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Research Article

On the Boundary Control of a Boussinesq System

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Abstract. A boundary control problem is considered for determining a canal depth function optimally for a canal system modeled by a nonlinear Boussinesq equation. By determining the optimal canal depth function, it is aimed to damp out the undesired waves in canal system filled up water. For achieving this aim, the existence and uniqueness of the solutions to system and controllability properties of the system is investigated. Optimal canal depth control function is obtained by means of a maximum principle, which is an elegant tool for transferring the optimal boundary control problem to solving a system of equations including initial-terminal-boundary conditions. The reason making this paper is important that optimal control function is gained without linearization of nonlinear term in the system. In order to show the correctness of the obtained theoretical results, several numerical examples are presented by MATLAB in graphical and table forms. Observing these tables and graphics, it is concluded that introduced boundary control algorithm is effective and has the potential for extending to other nonlinear control systems.

Keywords. Wellposedness; Boussinesq; Canal; Hamiltonian

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1. Introduction

In recent years, nonlinear evolution equations have acquired great attentions due to their properties on modeling of real world problems and explaining some nonlinear phenomena. One of these evolution equations is called Boussinesq equation, which is introduced by Joseph Boussinesq in 1872 [2] for modeling of long waves of the surface of water with a small amplitude. over the last two decades, Boussinesq equation is studied in various aspects by different researchers. These studies can be summarized as follows but not limited to: In [13], Wazwaz investigated the logarithmic-Boussinesq equation for Gaussian solitary waves and derived the Gaussian solitary wave solutions for the logarithmic-regularized Boussinesq equation. In [10], Shakhmurov obtained the existence and uniqueness of solution of the integral boundary value problem for abstract Boussinesq equations. Global well-posedness and long time decay of the 3D Boussinesq equations presented in [8]. Also, small global solutions to the damped two-dimensional Boussinesq equations obtained in [1]. The results on local well-posedness for the sixth-order Boussinesq equation are derived in [4]. In [3], damped infinite energy solutions of the 3D Euler and Boussinesq equations are presented. In [12], extended Boussinesq model to predict the propagation of waves in porous media is developed. The inertial and drag resistances are taken into account in the developed model. In [14], longtime dynamics of a damped Boussinesq equation is investigated. In [15], Fourier spectral approximation for the time fractional Boussinesq equation with periodic boundary condition is considered. In [7], Li considered the maxiumum principle for an optimal control problem governed by Boussinesq equations including integral type state constraints. Analysis and approximation of linear feedback control problems for the Boussinesq equations are studied in [6]. Regarding Boussinesq equations and related studies, the books [5, 9, 11] provide a general overview. Specifically, in this paper, optimal boundary control problem for canal system filled up water is considered. By determining the canal depth optimally, it is aimed to suppress the undesirable waves in the canal system. This system has also external excitation inducing water waves and the system is modeled by a nonlinear Boussinesq equation. For achieving the optimal canal depth function, maximum principle is employed and hence, optimal boundary control problem is reduced to solving of a nonlinear system of equations including terminal-boundary and initial conditions. The optimal canal depth control function is obtained without linearization of nonlinear term in the system. In order to indicate the effectiveness of the introduced control algorithm, several numerical examples are presented by means of `MATLAB`. This paper is organized as follows: in Second 2, mathematical formulation of the control problem is presented. In Section 3, optimal control problem is defined and a maximum principle is obtained. In Section 4, numerical results and discussions are presented.

2. Mathematical Formulation of the Control Problem

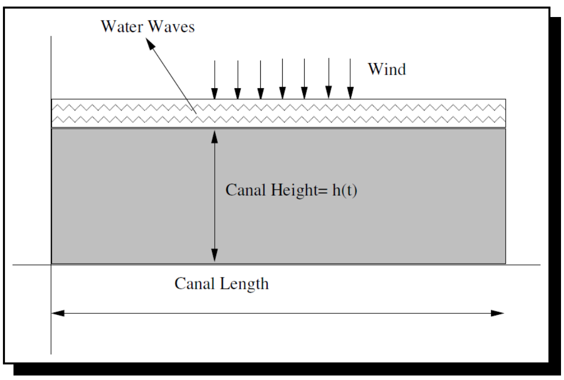

Figure 1. A canal system

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx} + f(t,x) \tag{2.1}$$

in which, state variable u is the elevation of the free surface of the water at $(t,x) \in \Omega = \{t \in (0,tf), x \in (0,\ell)\}$, t is time variable, t_f is predetermined terminal time, x is space variable, ℓ is the length of the canal, α, β are some constants in $\mathscr{R}^+$, f is the external excitation function. Eq. (2.1) is subject to the following boundary conditions

