed characteristic functions are very strong analytical tools for establishing characterizations of probability distributions. The present paper establishes a characterization of the distribution of a fundamental stochastic multiplicative model by making use of an integral equation based on three well known transformed characteristic functions.

1. Introduction

An extremely significant part of probability theory concentrates on the investigation of measurable functions, the random variables. Many of the most important problems concerning random variables can be analyzed in terms of distribution functions. The method of classical analysis provides an efficient approach for handling notable problems of this kind. It is very often advisable to consider, instead of distribution functions, characteristic functions which are the Fourier transforms of distribution functions. It is now universally recognized that characteristic functions constitute the most powerful tools for the investigation of distribution functions. The uniqueness theorem, the convolution theorem and the continuity theorem are the most significant theorems which describe the interrelationships between characteristic functions and distribution functions. These theorems account for the great importance of characteristic functions in probability theory [24].

Transformations of characteristic functions are generally accepted as very important area of the theory of characteristic functions [24]. For several decades there has been an increasing interest in transformations of characteristic functions, more precisely in operations which map a given characteristic function into a new characteristic function. These transformations offer very interesting information concerning the structure of characteristic functions. The investigation

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of the properties of a transformed characteristic function is of particular importance in probability theory. A very wide variety of characteristic functions of theoretical and practical interest arises as transformations of other characteristic functions.

The literature of transformations of characteristic function is very broad. The first transformation of characteristic function was due to Finneti [9]. Later, Khintchine [16] introduced a transformation by means of an integral which converts an arbitrary characteristic function into the characteristic function of a unimodal distribution. This transformation was studied by Girault [10] and was generalized by Loeffel [19]. Lukacs [20] introduced a transformation of an arbitrary characteristic function by means of two integrations. In Linnik [18] we find the renewal transformation of characteristic functions. The infinite divisibility of this transformation was studied by Steutel [28]. Johansen [14], following Khintchine [16], introduced a transformation which maps an infinitely divisible characteristic function into an infinitely divisible characteristic function. Transformations having similarities with the renewal transformation were introduced by Lukacs [22]. The majority of the above transformations is the result of mixing of characteristic functions. Research on the preservation of theoretical properties under mixing was mainly carried out by Goldie [11], Medgyessy [26], Holgate [12], and Steutel [29]. Moreover, research work on the preservation of theoretical properties under transformations of characteristic functions which are not the result of mixing of characteristic functions was mainly carried out by Jagers [13], Keilson and Steutel [15], Kubik [17], Lucaks [23], Medgyessy [26] and Artikis [2, 3, 4, 5, 6, 7]. The books of Lukacs and Laha [21], Ramachandran [27], Lukacs [24], Lukacs [25], Ushakov [31], Steutel and Van Harn [30] contain the majority of the results on transformations of characteristic functions.

Characterizations of probability distributions making use of transformations of characteristic functions constitute a fundamental research area of probability theory. Functional equations, integral equations and differential equations of transformed characteristic functions are the strong analytical tools for establishing such characterizations. The main purpose of the present paper is the incorporation of three transformations of characteristic functions in an integral equation for characterizing the distribution of a stochastic multiplicative model with particular theoretical and practical applicability.

2. **Certain Transformations of Characteristic Functions**

The present section of the paper mainly concentrates on the consideration of certain known transformations of characteristic functions corresponding to nonnegative random variables with finite mean.

Let X be a nonnegative random variable with characteristic function $\varphi_X(u)$ and finite mean μ, then

$$\varphi_U(u) = \frac{1}{i\mu u} \log \varphi_X(u), \tag{2.1}$$

is a characteristic function of a nonnegative random variable U with distribution function having a unique mode at the point 0 if, and only if, X is infinitely divisible [1]. The formula (2.1) is a transformation which maps the characteristic function $\varphi_X(u)$ into the characteristic function $\varphi_U(u)$. An integral equation incorporating the characteristic function $\varphi_U(u)$ and the characteristic function of a model, being the product of two nonnegative and independent random variables one of which is distributed as the random variable X and the other follows the standard uniform distribution, has been investigated by Artikis and Artikis [8]. Moreover, the formula

$$\varphi_X(u) = \exp\left(i\mu u \varphi_U(u)\right) \tag{2.2}$$

is a transformation which maps the characteristic function $\varphi_U(u)$ into the characteristic function $\varphi_X(u)$.