$$u(t,x) = 0, \;\; u_{xx}(t,x) = \hbar(t) \;\text{ at }\; x = 0, \ell \tag{2.2}$$

in which $\hbar(t)$ is the optimal canal depth control function and initial conditions

$$u(t,x) = u_0(x), \;\; u_t(t,x) = u_1(x) \;\text{ at }\; t = 0. \tag{2.3}$$

Let us assume following on the solution:

$$f, \hbar, u, \frac{\partial^{i+j} u}{\partial t^j \partial x^i} \in L^2(\Omega), \quad j = 0,1,2, \; i = 0,1,\ldots,4 \tag{2.4a}$$

$$u_0(x) \in H^1(0,\ell) = \left\{ u_0(x) \in L^2(0,\ell) : \frac{\partial u_0(x)}{\partial x} \in L^2(0,\ell) \right\}, \quad u_1(x) \in L^2(0,\ell), \tag{2.4b}$$

where $L^2(\Omega)$ denote the Hilbert space of real-valued square-integrable functions defined in the domain Ω in the Lebesque sense with usual inner product and norm defined by

$$\|\rho\|^2 = \langle \rho, \rho \rangle, \quad \langle \rho, \eta \rangle_\Omega = \int_\Omega \rho \eta \, d\Omega.$$

3. Optimal Control Problem

The main aim of this study is to optimally determine the canal depth control function $\hbar(t)$ and minimize the dynamic response of the water waves system at a predetermined terminal time t_f. Before defining the performance index functional of the system, let us define the admissible canal depth control function set as follows:

$$\hbar_{ad} = \{\hbar(t) | \hbar \in L^2(\Omega), \quad |\hbar(t)| \le \hbar_0 < \infty\}. \tag{3.1}$$

Then, the performance index functional of the system is given by as follows;

$$\mathscr{J}(\hbar) = \int_0^\ell [\vartheta_1 u^2(t_f,x) + \vartheta_2 u_t^2(t_f,x)] dx + \vartheta_3 \int_0^{t_f} \hbar^2(t) dt \tag{3.2}$$

in which $\vartheta_1, \vartheta_2 \ge 0$, $\vartheta_1 + \vartheta_2 \ne 0$ and $\vartheta_3 > 0$ are weighting constants. First integral on the left-hand side in Eq. (3.2) represents the modified dynamics response of the water waves system. First and second terms in this integral are quadratic functional of the displacement and velocity of the water wave, respectively. First term on the right-hand side in Eq. (3.2) is the measure of the total canal depth on the $(0,t_f)$. Then, optimal canal depth control problem is stated as follows:

$$\mathscr{J}(\hbar^\circ(t)) = \min_{\hbar(t) \in \hbar_{ad}} \mathscr{J}(\hbar(t)) \tag{3.3}$$

subject to the Eqs. (2.1)-(2.3). In order to achieve the maximum principle for obtaining optimal canal depth control function, let us introduce an adjoint variable $\upsilon \in L^*$, L^* is the dual to $L^2(\Omega)$ and has the same norm and inner product like in $L^2(\Omega)$. Adjoint system corresponding to Eqs. (2.1)-(2.3)is expressed as follows:

$$\upsilon_{tt} = -\alpha \upsilon_{xxxx} + \upsilon_{xx} \tag{3.4}$$

subjects to following boundary and terminal conditions, respectively:

$$v(t,x)=0,\ \ v_{xx}(t,x)=0,\ \ \text{at}\ \ x=0,\ell \tag{3.5}$$

$$-2\vartheta_1 u(t,x)=v_t(t,x),\ \ 2\vartheta_2 u_t(t,x)=v(t,x)\ \ \text{at}\ \ t=t_f. \tag{3.6}$$

The existence and uniqueness of the solutions to adjoint system defined by Eqs. (3.4)-(3.6) can be obtained similar to Eqs. (2.1)-(2.3). Maximum principle is derived a necessary condition for the optimal control function in terms of Hamiltonian functional. In case of some convexity assumptions, satisfied by Eq. (3.2), on performance index functional of the system, maximum principle is also sufficient condition for optimal control function. Maximum principle is elegant tool for obtaining the optimal control function and maximum principle sets up a direct relation between state variable and optimal control function via terminal conditions of adjoint system. Hence, maximum principle converts an optimal control problem to solving a system of equations, including state and adjoint variables, subjects to initial-boundary-terminal conditions. Let us derive the maximum principle as follows:

Theorem 3.1 (Maximum principle)**.** *The maximization problem is presented as follows:*

$$\textit{If}\quad \mathscr{H}[t;u^\circ,w^\circ,\hbar^\circ(t)]=\max_{\hbar(t)\in\hbar_{ad}}\mathscr{H}[t;u,v,\hbar(t)] \tag{3.7}$$

in which, Hamiltonian function is defined by

$$\mathscr{H}[t;u,w,\hbar(t)]=-\alpha\hbar(t)\Phi(t)-\vartheta_3\hbar^2(t)-\Psi(t), \tag{3.8}$$

$$\Phi(t)=v_x(t,1)-v_x(t,0),\quad \Psi(t)=\beta\int_0^\ell v(t,x)(u^2)_{xx}dx$$

then

$$\mathscr{J}[\hbar^\circ(t)]\le\mathscr{J}[\hbar(t)],\quad \forall\hbar(t)\ \in\hbar_{ad}, \tag{3.9}$$

where $\hbar^\circ(t)$ is the optimal canal depth control function.

Proof. Before giving the proof, let us introduce an operator as follows:

$$\varphi(u)=\varphi_1(u)+\varphi_2(u), \tag{3.10}$$

$$\varphi_1(u)=u_{tt},\quad \varphi_2(u)=\gamma u_{xxxx}-u_{xx}. \tag{3.11}$$

The deviations in the state variable and its derivatives with respect to the time variable are defined by

$$\Delta u=u-u^\circ,\quad \Delta u_t=u_t-u_t^\circ.$$

It is easy to see that

$$\varphi(u)=\beta(u^2)_{xx}\quad\text{and}\quad \varphi(u)-\varphi(u^\circ)=\beta(u^2)_{xx}-\beta((u^\circ)^2)_{xx}. \tag{3.12}$$

Due to linearity of operator φ, we can write following equality:

$$\varphi(\Delta u)=\beta(u^2)_{xx}-\beta((u^\circ)^2)_{xx}. \tag{3.13}$$

The operator

$$\varphi(\Delta u)=\beta(u^2)_{xx}-\beta((u^\circ)^2)_{xx}$$

is subject to the following boundary and initial conditions, respectively:

$$\Delta u(t,x)=0\ \ \text{and}\ \ \Delta u_{xx}(t,x)=\Delta\hbar(t)\ \ \text{at}\ \ x=0,\ell \tag{3.14}$$

and initial conditions

$$\Delta u(t,x) = \Delta u_t(t,x) = 0 \text{ at } t = 0. \tag{3.15}$$

Let us take into account the following functional

$$\iint_\Omega \left\{ v\varphi(\Delta u) - \Delta u \varphi(v) \right\} d\Omega = \beta \iint_\Omega v((u^2)_{xx} - ((u^\circ)^2)_{xx}) d\Omega. \tag{3.16}$$

Focus on the integral on the left side of Eq. (3.16);

$$\iint_\Omega \left\{ v\varphi(\Delta u) - \Delta u \varphi(v) \right\} d\Omega$$

$$= \iint_\Omega \left\{ v[\varphi_1(\Delta u) + \varphi_2(\Delta u)] - \Delta u[\varphi_1(v) - \varphi_2(v)] \right\} d\Omega \tag{3.17}$$

$$= \iint_\Omega \Big\{ \underbrace{[v\varphi_1(\Delta u) - \Delta u \varphi_1(v)]}_{I_1} + \underbrace{[v\varphi_2(\Delta u) - \Delta u \varphi_2(v)]}_{I_2} \Big\} d\Omega. \tag{3.18}$$