Let V be a nonnegative random variable with characteristic function $\varphi_V(u)$ and finite mean θ, then

$$\varphi_J(u) = \frac{\varphi_V(u) - 1}{i\theta u} \tag{2.3}$$

is the characteristic function of the renewal distribution corresponding to the distribution of the random variable V [18]. The formula (2.3) is a transformation which maps the characteristic function $\varphi_V(u)$ into the renewal characteristic function $\varphi_J(u)$.

The present paper makes use of the transformation in (2.2) and the transformation in (2.3) for providing an extension of the results established by Artikis and Artikis [8].

3. **Characterizing the Distribution a Stochastic Multiplicative Model**

The present section of the paper establishes a characterization of the distribution of the stochastic multiplicative model based on two nonnegative and independent random variables, one of which is exponentially distributed and the other follows the standard uniform distribution. Since an explicit evaluation of the distribution function of this model is not possible it is advisable to evaluate the corresponding characteristic function. The contribution of this section consists of providing an interesting extension of the results established by Artikis and Artikis [8].

Theorem. *Let W, T and L be independent random variables with W following the standard uniform distribution and T, L distributed as the random variable S with characteristic function*

$$\varphi_S(u) = \exp\left(i\kappa u \varphi_Y(u)\right), \quad \kappa > 0,$$

where $\varphi_Y(u)$ is the characteristic function of a nonnegative random variable Y with finite mean and distribution function $F_Y(y)$ having a unique mode at the point 0. The characteristic function of the random variable has the form

$$\varphi_Y(u) = \frac{\lambda}{iu}\log\left(\frac{\lambda}{\lambda - iu}\right),$$

where

$$\lambda = \frac{1}{\kappa}$$

if and only if, the random variable

$$C = (L+T)W$$

is equally distributed with the random variable

$$R$$

following the renewal distribution corresponding to the distribution of the random variable S.

Proof. Only the sufficiency condition will be proved since the necessity condition can be proved by reversing the argument.

It is readily shown that the independence of the random variables

$$W, \quad T, \quad L$$

implies the independence of the random variables

$$L+T, \quad W.$$

Hence it is also readily shown that the characteristic function of the random variable

$$C = (L+T)W$$

is given by

$$\varphi_C(u) = \int_0^1 \exp\left(2i\kappa uw\varphi_Y(uw)\right)dw. \tag{3.1}$$

Moreover, the characteristic function of the random variable R is given by

$$\varphi_R(u) = \frac{\exp\left(i\kappa u\varphi_Y(u)\right) - 1}{i\kappa u}. \tag{3.2}$$

From (3.1), (3.2) and the assumption that the random variables

$$R, \quad C = (L+T)W$$

are equal in distribution we get the integral equation

$$\frac{\exp\left(i\kappa u\varphi_Y(u)\right) - 1}{i\kappa u} = \int_0^1 \exp\left(2i\kappa uw\varphi_Y(uw)\right)dw$$

which can be written in the form

$$\frac{\exp\left(i\kappa u\varphi_Y(u)\right)-1}{i\kappa u}=\frac{1}{u}\int_0^u \exp\left(2i\kappa w\varphi_Y(w)\right)dw\,. \tag{3.3}$$

If we multiply both sides of the integral equation in (3.3) by $i\kappa u$ and then we differentiate we get the differential equation

$$\exp\left(i\kappa u\varphi_Y(u)\right)\frac{d}{du}i\kappa u\varphi_Y(u)=i\kappa\exp\left(2i\kappa u\varphi_Y(u)\right)\,. \tag{3.4}$$

which satisfies the boundary condition

$$\varphi_Y(0)=1\,.$$

Since the characteristic function

$$\varphi_S(u)=\exp(i\kappa u\varphi_Y(u))$$

has no real roots, then the differential equation in (3.4) can be written in the form

$$\exp\left(-i\kappa u\varphi_Y(u)\right)\frac{d}{du}i\kappa u\varphi_Y(u)=i\kappa\,. \tag{3.5}$$