Applying the integration by parts to each term in the integral in Eq. (3.18) and using boundary conditions given by Eq. (3.5), Eq. (3.14) and Eq. (3.15), following equalities are obtained;

$$I_1 = \iint_\Omega \left\{ [v\varphi_1(\Delta u) - \Delta u \varphi_1(v)] \right\} d\Omega = \int_0^\ell \left\{ [v(t_f,x)\Delta u_t(t_f,x) - \Delta u(t_f,x) v_t(t_f,x)] \right\} dx, \tag{3.19a}$$

$$I_2 = \iint_\Omega \left\{ [v\varphi_2(\Delta u) - \Delta u \varphi_2(v)] \right\} d\Omega = \alpha \int_0^{t_f} \Delta\hbar(t)[v_x(0,t) - v_x(1,t)] dt, \tag{3.19b}$$

$$I_1 + I_2 = \beta \iint_\Omega v((u^2)_{xx} - ((u^\circ)^2)_{xx}) d\Omega \tag{3.20}$$

Hence, by means of terminal conditions given by Eq. (3.6), following equality is obtained:

$$2\int_0^1 \left\{ \vartheta_1 u(t_f,x)\Delta u(t_f,x) + \vartheta_2 u_t(t_f,x)\Delta u_t(t_f,x) \right\} dx$$

$$= \int_0^{t_f} \left\{ -\alpha\Delta\hbar(t)[v_x(0,t) - v_x(1,t)] + \Delta\Psi(t) \right\} dt \tag{3.21}$$

Consider the difference of the performance index

$$\Delta\mathscr{J}[\hbar(t)] = \mathscr{J}[\hbar(t)] - \mathscr{J}[\hbar^\circ(t)] \tag{3.22}$$

$$= \int_0^\ell \left\{ \vartheta_1[u^2(t_f,x) - u^{\circ^2}(t_f,x)] + \vartheta_2[u_t^2(t_f,x) - u_t^{\circ^2}(t_f,x)] \right\} dx + \int_0^{t_f} \vartheta_3[\hbar^2(t) - \hbar^{\circ^2}(t)] dt \geq 0$$

Expanding $u^2(t_f,x)$ and $u_t^2(t_f,x)$ in Taylor series around $u^{\circ^2}(t_f,x)$ and $u_t^{\circ^2}(t_f,x)$, yields

$$u^2(t_f,x) - u^{\circ^2}(t_f,x) = 2u^\circ(t_f,x)\Delta u(t_f,x) + r, \tag{3.23a}$$

$$u_t^2(t_f,x) - u_t^{\circ^2}(t_f,x) = 2u_t^\circ(t_f,x)\Delta u_t(t_f,x) + r_t \tag{3.23b}$$

where $r = 2(\Delta u)^2$ + higher order terms > 0 and $r_t = 2(\Delta u_t)^2$ + higher order terms > 0. Substituting Eq. (3.23) into Eq. (3.22) gives

$$\Delta\mathscr{J}[\hbar(t)] = \int_0^1 \left\{ \vartheta_1[2u^\circ(t_f,x)\Delta u(t_f,x) + r] + \vartheta_2[2u_t^\circ(t_f,x)\Delta u_t(t_f,x) + r_t] \right\} dx$$

$$+ \int_0^{t_f} \vartheta_3[\hbar(t)^2 - \hbar(t)^{\circ^2}] dt \geq 0. \tag{3.24}$$

From Eq. (3.21) and because of $\vartheta_1 r + \vartheta_2 r_t > 0$, one obtains

$$\Delta \mathscr{J}[\hbar(t)] \geq \int_0^{t_f} \left\{ -\alpha\Delta\hbar(t)[v_x(0,t) - v_x(1,t)] + \Delta\Psi(t) + \vartheta_3[\hbar^2(t) - \hbar^{\circ 2}(t)] \right\} dt \geq 0 \tag{3.25}$$

which leads to

$$\alpha\Delta\hbar(t)\Phi(t) + \Delta\Psi(t) + \vartheta_3[\hbar^2(t) - \hbar^{\circ 2}(t)] \geq 0 \tag{3.26}$$

that is,

$$\mathscr{H}[t; u^\circ, w^\circ, \hbar^\circ] \geq \mathscr{H}[t; u, w, \hbar].$$

Hence, we obtain

$$\mathscr{J}[\hbar] \geq \mathscr{J}[\hbar^\circ], \quad \forall \hbar \in \hbar_{ad}.$$

Therefore, the optimal control function is given by

$$\hbar(t) = \frac{-\alpha\Phi(t)}{2\vartheta_3}. \tag{3.27}$$

□

4. Numerical Results and Discussions

In this section, obtained theoretical results are simulated by solving following system of nonlinear equations linked by initial-boundary-terminal conditions via MATLAB for indicating the effectiveness and robustness of the introduced control algorithm for damping excessive water waves in a canal system by optimally determined canal depth control function. But, solving following system of nonlinear equations is difficult in aspect of control. Therefore, the linearization of the nonlinear term u^2 is taken into account as a third order Taylor series expansion around the $t = 0$ in the computation steps.

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx} + f(t,x) \tag{4.1a}$$

$$u(t,x) = 0, \;\; u_{xx}(t,x) = \hbar(t), \;\; \text{at} \;\; x = 0, \ell, \tag{4.1b}$$

$$\hbar(t) = \frac{-\Phi(t)}{2v_3}, \;\; \Phi(t) = v_x(t,1) - v_x(t,0), \tag{4.1c}$$

$$u(t,x) = u_0(x), \;\; u_t(t,x) = u_1(x) \;\; \text{at} \;\; t = 0. \tag{4.1d}$$

$$v_{tt} = -\alpha v_{xxxx} + v_{xx} \tag{4.2a}$$

$$v(t,x) = 0, \;\; v_{xx}(t,x) = 0, \;\; \text{at} \;\; x = 0, \ell, \tag{4.2b}$$

$$-2\vartheta_1 u(t,x) = v_t(t,x), \;\; 2\vartheta_2 u_t(t,x) = v(t,x) \;\; \text{at} \;\; t = t_f, \tag{4.2c}$$

Also, it easy to see that the system given by Eq. (2.1) subjecting the Eqs. (4.1) can be reduced by ordinary differential equations. It is concluded by second order Picard's existence and uniqueness theorem that the system defined by Eqs. (4.1) has a unique solution around the $t = 0$. Assume that u is the unique solution to system given by Eqs. (4.1). It is concluded that when u is unique solution to system given by Eqs. (4.1), corresponding canal depth control function $\hbar(t)$ also must be unique for preserving the uniqueness of the solution to Eqs. (4.1). Then, the system is called as observable. By means of Hilbert Uniqueness method, it is easy concluded that observable is equal to controllable. Namely, the system Eqs. (4.1) is controllable.

Before evaluating the numerical results in tables and graphs, consider the optimal canal depth control function given by Eq. (3.27), in which, it is clear that as the value of ϑ_3 is decreasing, the value of the canal depth is increasing. As a conclusion of this situation, dynamic response of the excessive water waves given by first integral on the left side of the Eq. (3.2) is minimized by using minimum canal depth. Effectiveness of the introduced control actuation is examined in two cases. Both of two cases, t_f is taken into account as 5. Weighted coefficients are taken into account as $\vartheta_{1,2} = 1$ and $\vartheta_3 = 10^4$ and $\vartheta_3 = 10^{-2}$ for uncontrolled and controlled case, respectively. Canal length is $\ell = 1$. All figures are plotted in the middle of the canal, $x = 0.5$. In the first case, followings are taken into account:

$$\alpha = 0.2, \quad \beta = 1, \quad f(t,x) = xe^{-t}, \quad u_0(x) = 0, \quad u_1(x) = \sqrt{2}\cos(\pi x).$$

For Case a, Un/controlled displacements and velocities are given by Figures 2-3.