Integrating the differential equation in (3.5) with due regard to the above boundary condition we get that

$$\exp\left(-i\kappa u\varphi_Y(u)\right)=1-i\kappa u\,. \tag{3.6}$$

From (3.6) it follows that

$$\varphi_Y(u)=\frac{\lambda}{iu}\log\left(\frac{\lambda}{\lambda-iu}\right) \tag{3.7}$$

with

$$\lambda=\frac{1}{\kappa}\,.$$ □

It is of some importance to mention that the above characteristic function is derived as the particular case of the transformation (2.1) with X exponentially distributed.

It is also of some importance to represent the characteristic function in (3.7) as the limit of a sequence of renewal transformations. We have

$$\frac{\left(\frac{\lambda}{\lambda-iu}\right)^{\frac{1}{n}}-1}{\frac{iu}{n\lambda}}=\int_0^1\left(\frac{\lambda}{\lambda-iuw}\right)^{\frac{1}{n}}\left(\frac{\lambda}{\lambda-iuw}\right)dw,\quad n=1,2,\ldots\,. \tag{3.8}$$

From (3.8) and the dominated convergence theorem we get that

$$\lim_{n\to\infty}\frac{\left(\frac{\lambda}{\lambda-iu}\right)^{\frac{1}{n}}-1}{\frac{iu}{n\lambda}}=\int_0^1\frac{\lambda}{\lambda-iuw}dw \tag{3.9}$$

or equivalently

$$\lim_{n\to\infty} \frac{\left(\frac{\lambda}{\lambda - iu}\right)^{\frac{1}{n}} - 1}{\frac{iu}{n\lambda}} = \frac{\lambda}{iu} \log \frac{\lambda}{iu} \log \left(\frac{\lambda}{\lambda - iu}\right). \tag{3.10}$$

Hence the characteristic function in (3.7) can be represented as the limit of a sequence of renewal transformations corresponding to the gamma distribution with parameters $\frac{1}{n}$, λ.

From (3.9) and (3.10) it follows that the random variable Y can be written in the form of the multiplicative model

$$Y = DH, \tag{3.11}$$

where D, H independent random variables with D following the exponential distribution with parameter λ and H following the standard uniform distribution. The applicability, in theory and practice, of the stochastic multiplicative model, in (3.11) is generally recognized as particularly important [8]. Since the evaluation of the distribution function of this model is not possible, then the corresponding characteristic function constitutes an extremely powerful tool for theory and practice.

Closing, it seems to be of some practical importance to establish an interpretation in the risk severity reduction operations of the stochastic multiplicative model

$$C = (L + T)W,$$

incorporated by the present paper for characterizing the distribution of the stochastic multiplicative model in (3.11). We suppose that the random variable L denotes the severity of a risk and the random variable T denotes the severity of another risk. We also suppose that the risks are of the same type. Since the random variable W follows the standard uniform distribution, then the random variable C can be interpreted as the total risk severity after applying a risk severity reduction operation to these risks. The strong dependence of modern complex organizations on information systems substantially supports the applicability of the proposed interpretation of the random variable C, for describing and analyzing the total severity of two information risks of the same type after the application of a risk frequency reduction operation to such information risks. Electric and magnetic disturbance are examples of information risks of the same type.

4. **Conclusions**

The theoretical contribution of the paper consists of establishing a characterization of for the distribution function of a stochastic multiplicative model, based on two nonnegative and independent random variables one of which follows the exponential distribution and the other follows the standard uniform distribution. The establishment of the characterization makes use of an integral equation for certain well known transformed characteristic functions. The importance of the results of the paper is substantially supported by the extensive applicability of such a model.

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