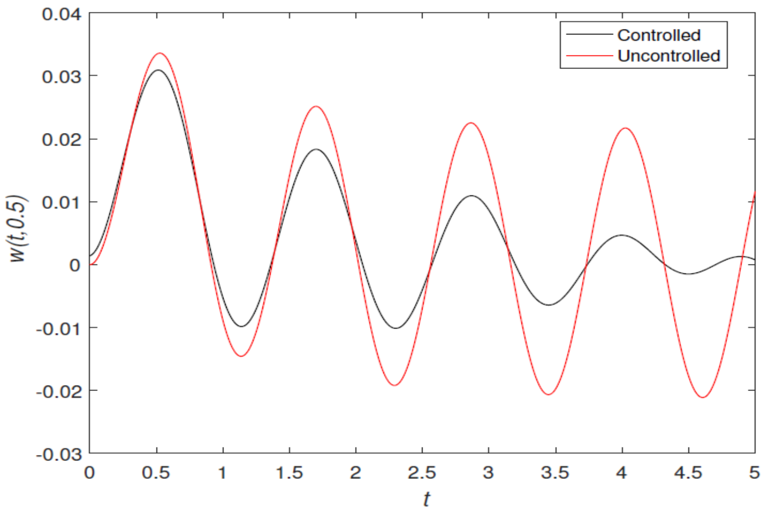

Figure 2. Uncontrolled and controlled displacements for Case a

In Figure 2, un/controlled displacements are plotted at the middle point of the canal system and undesired water waves are successfully damped out by means of applied control process.

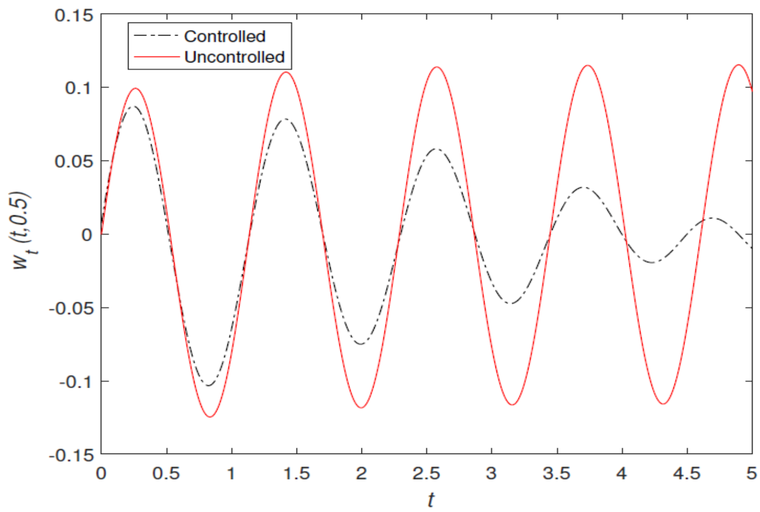

Figure 3. Uncontrolled and controlled velocities for Case a

Also, by observing Figure 3, same observation is obtained that the velocity of the undesired wave is suppressed as a result of control. For Case b, following coefficients and functions are considered:

$$\alpha = 0.5,\quad \beta = 0.25,\quad f(t,x) = \sqrt{2}\sin(\pi x),\quad u_0(x) = \sqrt{2}\sin(\pi x),\quad u_1(x) = \sqrt{2}\sin(\pi x).$$

Corresponding un/controlled displacements and velocities are plotted in Figures 4-5 for Case b.

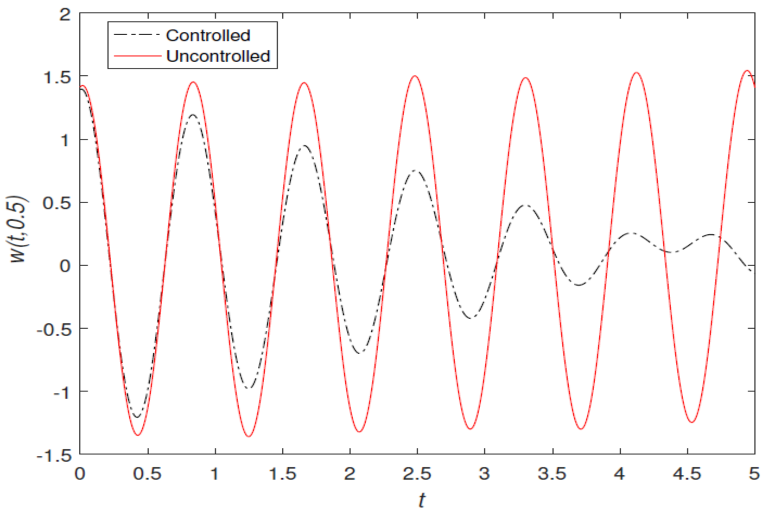

Figure 4. Uncontrolled and controlled displacements for Case b

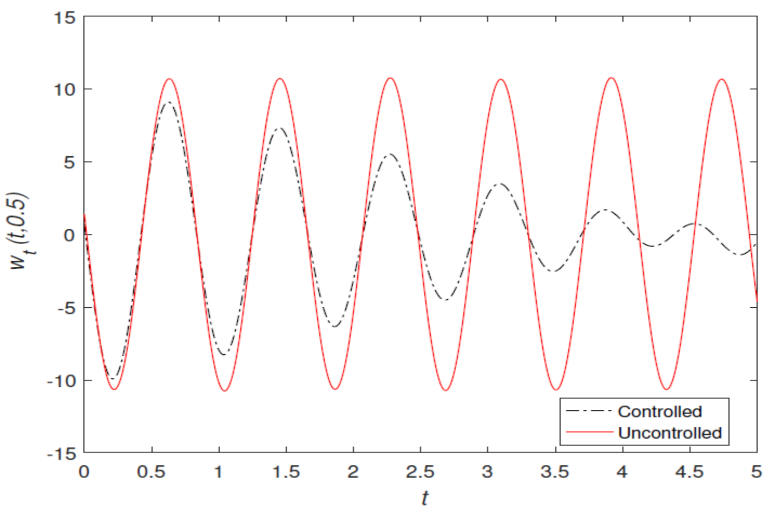

Figure 5. Uncontrolled and controlled velocities for Case b

As the value of the ϑ_3 decreases, corresponding canal depth increases and displacement and velocity of waves in the canal system are suppressed effectively for Case b. Let us give the dynamic response of the wave in the canal system as follows:

$$\mathscr{J}(u) = \int_0^1 [u^2(t_f,x) + u_t^2(t_f,x)]dx \tag{4.3}$$

and used canal depth accumulates over $(0, t_f)$:

$$\mathscr{J}(\hbar) = \int_0^{t_f} \hbar^2(t)dt. \tag{4.4}$$

The dynamic response of the wave in the canal system is given by table forms and it seemed from tables that as weighted coefficient ϑ_3 in canal depth control function decreases, dynamic response of the wave decreases due to an increasing in the value of canal depth control function. These observations show that introduced control actuation for water waves in a canal system is successful and effective.

Table 1. The values of $\mathscr{J}(w)$ and $\mathscr{J}(h)$ for different values of ϑ_3 in Case a.

ϑ_3	$\mathscr{J}_a(u)$	$\mathscr{J}_a(h)$
10^4	11.7	5.0 e-6
10^2	5.8	2.5 e-2
10^0	0.22	5.4
10^{-2}	0.12	1.24

Table 2. The values of $\mathscr{J}(w)$ and $\mathscr{J}(h)$ for $\beta = 0.01$ and different values of ϑ_3 in Case b.

ϑ_3	$\mathscr{J}_{0.01}(u)$	$\mathscr{J}_{0.01}(h)$
10^4	4.8 e-3	3.7 e-10
10^2	4.1 e-3	3.0 e-6
10^0	2.0 e-4	4.0 e-4
10^{-2}	4.7 e-5	5.8 e-4

5. Conclusion

In this study, optimal canal depth control problem is taken into account for suppressing excessive water waves in a canal system modeling by boundary control of a Boussinesq system. Optimal canal depth control function is obtained by means of a maximum principle, converts the optimal control problem to solving a system of nonlinear equations including state and adjoint variables, which are subjected to initial-boundary-terminal conditions. Optimal control function is gained without linearization of nonlinear term. Several numerical examples are given for indicating the effectiveness and capability of the introduced control algorithm. By observing the theoretical and applied results of this paper, it is concluded that introduced control algorithm is applicable for other nonlinear control systems for obtaining the optimal control function without linearization of nonlinear terms in the equation of motion. Hence, it is clear that the controlling of systems having nonlinear terms will be more realistic and control results will be more accurate and robust.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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